Scattering of discrete states in two-dimensional open string field theory

Branko Urošević
Physics Department, Brown University, Providence, Rhode Island 02912
(Received 9 August 1993)

This is the second in a series of papers devoted to open string field theory in two dimensions. In this paper we aim to clarify the origin and the role of discrete physical states in the theory. To this end, we study interactions of discrete states and generic tachyons. In particular, we discuss at length four point amplitudes. We show that the behavior of the correlation functions is governed by the number of generic tachyons involved and values of the kinematic invariants *s*, *t*, and *u*. The divergence of certain classes of correlators is shown to be the consequence of the fact that certain kinematic invariants are nonpositive integers in that case. Explicit examples are included. We check our results by the standard conformal technique.

PACS number(s): 11.17.+y

I. INTRODUCTION

This is the second in a series of papers devoted to open string field theory (SFT) in two dimensions. In the previous paper, Ref. [1], we have constructed Witten-like vertices for open string field theory and discussed the quantization of the theory in Siegel's gauge ($b_0=0$). We have used, then, the corresponding Feynman rules to calculate tachyon-tachyon scattering amplitudes. Our results were presented in the form of a sum over poles corresponding to tachyonic and (discrete) excited intermediate states. Residues of these poles were shown to match the results of Bershadsky and Kutasov, Ref. [2]. For a summary of our notation and conventions the reader may consult Ref. [1], Sec. II.

It is well known that in addition to the massless tachyon, string theory in two dimensions (2D) also has discrete higher string states. The presence of discrete states (DS's) in the spectrum of two-dimensional strings, i.e., states which are physical only for some particular values of momenta, has been known for quite some time now. They were first discovered in the matrix model approach (Refs. [3,4]), and rediscovered in the Liouville approach (Refs. [5–11]). In Ref. [5], it was shown that Becchi-Rouet-Stora-Tyutin (BRST) cohomology $H^{(*)}$ of the two-dimensional string is nontrivial for more than one ghost number. This is in sharp contrast to 26-dimensional string theory, where only $H^{(1)}$ is nontrivial (Ref. [12]). This novel feature of 2D string theory has its origin in the nontrivial background charge of the system. The question is then how the discrete states participate in the field theory. For example, it was even suggested recently (Ref. [13]) that in order to recover the results of Ref. [5] from the field-theoretic point of view, one would need to modify Witten's classical action so as to include more than one ghost number.

In this paper we would like to clarify the origin and the role of discrete states in the string field theory. We first show (Sec. II) that one can stay with the original Witten action. The modification is redundant since the proper object to consider in investigation of discrete states is a *gauge-fixed* action. In particular, choosing Siegel's gauge is then equivalent to restricting ourselves to the relative cohomology of the first-quantized BRST operator. To probe the dynamics of discrete states we study their scattering with generic tachyons (T). The naive counting of the degrees of freedom for string excitations in two dimensions gives $2-2=0$, so that the excited (discrete) states form a set of measure zero with respect to the functional integral measure. It is natural to expect, therefore, that the correlators involving these states may not be well defined. Indeed, earlier calculations of the on-shell scattering amplitudes have indicated that DS-DS amplitudes are intrinsically divergent for the closed (Ref. [14]) as well as open strings (Ref. [15]). It is then of certain importance to consider the calculation of the amplitudes in SFT. In Sec. V, we present a detailed analysis of the four-string scattering in the case when all four of the external states are from $H^{(1)}$. Our expansion is found in the form of a sum over poles. We have a simple criterion which allows us to determine whether an amplitude is well defined, diverges, or vanishes. Namely, we consider the three kinematic invariants *s*, *t*, and *u* (see Sec. IV). If some of them are integers we call the corresponding channels *degenerate*. A degenerate channel can either diverge (if the kinematic invariant is nonpositive) or vanish (if it is positive). We provide examples which illustrate how this works. Clearly, the less generic tachyons we have, the more channels will degenerate. In fact, while A_{TTTD} are always well defined and contain an infinite number of physical poles (which is the behavior one expects from string amplitudes), A_{TTDD} amplitudes have a more subtle behavior. Namely, there are three different subclasses of amplitudes of that type: A^{+}_{TTDD}, A^{-}_{TTDD}, and $A^{\rm deg}_{TTDD}$. The superscript denotes the sign of the degenerate kinematic invariant. The correlators of the first class are well defined and have a finite number of poles in two channels. Amplitudes of the second class diverge and have infinitely many poles in two channels. Class $A^{\rm deg}_{TTDD}$ is divergent and degenerates in all three channels. The same is true for the amplitudes of the type A_{DDDD}.

The paper is organized as follows. In Sec. II we show how discrete states originate from the field-theoretic point of view. In Sec. III we discuss three-point functions in the theory and show that they agree with the first-quantized results. In Sec. IV, we consider four-tachyon correlation functions A_{++--}. This is an example from which one can learn some important lessons. In particular, we there encounter, for the first time, the degeneracy of an amplitude. The amplitude is shown to vanish. Section V represents the main course of the paper. There, we present a classification of the four-point amplitudes with respect to their dynamical properties. We show that divergence occurs when, and only when, certain kinematic invariants are nonpositive integers. In particular, in Sec. V A we discuss A_{TTTD} class of amplitudes. Section V B is devoted to the analysis of the A_{TTDD} class. We conclude the section by proving that the amplitudes of the class A_{DDDD} always diverge (Sec. V C). We check our conclusions using the conformal technique. Finally, in Sec. VI we present a brief overview of the main results and outline some of the open problems and possible directions of future work.

II. DISCRETE STATES FROM STRING FIELD THEORY

As mentioned in the Introduction, the presence of DS's in the physical spectrum of two-dimensional strings is one of its most intriguing features. Their relevance was first discovered in context of matrix models (Refs. [3,4]). In the continuum approach, discrete studies are part of the spectrum which correspond to string excitations. They survive the gauge fixing due to the nontrivial background charge. In general, BRST analysis is a particularly elegant tool for determining physical degrees of freedom in a covariant way. In the first quantization, physical states are cohomology classes of the first-quantized BRST operator Q. Unlike critical string theory, where the only nontrivial cohomology group is $H^{(1)}$, two-dimensional strings have physical states for more than one ghost number (Refs. [5,11]). For the chiral sector, which corresponds to open strings, the relative cohomology ($b_0=0$) of Q is nontrivial for $g=0$, 1, and 2. Below we would like to discuss physical states from the field-theoretic point of view.

Let us, first, summarize the notation and definitions introduced in Ref. [1]. A string field is given by an arbitrary ket vector $|A\rangle=\sum_s|s\rangle a_s$ (see Ref. [1], Sec. III). The kets $|s\rangle$ belong to a one-string Fock space F and contain information about string excitations. Coefficient functions a_s, on the other hand, depend solely on the center-of-mass coordinates. Second quantization elevates the coefficient functions to the dynamical level.

