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# Dirty black holes: Entropy versus area

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Considerable interest has recently been expressed in the entropy versus area relationship for "dirty" black holes—black holes in interaction with various classical matter fields, distorted by higher derivative gravity, or infested with various forms of quantum hair. In many cases it is found that the entropy is simply related to the area of the event horizon: $S = kA_H/(4\ell_P^2)$. For example, the "entropy = (1/4) area" law *holds* for Schwarzschild, Reissner-Nordström, Kerr–Newman, and dilatonic black holes. On the other hand, the "entropy = (1/4) area" law *fails* for various types of (Riemann)$^n$ gravity, Lovelock gravity, and various versions of quantum hair. The pattern underlying these results is less than clear. This paper systematizes these results by deriving a general formula for the entropy:

$$S = \frac{kA_H}{4\ell_P^2} + \frac{1}{T_H}\int_\Sigma \{\varrho_L - \mathcal{L}_E\} K^\mu d\Sigma_\mu + \int_\Sigma s\ V^\mu d\Sigma_\mu .$$

($K^\mu$ is the timelike Killing vector, $V^\mu$ the four-velocity of a corotating observer.) If no hair is present the validity of the "entropy = (1/4) area" law reduces to the question of whether or not the Lorentzian energy density for the system under consideration is formally equal to the Euclideanized Lagrangian.

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## I. INTRODUCTION

For a variety of reasons, considerable interest has recently been expressed in the entropy versus area relationship for generic "dirty" black holes. (By a dirty black hole I mean a black hole possibly in interaction with various classical matter fields, possibly modified by higher curvature terms in the gravity Lagrangian [(Riemann)$^n$], or possibly infested with some version of quantum hair.) Some of these reasons are the following. (1) The low-energy point-field limit of string theory includes a dilaton field. The presence of the dilaton field modifies the Reissner-Nordström and Kerr-Newman black holes. (2) Despite the successes of string theory, a fully satisfactory theory of quantum gravity has proved elusive. Nevertheless, whatever the underlying quantum theory is, one would expect on general grounds that the low-energy theory should be describable by the Einstein-Hilbert action modified by higher-order terms in the Riemann tensor. (3) Quantum hair is a result of quantum fluctuations in the various low-energy quantum fields with which the black hole geometry interacts. As such, quantum hair is of interest independently of the details as to how one quantizes gravity.

In concordance with Bekenstein's original suggestion [1], in many cases it is found that the entropy is simply related to the area of the event horizon:

$$S = \frac{kA_H}{4\ell_P^2}. \tag{1}$$

On the other hand, in many other cases this simple relationship fails. The pattern, if any, underlying the various results is less than clear. Consider the following examples.

$S = (1/4)A$: The "entropy = (1/4) area" law *holds* for (1) Schwarzschild black holes [2,3], (2) Reissner-Nordström black holes [2,3], (3) Kerr-Newman black holes [2,3], (4) dilatonic black holes [4,5], (5) rotating dilatonic black holes [6], and (6) generic (Riemann)$^2$ gravity in $D = 4$ [7].

$S \neq (1/4)A$: The "entropy = (1/4) area" law *fails* for (1) specific examples of (Riemann)$^2$ gravity ($D \neq 4$) [8,9], (2) generic (Riemann)$^3$ gravity ($D$=4) [10], (3) specific examples of (Riemann)$^4$ gravity [11], (4) Lovelock gravity ($D \neq 4$) [12,13], and (5) various versions of quantum hair [14,15].

This paper systematizes these results by deriving a general formula for the entropy in terms of (1) the area of the event horizon, (2) the Lorentzian energy density in the classical fields surrounding the black hole, (3) the Euclideanized Lagrangian describing those fields, (4) the Hawking temperature, (5) the entropy density associated with the fluctuations [quantum hair, statistical hair], and finally (6) the metric. The derivation is particularly transparent, and the physical interpretation clear, if one temporarily restricts attention to the spherically symmetric case [zero angular momentum]. In terms of

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the shape function $b(r)$ and the anomalous redshift $\phi(r)$ the promised formula reads

$$S = \frac{kA_H}{4\ell_P^2} + \frac{1}{T_H}\int_\Sigma e^\phi\left(\varrho_L - \mathcal{L}_E\right)d^3r + \int_\Sigma \frac{s}{\sqrt{1-(b/r)}}d^3r. \qquad (2)$$

If no fluctuations are present, ($s = 0$, no quantum hair, no statistical-mechanics effects), the issue of the validity of the "entropy = (1/4) area" law reduces to the question of whether or not the Lorentzian energy density for the system under consideration is formally equal to the Euclideanized Lagrangian. As a rule of thumb, Lagrangians with quadratic kinetic terms satisfy the "entropy = (1/4) area" law. Lagrangians containing (curvature)$^2$ terms and higher typically do not.

The generalization to the case of nonzero angular momentum (axisymmetric geometry) is straightforward, requiring a little extra technical machinery in the form of the timelike and azimuthal Killing vectors, and a suitable invariant integration over the three-surface defined by taking a constant time slice.

The basic tools to be employed are the relationship between the thermodynamic functions and the partition function associated with the "Wick rotated" Euclidean section [3], and the Bardeen-Carter-Hawking mass theorem for geometries containing a timelike Killing vector [16]. The technical computations are actually relatively simple. Some care must be taken, however, in carefully navigating through a thicket of conceptual and definitional issues, and with various subtleties associated with the shift in signature.

Notation: Adopt units where $c \equiv 1$, but all other quantities retain their usual dimensionalities, so that in particular $G \equiv \ell_P/m_P \equiv \hbar/m_P^2 \equiv \ell_P^2/\hbar$. The metric signature is either $(-,+,+,+)$ or $(+,+,+,+)$ depending on context. The symbol $T$ will always denote a temperature. The stress-energy tensor will be denoted by $t^{\mu\nu}$, and its trace by $t$.

## II. LORENTZIAN TECHNIQUES

### The metric, horizon, and Hawking temperature

In any static spherically symmetric asymptotically flat spacetime the metric $g_L$ may without loss of generality be cast into the form

$$ds^2 = -e^{-2\phi(r)}\left[1 - b(r)/r\right]dt^2 + \frac{dr^2}{[1-b(r)/r]} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \qquad (3)$$

The function $b(r)$ will be referred to as the "shape function," while $\phi(r)$ will be referred to as the "anomalous redshift" [17]. Applying boundary conditions at spatial infinity permits one, without loss of generality, to set $\phi(\infty) = 0$ . Once this normalization of the asymptotic time coordinate is adopted, one may interpret $b(\infty)$ in terms of the asymptotic mass $b(\infty) = 2GM$. This metric has putative horizons at values of $r$ satisfying $b(r_H) = r_H$. Only the outermost horizon is of immediate interest.

The Hawking temperature of a black hole is given in terms of its surface gravity by $kT_H = (\hbar/2\pi)\kappa$. A brief computation yields [17]

$$\kappa = \frac{1}{2r_H}e^{-\phi(r_H)}\left[1 - b'(r_H)\right]. \qquad (4)$$

This formula receives most of its physical significance after $b'(r_H)$ and $\phi(r_H)$ are related to the distribution of matter by imposing the Einstein field equations.

The first two Einstein equations are [17]

$$b' = 8\pi G\,\rho\, r^2, \qquad (5)$$

$$\phi' = -\frac{8\pi G}{2}\frac{(\rho-\tau)r}{(1-b/r)}. \qquad (6)$$

Instead of imposing the third Einstein equation, observe that (as is usual) the third equation is redundant with the imposition of the conservation of stress energy. Thus one may take the third equation to be the anisotropic version of the Oppenheimer-Volkoff equation

$$\tau' = (\rho-\tau)[-\phi' + \tfrac{1}{2}\{\ln(1-b/r)\}'] - 2(p+\tau)/r. \qquad (7)$$

Taking $\rho$ and $\tau$ to be primary, one may formally integrate the Einstein equations:

$$b(r) = r_H + 8\pi G\int_{r_H}^{r}\rho\tilde r^2 d\tilde r = 2GM - 8\pi G\int_r^\infty \rho\tilde r^2 d\tilde r, \qquad (8)$$

$$\phi(r) = \frac{8\pi G}{2}\int_r^\infty \frac{(\rho-\tau)\tilde r}{(1-b/\tilde r)}d\tilde r. \qquad (9)$$

The transverse pressure $p$ is then determined via the anisotropic Oppenheimer-Volkoff equation. The Hawking temperature is

$$kT_H = \frac{\hbar}{4\pi r_H}\exp\left(-\frac{8\pi G}{2}\int_{r_H}^\infty \frac{(\rho-\tau)r}{(1-b/r)}dr\right) \times \left(1 - 8\pi G\,\rho_H\, r_H^2\right). \qquad (10)$$

Attempting to determine the entropy by integrating the thermodynamic relation $dM = T_H dS$ works well in simple cases, but in general quickly leads to an impenetrable morass. This is about as far as one can get using Lorentzian techniques. A different method of attack is called for.

## III. EUCLIDEAN TECHNIQUES

### A. The metric, horizon, and Hawking temperature

Another way of calculating the Hawking temperature is via the periodicity of the Wick rotated Euclidean-signature analytic continuation of the manifold [3]. Proceed by making the formal substitution $t \to -it$ to yield a fiducial Euclidean metric $g_E$:

$$ds_E^2 = +e^{-2\phi(r)}\,[1-b(r)/r]\,dt^2 + \frac{dr^2}{[1-b(r)/r]} + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \qquad (11)$$

In view of the $t$ independence of this metric, this Wick rotation preserves the mixed components of the Riemann and Ricci tensors:

$$[\text{Riemann}(g_E)]^\alpha{}_\beta{}^\gamma{}_\delta = [\text{Riemann}(g_L)]^\alpha{}_\beta{}^\gamma{}_\delta,$$

$$[\text{Ricci}(g_E)]^\alpha{}_\beta = [\text{Ricci}(g_L)]^\alpha{}_\beta,$$

$$R(g_E) = R(g_L).$$

As is usual, discard the entire $r < r_H$ region, retaining only the (analytic continuation of) that region that was outside the outermost horizon ( i.e., $r \geq r_H$). A Taylor series expansion about $r = r_H$ shows that the $(r,t)$ plane is a smooth two-dimensional manifold if and only if $t$ is interpreted as an angular variable with period

$$\tau_H = 4\pi r_H\, e^{\phi(r_H)}\, [1 - b'(r_H)]^{-1} = 2\pi/\kappa. \qquad (12)$$