The open string field theory can be described by Witten's action

$$W_{\rm cl}=\frac{1}{2}\int\left[A_1*QA_1+\frac{2}{3}A_1*A_1*A_1\right], \qquad (2.1)$$

where the subscript 1 means that this field has the ghost number $g=1$. The Witten's action is invariant under the gauge transformations ($g\Lambda=0$):

$$\Delta A_1=Q\Lambda+A_1*\Lambda-\Lambda*A_1 . \qquad (2.2)$$

To prove this, one has to take into account the properties of string field multiplication and integration known as Witten's axioms (Ref. [1], Eq. (3.3)). In a nutshell, they state that string fields should be treated in analogy to differential forms, with the first-quantized BRST charge (cf. Ref. [1], Eq. (2.17)) playing the role of the differential and the (first-quantized) ghost number g_s providing for grading. It was shown in Ref. [12], that it is advantageous to introduce yet another grading, Z_2 Grassmann grading, so that the coefficient functions a_s are Grassmann even or odd. Alternatively, one can define the *target space* ghost number $G_s=1-g_s$ so that the Grassmann parity of the coefficient functions is consistent with their ghost numbers: $(-)^{a_s}=(-)^{G_s}$. Then, the *total parity* or, simply, parity of a string field is $(-)^{g_s}(-)^{a_s}=-1$. We demand that all dynamical string fields are overall *odd*. This enables us to present a gauge-fixed action in a very simple way.

The problem of gauge fixing for the systems with reducible gauge symmetry, such as string field theory, can be treated successfully by the Batalin-Vilkovisky (BV) formalism. Recent developments in this field provide us with a geometric understanding of the BV formalism (Ref. [16]) and are closely connected to the promise of a background-independent formulation of string field theory (Refs. [17,18]). In that approach, one would like to postpone the fixing of the gauge as much as possible. To formulate the perturbation theory, on the other hand, we still need to choose one particular gauge. In SFT, the two most popular choices are the light-cone and covariant Siegel gauge ($b_0=0$). In this paper we work in Siegel's gauge, although alternative gauges are tempting to explore (see Ref. [13] as well as comments below). The gauge-fixed action reads

$$W_{\rm GF}=\frac{1}{2}\int[A*QA+\tfrac{2}{3}A*A*A-2(b_0\beta)*A], \qquad (2.3)$$

where the string field A contains all possible ghost numbers and the field β is a Lagrange multiplier enforcing the gauge condition. It can be proven, along the same lines as in Ref. [12], that $W_{\rm GF}$ satisfies the classical BV master equation, so that, at least on the tree level, it is a consistent gauge-fixed action.

A concrete realization of Witten's operations ([1], Eqs. (3.12)–(3.16)) is subject to stringent constraints:

$$ {}_{12\cdots n}\langle V|\left[\sum_{r=1}^{n}\alpha_0^{\mu r}+Q^{\mu}\right] = {}_{12\cdots n}\langle V|\delta^{(2)}\left[\sum_{r=1}^{n}p^{\mu r}+Q^{\mu}\right]=0 , \qquad (2.4)$$

$$ {}_{12\cdots n}\langle V|\left[\sum_{r=1}^{n}\sigma_0^{r}-3\right]={}_{12\cdots n}\langle V|\delta\left[\sum_{r=1}^{n}\lambda^{r}-3\right]=0 ,$$

where p^{μ} is the momentum (matter and Liouville) of the string state, λ is its ghost number, and Q^μ is the background charge of the matter-Liouville system (cf. Ref. [1], Eq. (2.2)). The first of these equations means that we are necessarily calculating the "bulk" amplitudes, i.e., that we are explicitly putting the cosmological constant to zero. The second one represents the ghost number conservation. Nonvanishing amplitudes involve the external states in which ghost numbers add up to 3, or alternatively, in which target space ghost numbers add up to zero. Another important property of the multistring vertices is that they are cyclically symmetric:

$$_{12,\ldots,n}\langle V| = {}_{n1,\ldots,n-1}\langle V| = \cdots = {}_{23,\ldots,1}\langle V| \ .$$

Now let us go back to the question of physical states. Naively, the problem consists of solving the free classical equation of motion (EOM) $QA_1=0$, modulo gauge transformations $\Delta A_1 = Q\Lambda$. It does reproduce part of the spectrum, namely, $H^{(1)}$, but the rest of the states ($g=0$ and 2) seem to be missing (classical field A_1 has ghost number $g=1$). To circumvent this problem, a modified classical action was proposed (Ref. [13]) which includes more than one ghost number. We would like to argue that this is not necessary. In fact, instead of the classical action, the analysis should be based on its *gauge-fixed* version:

$$W_{\text{lin}} = \frac{1}{2}\int [A * QA - 2(b_0\beta) * A] \ . \tag{2.5}$$

Note that gauge-fixed actions in Siegel's gauge ($b_0=0$) for both the original and the modified classical actions have the same form (2.5). The action (2.5) is invariant under the *target space* BRST transformations s:

$$\begin{aligned} sA_{\le 1} &= (QA)_{\le 1} \ , \\ sA_{\ge 2} &= (b_0\beta)_{\ge 2} \ , \\ s\beta &= 0 \ , \end{aligned} \tag{2.6}$$

where the subscripts denote, as usual, ghost numbers g of the states in question. One should bear in mind that s acts on coefficient functions a_s whereas Q acts on the states $|s\rangle$. From the relation $G=1-g$ one infers that the target space BRST transformation s increases the *target space* ghost number G by one unit. The Eq. (2.6) implies that, for $g\le 1$, $sA_1 = QA_0$, $sA_0 = QA_{-1}, \ldots$, whereas, for $g\ge 2$, one has that $sA_2 = b_0\beta_3$, $sA_3 = b_0\beta_4, \ldots$. It is easy to check that s is nilpotent off shell.

Let us now prove that solutions of the EOM corresponding to (2.5), modulo BRST transformations (2.6), exactly reproduce the physical spectrum of the theory. The target space BRST symmetry is a residual symmetry left after the fixing of the gauge. The EOM reads

$$QA_n - b_0\beta_{n+2} = 0 \ . \tag{2.7}$$

Substituting Eq. (2.7) into Eq. (2.6) one gets

$$sA_n = QA_{n-1} \tag{2.8}$$

for all $g=n$. It is now obvious that s and Q cohomology are exactly the same, so that $H^{(*)}$ is trivial except for $g=0$, 1, and 2. (The more accurate statement is that we have shown that the relative cohomologies coincide. To consider the absolute cohomology, one would have to find a different gauge (Ref. [13]) for which $b_0 \ne 0$. In that case, there would be, also physical states of $g=3$.)