Invoking the usual incantations [3], this periodicity in imaginary (Euclidean) time is interpreted as evidence of a thermal bath of temperature $kT_H = 1/\beta_H = \hbar/\tau_H$, so that the Hawking temperature is identified as

$$kT_H = \frac{\hbar}{4\pi r_H}\, e^{-\phi(r_H)}\, [1 - b'(r_H)]. \qquad (13)$$

This is the same result as was obtained by direct calculation of the surface gravity.

### B. Helmholtz free energy

The Helmholtz free energy of an arbitrary statistical-mechanical system is defined in terms of the partition function as

$$F = -kT\, \ln Z. \qquad (14)$$

For the particular case at hand, one writes the partition function as [3]

$$Z = \int \mathcal{D}(g,\Phi)\, \exp[-I_E(g,\Phi)/\hbar]. \qquad (15)$$

Here $\Phi$ denotes the generic class of matter fields: fermions, gauge bosons, Higgs particles, axions, dilatons, etc. The range of integration runs over all possible matter field configurations, and over some suitable class of Euclidean metrics. There is some confusion as to the class of Euclidean metrics which should be integrated over in general, but for the present problem it is sufficient to integrate over all Euclidean metrics $g$ that have the same topology as the fixed fiducial metric $g_E$, are asymptotically flat, and are periodic in imaginary time with period $\tau_H = 2\pi/\kappa = \hbar\beta = \hbar/kT_H$ [15]. By adopting background field techniques one can define an exact decomposition

$$Z = \exp[-I_E(g_E,\Phi_0)/\hbar]\; Z_{\text{fluctuations}}. \qquad (16)$$

Here $g_E$ is the fiducial background metric, $\Phi_0$ denotes the background matter fields, and $Z_{\text{fluctuations}}$ denotes the contributions to the partition function coming from quantum fluctuations around the fiducial background — these fluctuations can be described by the usual loop expansion.

(Anyone who is worried about the precise class of metrics to integrate over, or unhappy about invoking background field techniques can go straight from the definition of the partition function to the semiclassical limit. Doing so yields an approximation

$$Z \approx \exp[-I_E(g_E,\Phi_0)/\hbar]\; Z_{\text{one-loop}}. \qquad (17)$$

This version of the semiclassical limit handles only one-loop effects in linearized gravitational and matter fluctuations.)

Adopting either of these decompositions one may write

$$F = \frac{kTI_E}{\hbar} + F_{\text{fluctuations}}. \qquad (18)$$

The various contributions to the Euclidean action can be grouped into three distinct terms

$$I_E(g_E,\Phi_0) = -\frac{1}{8\pi G}\int_{\partial\Omega}[K]\sqrt{{}^3g_E}\; d^3x - \frac{1}{16\pi G}\int_\Omega R\sqrt{g_E}\; d^4x + \int_\Omega \mathcal{L}_E\sqrt{g_E}\; d^4x. \qquad (19)$$

These various terms are: (1) the gravitational surface term, to be integrated over the three-surface at spatial infinity (topology $S^2 \times S^1$), (2) the Einstein-Hilbert term, to be integrated over the entire Euclidean manifold (topology $S^2 \times D^1$), and (3) the Euclideanized "matter" Lagrangian. Higher-order geometrical terms [e.g., $(\text{Riemann})^2$], if present, are lumped into the "matter" Lagrangian.

The boundary term is easily evaluated:

$$-\frac{1}{8\pi G}\int_{\partial\Omega}[K]\sqrt{{}^3g_E}\; d^3x = -\frac{1}{8\pi G}\tau_H(-4\pi GM) = +\frac{M\tau_H}{2} = \frac{\hbar\beta M}{2}. \qquad (20)$$

To evaluate the Einstein-Hilbert term, one invokes the Einstein field equation $G_{\mu\nu} = 8\pi G\, t_{\mu\nu}$. In conformance with the conventions already established, the Euclidean stress energy tensor is defined by setting its mixed components equal to the mixed components of the Lorentzian stress-energy: $(t_E)^\mu{}_\nu = (t_L)^\mu{}_\nu$. Consequently, for the trace, $t_E = t_L$.

The subscripts $(E,L)$ will often be omitted if no confusion can arise. Thus

$$-\frac{1}{16\pi G}\int_\Omega R\sqrt{g_E}\,d^4x = -\frac{1}{16\pi G}\int_\Omega(-8\pi Gt)\sqrt{g_E}\,d^4x = +\frac{1}{2}\int_\Omega t\sqrt{g_E}\,d^4x$$
$$= +\frac{1}{2}\int_\Sigma t\,e^{-\phi}4\pi r^2 dr\,\tau_H = +\frac{\hbar\beta}{2}\int_\Sigma e^{-\phi}t\,d^3r. \qquad (21)$$

Here $\Sigma$ denotes a constant time hypersurface (topology $S^2\times\Re^+$). Similarly, the matter action can be rewritten as

$$\int_\Omega \mathcal{L}_E\sqrt{g_E}\,d^4x = \hbar\beta\int_\Sigma e^{-\phi}\mathcal{L}_E\,d^3r. \qquad (22)$$

Finally, the fact that the Helmholtz free energy is an extensive quantity justifies the introduction of a free-energy density associated with the fluctuations. This free-energy density $f$ is defined by

$$F_{\text{fluctuations}}\,\tau_H = \int_\Omega f\sqrt{g_E}\,d^4x. \qquad (23)$$

Equivalently,

$$F_{\text{fluctuations}} = \int_\Sigma e^{-\phi}f\,d^3r. \qquad (24)$$

Combining everything,

$$F = \frac{M}{2} + \int_\Sigma e^{-\phi}\left\{\frac{t}{2}+\mathcal{L}_E+f\right\}d^3r. \qquad (25)$$

### C. Bardeen-Carter-Hawking mass theorem

For a static spacetime, the existence of a timelike Killing vector, together with the use of the Einstein field equations, implies [16]

$$M = \frac{\kappa A_H}{4\pi G} - \int_\Sigma \{2t_\mu{}^\nu - t\delta_\mu{}^\nu\}K^\mu d\Sigma_\nu. \qquad (26)$$

This is a purely geometrodynamic statement in terms of the surface gravity, the area of the event horizon, and the stress-energy tensor. In view of the conventions adopted herein, this result holds equally well in Lorentzian or Euclidean signature. To keep subsequent formulas more transparent, I have reversed the orientation of the hypersurface $\Sigma$ relative to that adopted by Bardeen, Carter, and Hawking [16]. Thus, with my conventions, $K^\mu d\Sigma_\mu \mapsto +e^{-\phi}d^3x$ for the case of spherical symmetry. Using the relationship between surface gravity and the Hawking temperature, and using the explicit forms of the metric and the timelike Killing vector, permits this to be rewritten as

$$M = \frac{kT_HA_H}{2\ell_P^2} + \int_\Sigma e^{-\phi}\{2\rho+t\}\,d^3r. \qquad (27)$$

Resubstituting into the formula for the Helmholtz free energy in such a way as to eliminate the integral over the trace of the stress-energy tensor yields

$$F = M - \frac{kT_HA_H}{4\ell_P^2} + \int_\Sigma e^{-\phi}\{\mathcal{L}_E+f-\rho\}d^3r. \qquad (28)$$

### D. Thermodynamic relations

By definition $F=U-TS$. For an asymptotically flat geometry the internal energy $U$ is defined to be the asymptotic mass $M$. Eliminating $F$

$$S = \frac{kA_H}{4\ell_P^2} + \frac{1}{T_H}\int_\Sigma e^{-\phi}\{\rho-\mathcal{L}_E-f\}d^3r. \qquad (29)$$

This is almost the required form. To proceed, note that the $\rho$ occurring above is the *total* energy density, and that the way things have been defined, energy density can arise either from the classical matter fields surrounding the black hole, or from the quantum fluctuations, or both. This justifies a split:

$$\rho = \varrho_L + \varrho_f. \qquad (30)$$

But the energy density in the fluctuations, and the Helmholtz free-energy density in the fluctuations, are related by $f=\varrho_f - Ts$, where $s$ is the local entropy density in the fluctuations and $T$ is the *local* temperature. Because the whole system is at thermal equilibrium at a redshifted temperature $T_H$, the local temperature varies as

$$T = \frac{T_H}{\sqrt{g_{tt}}} = \frac{T_H\,e^{+\phi}}{\sqrt{1-(b/r)}}. \qquad (31)$$

Resubstituting everything yields the final result for the entropy:

$$S = \frac{kA_H}{4\ell_P^2} + \frac{1}{T_H}\int_\Sigma e^{-\phi}\{\varrho_L-\mathcal{L}_E\}d^3r + \int_\Sigma \frac{s}{\sqrt{1-(b/r)}}d^3r. \qquad (32)$$

This is a very pleasing result which accounts for all known violations of the "entropy = (1/4) area" law in a unified manner. Furthermore, the result immediately generalizes: instead of considering quantum fluctuations of the gravitational and matter fields I could just as easily have dumped a few particles outside the event horizon of the black hole and proceeded to do ordinary statistical mechanics in a fixed background geometry. Consequently, the fluctuations discussed in this paper can be thought of as being ordinary statistical-mechanics fluctuations as easily as quantum fluctuations. The entropy formula derived above applies equally well to dirty black holes, to classical field configurations, and to stars. (Subject to the present constraint of spherical symmetry.) Compare this to the discussion by Gibbons and Hawking [3]. Gibbons and Hawking discuss electrovac black holes and perfect fluid stars. There is no need in the present formulation for the effect of the fluctuations, or for the effect of the classical matter fields, to be constrained to mimic a perfect fluid — any generic stress-energy tensor will suffice.