The appearance of states with a ghost number 1 ($g=1$) is predictable since they exist in higher dimensions as well. In two dimensions, $H^{(1)}$ consists of a massless tachyon (of generic momenta) and a DS. Let us consider them in more detail (cf. Ref. [6]). For a generic tachyon T to be physical, the mass-shell condition should be satisfied ($Q=2\sqrt{2}$ for $c=1$):

$$\tfrac{1}{2}k^2 - \tfrac{1}{2}\beta(\beta+Q) = 1 \ , \tag{2.9}$$

where k is the matter and β is the Liouville momentum. Solving for β, we see that tachyons may have two different Liouville dressings:

$$k_\pm^\mu = (k, -Q/2 \pm k) \ . \tag{2.10}$$

In an important special case, k is an integer or half-integer multiple of $\sqrt{2}$. Such a tachyon is called a special or discrete tachyon. Discrete tachyons play a very important role. Namely, they are the highest weights of the underlying SU(2) symmetry of the spectrum. Starting from a discrete tachyon $V_{s,s}^\pm$ (here s is a non-negative integer or half-integer),

$$V_{s,s}^\pm = ce^{i\sqrt{2}sx}e^{(-\sqrt{2}\pm\sqrt{2}s)\varphi} \ , \tag{2.11}$$

one can construct all of the $g=1$ discrete states by applying the raising or lowering SU(2) operators

$$H_\pm = \frac{1}{2\pi i}\oint e^{\pm i\sqrt{2}x}$$

to (2.11) a certain number of times:

$$W_{s,n}^\pm \propto (H_-)^{s-n}V_{s,s}^\pm \ . \tag{2.12}$$

As we can see, discrete states also have positive or negative dressings, which correspond to positive or negative energies. In theories with a nonvanishing cosmological constant, or in a nontrivial background (black holes), the sign of the energy is important. The nonzero cosmological constant introduces the Liouville wall at $-\infty$. This means that the wave functions corresponding to the negative energy states cannot be normalized—they are "wrongly dressed." In our case the cosmological constant is absent so, dynamically, there is not much difference between the two dressings. Even in that case, however, there is a difference in the interpretation of the two branches of states (Ref. [6]). In fact, the positive branch of discrete states can be viewed as singular gauge transformations while there is no such interpretation for the negative branch. Neither of them are, of course, gauge artifacts and are physical degrees of freedom.

Matter and Louiville fields enter Eq. (2.12) on a different footing, since only the matter part contributes to excitations. In fact, it can be proved that such a gauge is a legitimate one. On the other hand, a gauge which would have only Liouville excitations is not.

For each DS of $g=1$ there are "partners" of $g=0$ ("chiral ground ring") and $g=2$ (Ref. [9]). These states play an important role in the spectrum-generating symmetry of the theory (W_∞) and their dynamics will be studied elsewhere. In what follows, we are interested in the scattering of $g=1$ states only.

III. FEYNMAN RULES AND THREE-POINT FUNCTIONS

In this section we summarize the Feynman rules and discuss the three-point functions in the theory. After inserting the coupling constant g and integrating over the Lagrange multiplier field β, (2.3) becomes

$$W_{\rm gf}=\frac{1}{2}\sum_{s,l} K_{sl}a_l a_s+\frac{g}{3}\sum_{s,l,m} V_{slm}a_m a_l a_s(-)^{a_l} , \qquad (3.1)$$

where the kinetic term involves

$$K_{sl}\equiv {}_{21}\langle V\|s\rangle_1 c_0(L_0-1)|l\rangle_2$$

and the interaction vertices are given by

$$V_{slm}={}_{321}\langle V\|s\rangle_1|l\rangle_2|m\rangle_3 .$$

Kinetic matrices K_{sl} are invertible and their inverses D_{sl} are the free propagators of the corresponding coefficient fields. Since in this work we are interested in tree amplitudes (the loop corrections are, undoubtedly, very interesting but more complicated since they necessarily involve closed strings, Ref. [12]), free propagators give the two-point functions. Three-point functions are obtained in the standard fashion, using Wick's theorem:

$$A_3(s,l,m)=-g(-)^{a_l}V_{slm}[1+(-)^{R_s+R_l+R_m}] , \qquad (3.2)$$

where by R we have denoted the *fermion number* of the corresponding state. It is an eigenvalue of the level operator which, acting on conformal and physical vacua, respectively, gives $R|0\rangle=|0\rangle, R|\Omega\rangle=0$ (cf. Ref. [1], Sec. II). In deriving (3.2) we have used the Grassmannian parity of the coefficient functions as well as cyclic symmetry of the vertices. This completes the derivation of the Feynman rules.

Let us now consider the three-point functions, Eq. (3.2). Ghost number conservation allows for two different types of on-shell correlators: either all three particles have $g=1$, or each of them has a different ghost number (i.e., 0, 1, and 2). One notices that $g=1$ states are involved in both cases. Consider in more detail the three-point functions where all three states are from $H^{(1)}$. There are two distinct classes of amplitudes of that type: V_{TTD} and V_{DDD}. Here, we have denoted a generic on-shell tachyon by T and all $g=1$ discrete states, including the discrete tachyons, by D. The fact that there is no vertices of the type V_{TTT} or V_{TDD} follows from the momentum conservation. Let us prove that the class V_{TTT} is empty (the second claim is obvious). The momentum conservation gives

$$\begin{aligned} &k_1+k_2+k_3=0 ,\\ &-\sqrt{2}\pm k_1\pm k_2\pm k_3=0 , \end{aligned} \qquad (3.3)$$

where + (−) corresponds, as usual, to the positive (negative) chirality. Clearly, we cannot take all signs in the second equation to be the same. So, we can express two of the momenta entering with the same sign in terms of the third one (using the first equation) and plug it back into the second equation. In this way the third momentum is completely determined—it corresponds to a discrete tachyon. We have shown, therefore, that the amplitude does not belong to V_{TTT} but, rather, to V_{TTD}. It is customary to normalize the three-tachyon amplitudes to be 1.

Consider, now, the most general scattering of one DS and two generic tachyons. In that case the momentum conservation gives

$$\begin{aligned} &\sqrt{2}n_1+k_2+k_3=0 ,\\ &\sqrt{2}(s_1-1)\pm k_2\pm k_3=0 . \end{aligned} \qquad (3.4)$$

Here, the index 1 is reserved for the discrete state, while 2 and 3 label the tachyons. Up to now, the chiralities of the tachyons were arbitrary. However, for k_i to be nondiscrete, the determinant of the system (3.4) must vanish. This means that both signs in (3.4) should be the same. Let us, for definiteness, take the plus sign. Then, the compatibility dictates that $n_1=s_1-1$. A simple example of the state of that type is

$$W^+_{3/2,1/2}=\left[-(\partial x)^2-\frac{i}{\sqrt{2}}\partial^2 x\right]\exp\left[i\frac{\sqrt{2}}{2}x\right] \times\exp\left[\frac{\sqrt{2}}{2}\varphi\right] .$$

The calculation of the correlation function $A_3=\langle W^+_{3/2,1/2}T^+_{k_2}T^+_{k_3}\rangle$ performed by the standard conformal technique gives

$$A_3=(k_2)^2+k_2/\sqrt{2} . \qquad (3.5)$$

To arrive at (3.5) we have fixed, using the SL(2,R) symmetry, the three points on the boundary to be $z_1=-0$, $z_2=1$, and $z_3=\infty$. Also, we have used the momentum conservation (3.4). Let us calculate the same amplitude from the field theoretic point of view. From (3.2) one has that

$$\begin{aligned} A_3&=-g(-)^{a_l}V_{slm}[1+(-)^{R_s+R_l+R_m}]\\ &=-2gV_{WTT}=-2g[e^{-2N_{00}}(N^{12}_{10})^2(k_2-k_3)^2+N^{11}_{11}-N^{11}_{20}]\\ &=-2g\left[\frac{27}{16}\left[\frac{4}{27}4(k_2)^2+2\sqrt{2}k_2+\frac{1}{2}\right]-\frac{5}{27}+\frac{3}{27}\right]\\ &=-2g\left[(k_2)^2+\frac{1}{\sqrt{2}}k_2\right] , \end{aligned} \qquad (3.6)$$

where, in the third line, we have used the explicit expressions for the Neumann coefficients (see Ref. [19]), and expressed, using Eq. (3.4), k_3 in terms of k_2. It is evident that, apart from an overall normalization factor, we have obtained the same amplitude. As explained above, the normalization is fixed by requiring that the three tachyon amplitudes are equal to unity, so that $-2g=1$.

In exactly the same way one can calculate V_{DDD} amplitudes. For A_{+++}, for example, the momentum conservation gives

$$n_1+n_2+n_3=0\ ,\qquad s_1+s_2+s_3=1\ , \tag{3.7}$$

and similarly for the other possible chiralities. Using the associativity of the operator product expansion (OPE), a three-point function can be represented as a linear combination of the two-point functions. Coefficients in the expansion are proportional to the Clebsch-Gordan coefficients, Ref. [6]. To be nonzero, two-point functions should pair the states with their reflected states: $|A\rangle_1^r \equiv {}_{12}\langle V \| A\rangle^2$. They are normalized to the momentum δ function from which follows that the three point functions are equal to the OPE coefficients (Ref. [6]). Such an amplitude is just a number (as opposed to an entire function in momenta, as it is the case for V_{TTD}). As an illustration, consider the same example as before, but instead of choosing both plus signs in the second of Eqs. (3.4), let us take the second one to be a minus sign. Then, $k_2=-1/\sqrt{2}$, $k_3=0$ and the amplitude vanishes, since

$$(k_2)^2+\frac{1}{\sqrt{2}}k_2=0\ .$$

This result holds, obviously, in both the first- and the second-quantized approaches. Such an agreement between the two approaches clearly exists for all of the three-point functions.

IV. AN INSTRUCTIVE EXAMPLE: FOUR-TACHYON SCATTERING

As an introduction to our discussion of the four-point scattering amplitudes, we would like in this section to consider an example which shows some of the peculiar properties of the two-dimensional strings. Let us introduce two-dimensional counterparts of the kinematic invariants in four dimensions:

$$s=\frac{1}{2}\left[k_1+k_2+\frac{Q}{2}\right]^2,\quad t=\frac{1}{2}\left[k_1+k_3+\frac{Q}{2}\right]^2,\quad u=\frac{1}{2}\left[k_1+k_4+\frac{Q}{2}\right]^2 .$$

It is well known that for an arbitrary number of space-time dimensions $s+t+u$ is an invariant quantity, determined solely by the masses of the external particles in question. In two dimensions, for four tachyons, one has that $s+t+u=1$ [see (5.22)]. The total amplitude is the sum over s, t, and u channels: $A_4^{(\mathrm{tot})}=A_4^{(s)}+A_4^{(t)}+A_4^{(u)}$. As a function of kinematic variables, an amplitude can be, generically, decomposed into the singular (which has simple poles in kinematic invariants) and the regular parts: $A=A_{\mathrm{sing}}+A_{\mathrm{reg}}$. It is important to note that only A_{sing} is physically relevant, since the physical information is contained in the residues of the poles. Two amplitudes agree if their singular parts agree. We will often be sloppy and suppress the regular parts altogether.

For the s channel, for example, one obtains (here $a,\ldots,d$ stand for first, . . . , fourth external string states, respectively)

$$A_4^{(s)}=-g^2V_{bla}(-)^a[1+(-)^{R_a+R_l+R_b}]D_{ml}V_{dmc}\times(-)^d[1+(-)^{R_c+R_m+R_d}]\ . \tag{4.1}$$

The summation over the repeated indices l and m is implied in (4.1). For the four-tachyon scattering, one has the kinematic-independent expression [cf. Ref. [1], Eq. (5.5)]

$$A_s^{(4)}\propto g^2\left[\frac{16}{27}\right]^{(k_1+k_2+Q/2)^2/2}\left[\frac{1}{\frac{1}{2}(k_1+k_2+Q/2)^2}+\frac{\frac{4}{27}(k_1-k_2)\cdot(k_3-k_4)}{\frac{1}{2}(k_1+k_2+Q/2)^2+1}+\frac{\frac{2}{81}(k_1+k_2+Q)\cdot(k_1+k_2)+\frac{8}{729}[(k_1-k_2)\cdot(k_3-k_4)]^2}{\frac{1}{2}(k_1+k_2+Q/2)^2+2}+\frac{\frac{124}{729}-\frac{10}{729}[(k_1-k_2)^2+(k_3-k_4)^2]}{\frac{1}{2}(k_1+k_2+Q/2)^2+2}+\cdots\right]\delta\left[\sum_{i=1}^{4}k_i+Q\right] . \tag{4.2}$$

To obtain t (u) channel contributions one, simply, substitutes $2\leftrightarrow 3$ ($2\leftrightarrow 4$) in (4.2).

In Ref. [1], we have analyzed the situation where three of the external tachyons are of one chirality and the fourth one is of the opposite. In particular, we have shown that the amplitude (4.2) reproduces the Bershadsky-Kutasov amplitude (Ref. [2]) in that kinematic region, if one compares the residues of the two expressions (see Ref. [1], Sec. V).

Let us now consider our main topic in this section, that is, a four-tachyon amplitude A_{++--}. Bershadsky and Kutasov have argued that the total amplitude in that case should be zero. We would like to show how the same result appears from a field-theoretic point of view

For that particular kinematics one has, for the matter momentum, $k_1+k_2=-(k_3+k_4)=\sqrt{2}$. It is easy to see that this implies that $s=1$ and that, therefore, $t+u=0$. This proves to be crucial for the vanishing of the amplitude. To see this, let us first calculate the t channel contribution. We obtain

$$A_4^{(t)} = -g^2 \left[\frac{16}{27}\right]^t \left[\frac{1}{t} + \frac{\frac{120}{729} + \frac{76}{729}t + \frac{32}{720}(t+1)^2}{t+2} + \cdots \right] . \tag{4.3}$$

The ellipsis stands for higher level contributions. The u channel amplitude is obtained from (4.3) by $t \leftrightarrow u$. It is straightforward to check that almost all poles in $A_4^{(t)}$ ($A_4^{(u)}$) are *fake*, i.e., that they have zero residues. The only exception is the pole $t=0$ ($u=0$). Because of that, taken separately, singular parts of $A_4^{(t)}$ and $A_4^{(u)}$ do not vanish. This is not true for their sum, however, due to $t+u=0$. In fact,

$$A_4^{(t)} + A_4^{(u)} \propto -g^2 \left[\left[\frac{16}{27}\right]^t \frac{1}{t} + \left[\frac{16}{27}\right]^u \frac{1}{u} \right]$$
$$= -g^2 \left[\left[\frac{16}{27}\right]^t - \left[\frac{16}{27}\right]^{-t} \right] \frac{1}{t} . \tag{4.4}$$