In adding statistical-mechanical hair to the system, one may also wish to include discussion of the effect of the chemical potential. There are two compensating modifications. First, note that for the system as a whole $F = M - TS - \mu_\infty N$. Here $\mu_\infty$ is the chemical potential as measured at asymptotic infinity, and $N$ is the total number of particles. Second, for the statistical-mechanical hair, $f = \rho_f - Ts - \mu n$. Here $\mu$ is the locally measured chemical potential, and $n$ is the local number density. Because the whole system is taken to be in chemical equilibrium, the local chemical potential must be a constant up to a redshift factor: $\mu = \mu_\infty/\sqrt{g_{tt}}$. The putative additional contribution to the entropy is proportional to

$$\mu_\infty N - \int_\Sigma e^{-\phi}\, \mu\, n\, d^3r = \mu_\infty N - \mu_\infty \int_\Sigma n\sqrt{g_3}\, d^3x = 0\ . \tag{33}$$

The formula for the entropy is not disturbed by the addition of a chemical potential to the system.

Another immediate generalization is that to an arbitrary static, asymptotically flat, but not spherically symmetric spacetime. The metric is

$$ds^2 = -e^{-2\Psi}dt^2 + g_{ij}dx^i dx^j \tag{34}$$

and the entropy becomes

$$S = \frac{kA_H}{4\ell_P^2} + \frac{1}{T_H}\int_\Sigma e^{-\Psi}\{\varrho_L - \mathcal{L}_E\}\sqrt{g_3}d^3x + \int_\Sigma s\sqrt{g_3}d^3x. \tag{35}$$

A subtlety is that because I have not placed any energy conditions on the stress tensor one cannot now invoke the usual proof that the Hawking temperature is a constant over the horizon. Instead, constancy of the Hawking temperature over the horizon is now enforced by the assumption that the system is in thermal equilibrium.

A striking feature of the entropy formula is the existence of an anomalous contribution associated with the interplay between certain types of classical field and the existence of the heat bath. Explicitly,

$$S_{\text{anomalous}} = \frac{1}{T_H}\int_\Sigma e^{-\phi}\{\varrho_L - \mathcal{L}_E\}d^3r = \frac{k}{\hbar}\int_\Omega \{\varrho_L - \mathcal{L}_E\}\sqrt{g_E}d^4x. \tag{36}$$

In many cases this anomalous entropy vanishes. In many other cases it does not.

## IV. THE ANOMALOUS ENTROPY

### A. Lagrangians containing only first-order time derivatives

#### *1. Quadratic kinetic energy*

Consider a Lorentzian Lagrangian that is quadratic in first-order time derivatives. Such a Lagrangian may, without loss of generality, be cast in the form

$$\mathcal{L}_L = \frac{1}{2}g_{ab}(\Phi)\ \dot\Phi^a\dot\Phi^b - V(\Phi). \tag{37}$$

The Lorentzian energy density is

$$\varrho_L = \pi_a\dot\Phi^a - \mathcal{L}_L = \frac{1}{2}g_{ab}(\Phi)\ \dot\Phi^a\dot\Phi^b + V(\Phi). \tag{38}$$

On the other hand, the Euclideanized Lagrangian is defined by $\mathcal{L}_E \equiv -\mathcal{L}_L(t \mapsto -it)$. For the case under consideration,

$$\mathcal{L}_E = \frac{1}{2}g_{ab}(\Phi)\ \dot\Phi^a\dot\Phi^b + V(\Phi) = \varrho_L. \tag{39}$$

Consequently, the anomalous entropy vanishes, and modulo the effects of quantum and statistical hair, "entropy = (1/4) area."

Examples of this behavior are the electrovac black holes (Schwarzschild, Reissner-Nordström, and Kerr-Newman [2,3]), as well as the various variations on the theme of the dilatonic black hole [4–6]. This observation also applies to the Lagrangian of the standard model of particle physics, modulo minor technical fiddles with the Fermi fields. The recent general discussion of the "entropy = (1/4) area" law by Moss [18] took the quadratic nature of the kinetic terms as a basic assumption. Consequently, that analysis failed to detect the anomalous $\varrho_L - \mathcal{L}_E$ term.

#### *2. Generic kinetic energy*

Still restricting attention to Lagrangians that are first order in time derivatives, suppose the kinetic energy term to be generic (subject only to time reversal invariance). Then suppressing field indices one may write

$$\mathcal{L}_L = K(\dot\Phi^2, \Phi) - V(\Phi). \tag{40}$$

The Lorentzian energy density is

$$\varrho_L = \pi\dot\Phi - \mathcal{L}_L = K'(\dot\Phi^2,\Phi)\ [2\dot\Phi]\ \dot\Phi - K(\dot\Phi^2,\Phi) + V(\Phi). \tag{41}$$

On the other hand, the Euclideanized Lagrangian is

$$\mathcal{L}_E = -K(-\dot\Phi^2, \Phi) + V(\Phi). \tag{42}$$

In the difference, $\varrho_L - \mathcal{L}_E$, the potential energy cancels

$$\varrho_L - \mathcal{L}_E = [2\dot\Phi^2]\ K'(\dot\Phi^2, \Phi) - K(\dot\Phi^2, \Phi) + K(-\dot\Phi^2, \Phi). \tag{43}$$

This looks like a mess. Fortunately, *if* the field $\Phi$ is a physical field, one can use the static nature of the spacetime to deduce $\dot\Phi = 0$. In this case

$$\varrho_L - \mathcal{L}_E = 0 - K(0, \Phi) + K(0, \Phi) = 0, \tag{44}$$

and the "entropy = (1/4) area" law follows.