As expected, the expression (4.4) is regular at $t=0$. Since $s=1$, $A^{(s)}$ is a regular function as well. Thus, the singular part of $A^{(\mathrm{tot})}$ vanishes. We can add to $A^{(\mathrm{tot})}$ an arbitrary entire function without changing the physics. If we choose $A^{(\mathrm{reg})}$ to vanish, then $A^{(\mathrm{tot})}=0$ as well. This completes the proof. In exactly the same way one can prove that A_{+-+-} vanishes and that, more generally, $A_{n_1,m_1,\ldots,n_k,m_k}=0$ [where n_i (m_i) is the number of consecutive $+$ ($-$) for $k \geq 2$.

To summarize, there are a couple of important messages from this simple example. First, to be able to draw physical conclusions, one is to consider all possible channels and not only one (as is sometimes the habit). Second, physical information is contained in residues of the poles. The peculiarity of two dimensions is that, there, due to special kinematic restrictions, poles in kinematic variables can "degenerate"—instead of a variable we get an integer in the denominator. If this number is positive (as it was the case in this example) the amplitude becomes an entire function (or a number). If the number is nonpositive (see below), one can anticipate the existence of an unbounded contribution to the sum. As we shall see below, this is the field-theoretic origin of the divergence of 2D amplitudes discovered in Ref. [14] in the conformal approach.

V. FOUR-POINT AMPLITUDES INVOLVING DISCRETE STATES

In this section we analyze four-point amplitudes involving DS's as well as tachyons. We are interested in correlation functions where all four asymptotic states are from $H^{(1)}$. As in the three-point case, four-point amplitudes are severely restricted by the momentum conservation law and can be classified in the similar fashion as it was done in Sec. III for the three-point functions. There are three different classes of correlators: A_{TTTD}, A_{TTDD}, and A_{DDDD}. Note that the four-tachyon amplitude A_{+++-} belongs to A_{TTTD} rather than A_{TTTT} since the negative chirality tachyon is fixed by the kinematics to be $W^-_{1,-1}$. Truly belonging to A_{TTTT} class would be the amplitude considered in Sec. IV, A_{++--}, but it vanishes. Clearly, there is no amplitude of the type A_{TDDD} either. Before proceeding to the detailed analysis of each one of the three classes, some general comments are in order.

A typical s channel contribution to the four-point amplitude is [see (4.1)]

$$A_4^{(s)} \propto -2g^2 \sum_l V_{bla} D_l V_{dlc} = \sum_{n \geq 0} \frac{A_n(t)}{s+n} , \tag{5.1}$$

and similarly for t and u channels. In the previous section we have shown that s and, therefore, propagator $D_l \propto 1/(s+n)$ can "degenerate" and become a number (instead of a function of momenta) for some particular kinematics. In that case, s was a positive integer ($s=1$) and the amplitude was shown to vanish. Quite generally, if the amplitude degenerates in some channel, that channel either does not contribute or the amplitude diverges. This is determined by the sign of the degenerate kinematic invariant. To see this, one should bear in mind that $n \geq 0$, so that if s is a positive number (in the degenerate case it is always an integer, cf. Sec. V C) amplitude (5.1) is an entire, bounded for finite k, function of t and, as such, it is irrelevant. If $s \leq 0$, on the other hand, there is always an n such that $s+n$ vanishes. This leads to the appearance of an unbounded term in the amplitude. This is the origin of divergencies in two dimensions.

One can arrive at the same conclusions using the conformal approach. Consider a generic correlator contribution

$$\propto \int_0^1 dx\, x^{k_1\cdot k_2+n_1}(1-x)^{k_2\cdot k_3+n_2} .$$

Here, n_1 and n_2 are integers. Note that s can be rewritten as

$$s = \frac{1}{2}\left[k_1 + \frac{Q}{2}\right]^2 + \frac{1}{2}\left[k_2 + \frac{Q}{2}\right]^2 + k_1\cdot k_2 + 1 .$$

Since the first two summands are the masses of the corresponding particles (integers in units of Regge slope) we see that s and $k_1\cdot k_2$ differ by, at most, an integer. Similarly, $k_2\cdot k_3$ ($k_1\cdot k_3$) differ from u (t) by, at most, an integer. In higher dimensions the amplitude is, in general, a meromorphic function of kinematic invariants and is, therefore, well defined. In two dimensions, when degeneracy occurs, the exponents of x and/or $1-x$ can be negative integers. In that case the amplitude is clearly ill defined. The value of the exponent, on the other hand, is determined by the value of the kinematic invariant in question. This proves the claim.

As an application, we check that the whole class of amplitudes A_{TTTD} is well defined by simply showing that all three channels do not degenerate (Sec. V A). The opposite extreme is the amplitudes of the type A_{DDDD} (see Sec. V C). They degenerate in all three channels. What is more, at least one of the kinematic invariants is nonpositive. Thus, the amplitudes of that class always diverge. In between the two extremes is the class A_{TTDD} (Sec. V B). The correlators of that type have, at least, one degenerate channel. Some amplitudes from A_{TTDD} are divergent while the others are well behaved. In what follows, we analyze in detail the dynamical properties of the four-point amplitudes and present their classification.

A. Amplitudes involving one DS

Let us begin with the A_{TTTD} class. It contains, among the others, the four-tachyon amplitude A_{+++-}. The properties of this amplitude are well established (see Ref. [1], Sec. V). In this subsection, we would like to clarify some of the properties of A_{TTTD} class as a whole. In particular, we prove that an arbitrary amplitude belonging to it is well defined.

When a correlator involves an arbitrary discrete state $W^{\pm}_{s,n}$ and three generic tachyons, the momentum conservation reads

$$\begin{aligned} &\sqrt{2}n_1+k_2+k_3+k_4=0 \ , \\ &-2\sqrt{2}\pm\sqrt{2}s_1\pm k_2\pm k_3\pm k_4=0 \ . \end{aligned} \tag{5.2}$$

If an amplitude is to belong to the class above, tachyons should be all of the same chirality and the following consistency condition must be valid:

$$n_1=\pm(s_1-2) \ . \tag{5.3}$$

Here, the upper (lower) sign corresponds to the positive (negative) chirality tachyons, respectively. It is easy to check that

$$s=\frac{1}{2}\left[k_1+k_2+\frac{Q}{2}\right]^2=\mp(\sqrt{2}k_2+2n_1)-1$$

with the similar expressions for t and u. They are obtained from s by substituting $k_2\leftrightarrow k_3$ ($k_2\leftrightarrow k_4$). It is evident that none of the channels degenerates. In accordance with the discussion above, this means that all amplitudes of the type A_{TTTD} are well-defined. Note that $\mp 2n_1-1$ are *integers* so that the possible poles have the structure $k=\pm N/\sqrt{2}$, where N is an integer. Dynamical properties of the A_{TTTD} class can be summarized as follows: amplitudes of that class are meromorphic functions in discrete momenta. There are no degenerate channels in this case. Each amplitude of this class has an infinite number of proper physical poles.

As a simple example, take the discrete state to be $W^{+}_{3/2,\mp 1/2}$. Here, the upper-sign state is coupled to the positive-chirality tachyons and vice versa. One has

$$s=\mp\sqrt{2}k_2, \quad t=\mp\sqrt{2}k_3, \quad u=\mp\sqrt{2}k_4 \ . \tag{5.4}$$

Consider in more detail the first poles in the s channel corresponding to the lower sign in Eq. (5.4). The analysis of t and u channels goes along the same lines–one is just to exchange the labels as indicated above. The first potential pole is at $k_2=0$. The corresponding residue is

$$A_0=2(N^{12}_{10})^2\left[2k_2+\frac{1}{\sqrt{2}}\right]^2-2N^{11}_{20}+2N^{11}_{11}=0 \ , \tag{5.5}$$

where we have used $k_2=0$, the momentum conservation, and the explicit expressions for the Neumann coefficients. Of course, the same conclusion can be reached without any calculations since the residue is proportional to the vertex containing $W^{+}_{3/2,1/2}$ and the two discrete tachyons. Such a vertex, as it was shown in Sec. III, vanishes. Next potential contribution is $n=1$ ($k_2=-1/\sqrt{2}$). In that case,

$$\begin{aligned} A_1&=2\left[\frac{16}{27}\right]^{\sqrt{2}k_2-1}2(N^{12}_{10})^2N^{12}_{11}(k_3-k_4)\left[2k_2+\frac{1}{\sqrt{2}}\right]+2\sqrt{2}(k_3-k_4)(N^{12}_{10})^2\left[(N^{12}_{10})^2\left[2k_2+\frac{1}{\sqrt{2}}\right]^2+N^{11}_{11}-N^{11}_{20}\right] \\ &=-4\sqrt{2}(\tfrac{27}{16})^2(N^{12}_{10})^2N^{12}_{11}k_3=-\sqrt{2}k_3 \ . \end{aligned} \tag{5.6}$$

The next one is the pole at $k_2=-\sqrt{2}$, with the residue

$$A_2=\sqrt{2}k_3-2(k_3)^2 \ . \tag{5.7}$$

In the same way we can calculate the higher orders. It is clear that the residues are entire functions in k_3 and that they are, in general, nonvanishing.

To check our conclusions, let us calculate the amplitude using the conformal technique. To this end, consider the correlation function

$$\begin{aligned} A&=\int_0^1 dx\,\langle W^{+}_{3/2,1/2}e^{ik_2\cdot\phi}ce^{ik_3\cdot\phi}ce^{ik_4\cdot\phi}\rangle \\ &=\int_0^1 dx\, x^{k_1\cdot k_2}(1-x)^{k_2\cdot k_3}\left[\frac{k_2/\sqrt{2}+(k_2)^2}{x^2}+2\frac{k_2k_3}{x}+\frac{k_3}{\sqrt{2}}+(k_3)^2\right] \ . \end{aligned} \tag{5.8}$$

In passing from the first to the second line in Eq. (4.8) we have used Wick's theorem, fixed the SL$(2,R)$ gauge by choosing $z_1=0$, $z_3=1$, and $z_4=\infty$, and integrated over the $z_2=x$ $(0\le x\le 1)$. We have also used the on-shell conditions. For the kinematics in question $k_1\cdot k_2=1+\sqrt{2}k_2$ and

$$k_2\cdot k_3=-2-\sqrt{2}(k_2+k_3) \ .$$

It is straightforward to see that the amplitude is well defined on the entire complex plane k_2, excluding the discrete set of points $\sqrt{2}k_2+N=0$, where it has the simple poles. The explicit expression can be easily found:

$$A = \left[\frac{k_2}{\sqrt{2}} + (k_2)^2 \right] \frac{\Gamma(\sqrt{2}k_2)\Gamma(k_2 \cdot k_3 + 1)}{\Gamma(\sqrt{2}k_2 + k_2 \cdot k_3 + 1)} + \left[\frac{k_3}{\sqrt{2}} + (k_3)^2 \right] \frac{\Gamma(\sqrt{2}k_2 + 2)\Gamma(k_2 \cdot k_3 + 1)}{\Gamma(\sqrt{2}k_2 + k_2 \cdot k_3 + 3)} + 2k_2k_3 \frac{\Gamma(\sqrt{2}k_2 + 1)\Gamma(k_2 \cdot k_3 + 1)}{\Gamma(\sqrt{2}k_2 + k_2 \cdot k_3 + 2)} . \tag{5.9}$$

Positions of the poles are the same as predicted from SFT. Let us compare the residues. To calculate them we use the well-known relation $\Gamma(z) = \Gamma(z+1)/z$. The residue corresponding to $k_2 = 0$ vanishes. Note that only the first summand in (5.9) has a potential pole for that value of k_2 since $\Gamma(\sqrt{2}k_2) \propto 1/\sqrt{2}k_2$, but it is killed by the factor $k_2(k_2 + 1/\sqrt{2})$ which multiplies it. The next residue corresponds to $k_2 = -1/\sqrt{2}$. In that case, the potential contribution from the first summand, is again, suppressed by the prefactor. However, the third summand contributes to the residue which reads $2k_2k_3 = -\sqrt{2}k_3$. Starting from $n \geq 2$ all three summands in (5.9) begin contributing to the residues, which are polynomials in k_3. Again, one readily checks that the values of the residues calculated from the amplitude (5.9) match the ones calculated from SFT.

It is rather amusing to observe somewhat special role of the $W^{+}_{3/2, \pm 1/2}$ states. Namely, these are the only two states compatible *simultaneously* with V_{TTD} and A_{TTTD}. In fact, the compatibility conditions are

$$n = \pm(s-1) , \quad n = \pm(2-s) . \tag{5.10}$$

Each of the two systems of equations has a solution, namely, $s = \frac{3}{2}$, $n = \pm\frac{1}{2}$. These values correspond to the above-mentioned states. It is unclear, however, whether this peculiar property of the $W^{+}_{3/2, \pm 1/2}$ states has some deeper physical meaning.