### B. Lagrangians containing arbitrary-order time derivatives

Independent of the order of time derivatives appearing in the Lagrangian, the stress-energy tensor may be defined by

$$t^{\mu\nu}(x) = -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}(x)}\left[\int_\Omega \sqrt{-g}\ \mathcal{L}_L\right]. \tag{45}$$

More explicitly,

$$t^{\mu\nu} = -2\frac{\delta \mathcal{L}_L}{\delta g_{\mu\nu}} + g^{\mu\nu}\mathcal{L}_L. \tag{46}$$

Here the symbol $\delta\mathcal{L}_L/\delta g$ denotes $\partial\mathcal{L}_L/\partial g$ plus whatever terms arise from integrating by parts. Now $\varrho_L = t^{\hat 0\hat 0} = t^{tt}/|g^{tt}| = t^{tt}|g_{tt}|$, so

$$\varrho_L = -2g_{tt}\frac{\delta\mathcal{L}_L}{\delta g_{tt}} - \mathcal{L}_L\ . \tag{47}$$

If one is interested in only physical fields, the static nature of the spacetime implies, via the vanishing of all time derivatives, $\mathcal{L}_E \equiv -\mathcal{L}_L(t \mapsto -it) = -\mathcal{L}_L$. Consequently,

$$\varrho_L - \mathcal{L}_E = -2g_{tt}\frac{\delta\mathcal{L}_L}{\delta g_{tt}}. \tag{48}$$

The generic breakdown of the "entropy = (1/4) area" law in higher-order gravity theories is thus manifest. Typically the variation with respect to $g_{tt}$ will produce terms such as $R_{t\bullet t\bullet}$ or such as $R_{t\bullet\bullet\bullet}R_{t\bullet\bullet\bullet}$. Without the presence of an accidental zero, the failure of the "entropy = (1/4) area" law follows. In agreement with these observations, the law fails for (Riemann)$^2$ gravity ($D \neq 4$) [8,9] (Riemann)$^3$ gravity ($D = 4$) [10], (Riemann)$^4$ gravity [11], and Lovelock gravity ($D \neq 4$) [12,13].

Accidental zeros of the type alluded to above preserve the "entropy = (1/4) area" law for (Riemann)$^2$ gravity in $D = 4$ $[\mathcal{L} = R + a_1 R^2 + a_2 R_{\mu\nu}R^{\mu\nu} + a_3 R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}]$. To see this, note that in four dimensions the Gauss-Bonnet formula for the Euler characteristic allows one to rewrite $\int(\text{Riemann})^2$ as a topological invariant plus a linear combination of $\int(\text{Ricci})^2$ and $\int R^2$. This system has been analyzed by Whitt [7]. The modifications to the equations of motion are proportional to the Ricci tensor, with the result that the Schwarzschild solution remains a solution of the (Riemann)$^2$ system.

### C. Topological Lagrangians

If the Lagrangian contains a topological piece, its contribution to the anomalous entropy can be calculated trivially. For instance, in $D = 4$ consider the Gauss-Bonnet and Pontrjagin terms

$$\mathcal{L}_L = \frac{\alpha}{32\pi^2}\{R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 2R^{\alpha\beta}R_{\alpha\beta} + R^2\} + \frac{\beta}{8\pi^2}\{F^{\mu\nu}\tilde F_{\mu\nu}\}. \tag{49}$$

For such topological terms the energy density $\varrho_L$ is zero by definition. The anomalous entropy reduces to

$$S_{\text{anomalous}} = -\frac{1}{T_H}\int_\Sigma e^{-\phi}\mathcal{L}_E\ d^3r = -\frac{k}{\hbar}\int_\Omega \mathcal{L}_E\ \sqrt{g_E}d^4x = -k\{\alpha\chi + \beta p\}. \tag{50}$$

This is a simple fixed offset to the entropy generated by the Euler characteristic and Pontrjagin index of the manifold. This result is not exactly surprising and could have been easily deduced from the original definition of the Helmholtz free energy. If $Z_0$ denotes the partition function excluding topological effects $F = -kT\ \ln Z = kT\{\alpha\chi + \beta p\} - kT\ \ln Z_0$.

## V. AXISYMMETRIC SPACETIMES

The discussion up to the present has, for simplicity, only discussed the spherically symmetric case. To relax this constraint to merely require axial symmetry is not particularly difficult. (One needs to do this in order to be able to discuss black holes possessing angular momentum.)