B. Amplitudes involving two DS's

Let us focus, now, on the properties of A_{TTDD} class. We clarify, first, the conditions under which an amplitude belongs to that class. If we are given two arbitrary DS's and two generic tachyons, the conservation of momentum tells us that

$$\sqrt{2}n_1 + k_2 + k_3 + \sqrt{2}n_4 = 0 , \quad -2\sqrt{2} \pm \sqrt{2}s_1 \pm k_2 \pm k_3 \pm \sqrt{2}s_4 = 0 , \tag{5.11}$$

where "1" and "4" label the discrete states and "2" and "3" the generic tachyons. It is clear that the tachyons should be of the same chirality. For concreteness, let us take it to be negative. Then, the consistency requires that

$$n_1 + n_4 = 2 \mp s_1 \mp s_4 . \tag{5.12}$$

Since the momenta k_1 and k_4 are fixed (discrete), the amplitude degenerates in, at least, one channel—the u channel. Whether or not it degenerates in the other two can be easily determined. Since

$$k_1 \cdot k_2 = \sqrt{2}(n_1 \pm s_1 - 1)k_2 + 2(\pm s_1 - 1) , \quad k_1 \cdot k_3 = \sqrt{2}(n_1 \pm s_1 - 1)k_3 + 2(\pm s_1 - 1) , \quad k_2 \cdot k_3 = 2(n_1 + n_4 - 1) , \tag{5.13}$$

s and t channels (simultaneously) degenerate if and only if $n_1 + s_1 - 1 = 0 = n_4 + s_4 - 1 = 0$. In that case, $k_1 \cdot k_2 = k_1 \cdot k_3 = 2(s_1 - 1)$. We refer to such an amplitude as *totally degenerate.* To determine the DS involved in a totally degenerate correlator we use the fact that $-s \leq n \leq s$. It is easy to see that both discrete states should be taken to be $W^{+}_{1/2,1/2}$. Then, $k_1 \cdot k_2 = -1$ and $k_2 \cdot k_3 = 2(n_1 + n_4 - 1) = 0$, so the amplitude reads

$$A_{\text{deg}} = \int_0^1 \frac{dx}{x} . \tag{5.14}$$

The integrand on the right-hand side is, evidently, ill defined. The same result can be conjectured utilizing the SFT result for the scattering of four tachyons, Eq. (4.2). The first term in the expansion for A^{s}_{deg} is $1/s$ and, since $s = 0$, it diverges. Channel t behaves in the same way. Since $u = 1$, channel u, on the other hand, does not contain any unbounded summands. Divergence of the integral (5.14), therefore, shows up in the field-theoretic approach through the presence of unbounded summands in the amplitude.

Degeneracy in all three channels of A_{TTDD} is not, however, the typical property of that class. Much more common is the situation where only one channel (for example, u) denerates, while the other two (s and t) do not. In that case, as explained above, dynamics is determined by the sign of u, or, equivalently, by the value of the product

$$k_2 \cdot k_3 = 2(n_1 + n_4 - 1) = u - 1 .$$

From the conformal field theory point of view, that sign determines the convergence properties on the upper limit of the Koba-Nielsen integral. One can, thus, subdivide the correlators A_{TTDD} into the three subclasses: A^{deg}_{TTDD}, A^{+}_{TTDD}, and A^{-}_{TTDD}. The totally degenerate class A^{deg}_{TTDD} has been discussed above. Let us, therefore, focus on the other two.

Amplitudes $A^{\pm}_{TTDD}$ contain physical intermediate particles. Note that s channel poles, for example, originate from the expression $s+n$ in the denominator [see Eq. (5.1)]. We have

$$s = \tfrac{1}{2}(k_1 + k_2)^2 = \tfrac{1}{2}(\sqrt{2}n_1 + k_2)^2 - \tfrac{1}{2}[\sqrt{2}(1 \mp s_1) + k_2]^2 = [n_1 - (1 \mp s_1)]\sqrt{2}k_2 + [n_1 + (1 \mp s_1)] , \tag{5.15}$$

and although n_1 and s_1 can be half-integers, their linear combinations $n_1 - (1 \mp s_1)$ and $n_1 + (1 \mp s_1)$ are always *integers.* The main difference between the two classes stems from the fact that, for A^{+}_{TTDD}, degenerate channel is a bounded, entire function ($u \geq 1$), while A^{-}_{TTDD} has an unbounded contribution ($u \leq 0$). Thus, A^{+}_{TTDD} correlators are well defined while A^{-}_{TTDD} are not.

To see this in more detail, consider first the A^{+}_{TTDD} class. A generic contribution is of the form

$$A=\int_0^1 x^a(1-x)^n dx \;, \tag{5.16}$$

where a is a variable and n is a (non-negative) integer constant. It is well known that the integral of the type $\int_0^1 dx\, x^{a-1}$ can be analytically continued, for all $a\neq 0$, to $\int_0^1 dx\, x^{a-1}=1/a$. The integral (5.16) is of the that type (one is just to use Newton's binomial expansion formula). So, amplitudes of the class A^{+}_{TTDD} are well defined and have finite number of poles in two different channels.

For example, let us take the two discrete states to be $W^{+}_{3/2,1/2}$ and $W^{-}_{1/2,1/2}$. In that case, the correlation function $\langle W^{+}_{3/2,1/2}T^{-}_{k_2}T^{-}_{k_3}W^{-}_{1/2,1/2}\rangle$ is

$$\langle W^{+}_{3/2,1/2}T^{-}_{k_2}T^{-}_{k_3}W^{-}_{1/2,1/2}\rangle = \int_0^1 dx\, x^{\sqrt{2}k_2+1}\left[\frac{k_2/\sqrt{2}+(k_2)^2}{x^2}+2\frac{k_2k_3}{x}+\frac{k_3}{\sqrt{2}}+(k_3)^2\right] , \tag{5.17}$$

where we have used the fact that $k_1\cdot k_2=\sqrt{2}k_2+1$, $k_2\cdot k_3=0$ or, alternatively, that $s=\sqrt{2}k_2$, $u=1$ in that kinematics. The amplitude (5.17) becomes

$$A^{s}\propto\frac{1}{\sqrt{2}k_2+1} \;. \tag{5.18}$$

Here, as usual, we have left out the finite part of the amplitude.

The same result can be obtained from the field-theoretic point of view. In fact, one can use (5.5)–(5.7) to show that, in that case, $A_1=1$ and $A_{n\neq 1}=0$. In exactly the same way we find that the t channel amplitude is $A^{t}=1/(\sqrt{2}k_3+1)$ and that the u channel contribution is of the form: $\sum_{n=0}[A_n/(n+1)]$, where $A_0=0$,

$$A_1=\tfrac{27}{16}\tfrac{1}{2}[8(N^{12}_{10})^4+3(N^{12}_{10})^2N^{12}_{11}]\sqrt{2}(k_3-k_2)$$
$$=\frac{10\sqrt{2}}{27}(k_3-k_2) \;,$$

and so on. We see that A^{u} is an entire bounded function of momenta. The amplitude has *finite* number (one in each channel) of poles in two different channels. One can readily see that the same holds for any A^{+}_{TTDD} correlator.