In a stationary axisymmetric asymptotically flat spacetime there is a unique translational Killing vector $K^\mu$ which is timelike and normalized to $K^\mu K_\mu = -1$ near spatial infinity. By abuse of language, this is often referred to as the timelike Killing vector. There is also a unique rotational Killing vector $\tilde K^\mu$ normalized by demanding that its orbits be closed curves with parameter length $2\pi$ [16].

The fundamental formula for the Helmholtz free energy in terms of the Euclidean action is recast as

$$F = \frac{M}{2} + \int_\Sigma \left\{\frac{t}{2} + \mathcal{L}_E + f\right\}\ K^\mu d\Sigma_\mu. \tag{51}$$

Here $\Sigma$ is a spacelike hypersurface, tangent to the azimuthal Killing vector $\tilde K$. The induced three-metric has volume form $d\Sigma_\mu$. By construction, $\tilde K^\mu d\Sigma_\mu = 0$.

On the other hand, one form of the Bardeen-Carter-Hawking mass formula now reads [16]

$$M = \frac{\kappa A_H}{4\pi G} + 2\Omega_H J_H - \int_\Sigma \{2t_\mu{}^\nu - t\delta_\mu{}^\nu\}\ K^\mu d\Sigma_\nu. \tag{52}$$

The extra contribution involves the angular momentum of the black hole $J_H$, and the angular velocity of the event horizon $\Omega_H$. The angular momentum of the black hole is defined by

$$J_H = +\frac{1}{8\pi G}\int_{\text{horizon}} \tilde{K}^{\mu;\nu} d\Sigma_{\mu\nu}. \tag{53}$$

To proceed, it is advantageous to further massage the term $\int t_\mu{}^\nu K^\mu d\Sigma_\nu$. Note that the stress energy surrounding the black hole should be rotating "with" the black hole. This notion may be formalized by requiring the stress energy tensor to possess a timelike unit eigenvector $V^\mu$, with corresponding eigenvalue $\rho$. Explicitly

$$t^\mu{}_\nu V^\nu = -\rho V^\mu. \tag{54}$$

This, in fact, defines the comoving energy density. An observer with four-velocity $V^\mu$ sees no energy flux. By the assumed axial symmetry the four-velocity must be of the form

$$\lambda V^\mu = K^\mu + \omega \tilde{K}^\mu. \tag{55}$$

($\lambda$ is a normalizing factor.) This indicates that, as expected, the stress energy surrounding the hole is rotating "with" it. The value of formalizing these notions in this indirect manner is that one is no longer restricted to the case of a perfect fluid. (cf. [3,16].) For the discussion at hand, one is interested only in a system in internal equilibrium. Hence one sets $\omega = \Omega_H$. (Everything rotates at the same angular velocity throughout the system.) Repeatedly using the fact that $\tilde{K}^\mu$ is tangent to the hypersurface $\Sigma$,

$$\begin{aligned}\int_\Sigma t_\mu{}^\nu K^\mu d\Sigma_\nu = \int_\Sigma t_\mu{}^\nu(\lambda V^\mu - \Omega_H \tilde{K}^\mu) d\Sigma_\nu &= \int_\Sigma (-\lambda\rho V^\nu - \Omega_H t_\mu{}^\nu \tilde{K}^\mu) d\Sigma_\nu \\ &= -\int_\Sigma \rho(K^\mu + \Omega_H \tilde{K}^\mu) d\Sigma_\mu - \Omega_H \int_\Sigma t_\mu{}^\nu \tilde{K}^\mu d\Sigma_\nu \\ &= -\int_\Sigma \rho K^\mu d\Sigma_\mu - \Omega_H J_{\text{matter}}. \end{aligned} \tag{56}$$

The angular momentum of the matter, $J_{\text{matter}}$, is defined in the usual manner [16]

$$J_{\text{matter}} = +\int_\Sigma t_\mu{}^\nu \tilde{K}^\mu d\Sigma_\nu \,. \tag{57}$$

For the case of interest (internal equilibrium, $\omega = \Omega_H$), the Bardeen-Carter-Hawking mass theorem now reads

$$M = \frac{\kappa A_H}{4\pi G} + 2\Omega_H J_{\text{total}} + \int_\Sigma \{2\rho + t\}\, K^\mu d\Sigma_\mu. \tag{58}$$

As was previously also the case, one can eliminate the integral over the trace of the stress energy. Combining the above

$$\begin{aligned} F = M - \frac{kT_H A_H}{4\ell_P^2} - \Omega_H J_{\text{total}} \\ + \int_\Sigma \{(\mathcal{L}_E + f) - \rho\} K^\mu d\Sigma_\mu. \end{aligned} \tag{59}$$

The relationship between the Helmholtz free energy and the other thermodynamic quantities is also modified. Including the effects of angular momentum and a chemical potential, $F = M - TS - \Omega_H J_{\text{total}} - \mu_\infty N$. Here $\Omega_H$ is again promoted to the status of the angular velocity of the entire heat bath — not just the angular velocity of the horizon. Eliminating $F$,

$$S = \frac{kA_H}{4\ell_P^2} - \frac{\mu_\infty N}{T_H} + \frac{1}{T_H}\int_\Sigma \{\rho - (\mathcal{L}_E + f)\,\} K^\mu d\Sigma_\mu. \tag{60}$$

To proceed, repeat the previous trick of splitting the total energy density into contributions from the fields and from the fluctuations: $\rho = \varrho_L + \varrho_f$. The energy density in the fluctuations, and the Helmholtz free-energy density in the fluctuations, being local quantities, are still related by $f = \varrho_f - Ts - \mu n$. Because the whole system is at thermal equilibrium, the local temperature and local chemical potential are redshifted by the normalization parameter $\lambda = ||K + \Omega_H \tilde{K}||$:

$$T = \frac{T_H}{\lambda}, \qquad \mu = \frac{\mu_\infty}{\lambda}. \tag{61}$$

Then

$$\begin{aligned}\int_\Sigma \{\varrho_f - f\} K^\mu d\Sigma_\mu = \int_\Sigma \{Ts + \mu n\} K^\mu d\Sigma_\mu = \int_\Sigma \{T_H s + \mu_\infty n\}(K^\mu/\lambda) d\Sigma_\mu &= \int_\Sigma \{T_H s + \mu_\infty n\} V^\mu d\Sigma_\mu \\ &= T_H \int_\Sigma s V^\mu d\Sigma_\mu + \mu_\infty N. \end{aligned} \tag{62}$$

Resubstituting everything yields the final result for the entropy

$$S = \frac{kA_H}{4\ell_P^2} + \frac{1}{T_H}\int_\Sigma \{\varrho_L - \mathcal{L}_E\} K^\mu d\Sigma_\mu + \int_\Sigma sV^\mu d\Sigma_\mu. \tag{63}$$

This final result now applies to stationary asymptotically flat axisymmetric spacetimes. The additional technical machinery required to go beyond spherical symmetry boils down to the introduction of appropriate volume forms on the constant time hypersurface $\Sigma$, together with a suitable definition of the energy density in terms of a corotating observer.

The present version of the analysis also makes it clear that there is nothing special about $(3+1)$ dimensions. The entropy formula continues to hold — with suitably defined volume forms — in arbitrary dimensionality.

## VI. DISCUSSION

In summary, this paper has exhibited a general formalism for calculating the entropy of stationary axisymmetric asymptotically flat dirty black holes. The formalism serves to tie together and explain in a unified manner a number of otherwise seemingly accidental results scattered throughout the literature. The total entropy can be cleanly separated into contributions from: (1) the horizon, (2) quantum or statistical hair, and (3) an anomalous term.

The anomalous entropy is

$$S_{\text{anomalous}} = \frac{1}{T_H}\int_\Sigma \{\varrho_L - \mathcal{L}_E\} K^\mu d\Sigma_\mu = \frac{k}{\hbar}\int_\Omega \{\varrho_L - \mathcal{L}_E\}\sqrt{g_E}\, d^4x. \tag{64}$$

It is certainly a peculiar object, depending as it does on both the temperature and on the classical background fields surrounding the black hole. The vanishing or non-vanishing of this term correctly reproduces all known violations and all known verifications of the naive "entropy = (1/4) area" law.

The effects of various types of Lagrangian can be summarized by a rule of thumb: Lagrangians with quadratic kinetic terms do not contribute to the anomalous entropy. Lagrangians containing $(\text{curvature})^2$ terms and higher typically do contribute to the anomalous entropy.

This suggests the following physical picture. Start with the standard model Lagrangian $\mathcal{L}_0$. It does not contribute to the anomalous entropy. Integration over the quantum fluctuations yields some quantum hair — call it $s_0$. Now introduce some energy scale $\Lambda$ and integrate out the fast modes. This yields some effective Lagrangian $\mathcal{L}_{\text{eff}}(\Lambda)$. Introducing this effective Lagrangian into the partition function and integrating out the remaining slow modes will yield modified quantum hair, call it $s_{\text{eff}}(\Lambda)$. But the effective Lagrangian will contain $(\text{curvature})^2$ terms and higher — and these terms will contribute to the anomalous entropy. Now the total entropy should not depend on where one places the division $(\Lambda)$ between fast and slow modes (after all, it is the same physical theory no matter how one divides it up). This suggests that occurrence of anomalous entropy is to a large extent due to the use of effective Lagrangians, and that moving the division line between fast and slow modes merely shifts entropy to and fro between the anomalous term and the quantum fluctuations. From this point of view, all known violations of the "area = (1/4) entropy" law can be interpreted as probing the effect of otherwise uncontrollable high-frequency quantum fluctuations by resorting to the use of some low-energy effective Lagrangian. This physical picture has implications external to the topic of black hole physics insofar at it indicates the existence of a general scheme for associating a quantum-mechanical entropy with an effective Lagrangian. Naturally, if the fundamental theory contains higher curvature terms, some of the anomalous entropy should be thought of as intrinsic.

As to the future, I would really like to see an explanation for this result phrased completely in terms of Lorentzian signature techniques. The Hawking temperature is already well understood from a purely Lorentzian point of view, and a similar understanding of the entropy is clearly desirable.

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