Quite different is the situation when A^{-}_{TTDD} is considered. In that case $u\leq 0$ and the amplitudes have the form

$$A=\int_0^1\frac{x^a}{(1-x)^n}dx \tag{5.19}$$

for a positive integer n. Such an integral clearly diverges on the upper limit ($x=1$). To illustrate the situation, consider the correlation function

$$\langle W^{+}_{1,1}T^{-}_{k_2}T^{-}_{k_3}W^{+}_{1/2,-1/2}\rangle=\int_0^1\frac{x^{\sqrt{2}k_2}}{1-x}dx \;.$$

This amplitude is of the type (5.19) with $n=1$ and $a=\sqrt{2}k_2$. To make some sense out of that expression, let us perform a simple trick. It consists of expanding $1/(1-x)$ in power series and formally integrating term by term. We say "formally" since the geometric series diverges for $x=1$, so that we do not have, strictly speaking, the right to exchange the order of summation and integration. Nevertheless, let us do just that. Then, the amplitude yields

$$\langle W^{+}_{1,1}T^{-}_{k_2}T^{-}_{k_3}W^{+}_{1/2,-1/2}\rangle=\int_0^1\frac{x^{\sqrt{2}k_2}}{1-x}dx=\sum_{n=0}^{\infty}\frac{1}{\sqrt{2}k_2+n+1}=\sum_{n=0}^{\infty}\frac{1}{s+n} \;, \tag{5.20}$$

where, in the last step, we have used the fact that $s=\sqrt{2}k_2+1$. It is tempting to check the validity of this formula employing the SFT approach. Since the example in question is nothing but the scattering of four tachyons (some of them of the discrete momenta), one can, once again, use the general expression for the four-tachyon amplitude (4.2). Clearly, poles have the same position in both approaches. The real question is whether they have the same residues. From (5.20) we see that the amplitude has the unusual property—all residues are equal to unity. Let us show that this is the case from the field theory approach. In fact, the first three residues are

$$A_0=(\tfrac{16}{27})^0=1 \;,$$
$$A_1=(\tfrac{16}{27})^{-1}\tfrac{4}{27}(k_1-k_2)\cdot(k_3-k_4)=\tfrac{27}{16}\times\tfrac{4}{27}\times 4=1 \;, \tag{5.21}$$
$$A_2=\left[\frac{27}{16}\right]^2\left[-\frac{4}{81}+\frac{36\times 8}{729}+\frac{124}{729}-\frac{12\times 10}{729}\right]=1 \;,$$

where, in the last line, we have presented, as separate summands, the numerical values of the four terms contributing to the residue at $s=-2$ in (4.2). Although by now reader should have been convinced that the two approaches give the same results, it is nice to see how all these messy coefficients add up to 1. We see that such an amplitude has infinitely many physical poles. Their residues are numbers. The same properties has t channel. On the other hand, $u=0$ so that u channel degenerates and contains an unbounded contribution $1/u$. This is where one can trace the "bad" behavior of the integral. Thus, although the total amplitude is ill defined, it has a well-defined "piece" from which one can draw physical conclusions. "Changing the order of summation and integration" is, thus, the operation which *removes* the divergence from the pole expansion. One can clearly generalize the above discussion to any amplitude of the A^{-}_{TTDD} class. Therefore, A^{-}_{TTDD} amplitudes have infinitely many physical poles in two different channels and are divergent in the third.

C. The amplitudes involving four DS's

To finish up this section, let us briefly comment on the third class of correlators, A_{DDDD}. In that case, all three channels degenerate, and the amplitudes are ill defined. To prove it, it is enough to show that for every amplitude of that class there is at least one nonpositive kinematic invariant. This follows from the fact that

$$s+t+u=\sum_{i=1}^{4}\frac{1}{2}\left[k_i+\frac{Q}{2}\right]^2+1=1-\sum_{i=1}^{4}R_i \ , \qquad (5.22)$$

where R_i are the levels of the four asymptotic states (see Sec. III). Since $R_i \geq 0$, and the equality holds only for tachyons, we have that

$$s+t+u\leq 1 \ . \qquad (5.23)$$

If we can show that the kinematic invariants are integers then we are, obviously, done. Let us, for example, show that s is an integer (the other two cases are treated similarly). We know that s can be represented as

$$s=\frac{1}{2}\left[k_1+\frac{Q}{2}\right]^2+\frac{1}{2}\left[k_2+\frac{Q}{2}\right]^2+k_1\cdot k_2+1 \ .$$

Since the masses of the external particles are integers, it is enough to show that $k_1\cdot k_2$ is an integer. If we denote by $\sqrt{2}n_i$ and $-\sqrt{2}\pm\sqrt{2}s_i$ the matter and Liouville momenta of the ith string, the product can be rewritten as

$$k_1\cdot k_2=2n_1n_2-2s_1s_2-(2\mp 2s_1\mp 2s_2) \ .$$

The term in parentheses is clearly an integer (since n_i are s_i are integers or half-integers). Also, although $2n_1n_2$ and $2s_1s_2$, taken separately, may be half-integers, their difference is always an integer. Thus, we have proven that, whenever degenerate, kinematic invariants are *integer valued*. By the same token, using (5.23), we have proven that at least one of the kinematic invariants is nonpositive. Since every amplitude of the class A_{DDDD} has at least one divergent channel, they are all ill defined.

VI. CONCLUDING REMARKS

In the present work we have aimed to better understand the dynamics of discrete states in open string field theory. In particular, we have shown that the origin of divergencies in 2D is rather simple to understand. Namely, they stem from the fact that, for certain kinematics, kinematic invariants become nonpositive integers. In that case, the amplitude, presented as sum over poles, has an unbounded contribution tantamount to the divergence of the corresponding Koba-Nielsen integral. We have seen, also, that, from the dynamical point of view, discrete tachyons are not at all different from the other discrete states. They differ dramatically, on the other hand, from the generic tachyons. As far as convergence of amplitudes is concerned one can state a simple rule of thumb: the more generic tachyons, the better the convergence.

There are several important questions which our discussion left open or partially unanswered. First, our formalism is adapted only for calculations of the bulk amplitudes, i.e., cosmological constant is absent in this approach. Clearly, this makes a difference since presence of the cosmological constant leads to additional divergences, due to the charge screening integrals (Ref. [14]). Knowledge of bulk amplitudes allows one, in principle, to deduce nonbulk correlation functions using the method outlined in Ref. [7]. However, it is clearly advantageous to calculate them directly. To do that, one would have to adapt the formalism of Ref. [1] so as to include nontrivial cosmological constant. This is an important problem which still needs to be solved. We have, at present, concentrated on the correlation functions involving states from $H^{(1)}$ only. This is, certainly, a very important class of amplitudes since $H^{(1)}$ contains, among the rest, generic tachyons. However, one would definitely like to know how the behavior of the correlation functions changes if some or all of the discrete states are from $H^{(0)}$ and $H^{(2)}$ (these are the other two relative cohomoloy classes; see Sec. III). The work on this subject is in progress and will be reported elsewhere. The related problem is the role which W_∞ symmetry plays in string field theory.

It is important to better understand the question of gauge fixing in 2D SFT. As we have shown in Sec. III, choosing the Siegel's gauge is equivalent to restricting the physical spectrum of open strings to their relative cohomology. In Ref. [13], search for different gauges, which would allow ghost number three states, is advocated. How to choose a gauge fixed action is not just an academic problem since this is *the* action upon which Feynman rules are constructed. While Siegel's gauge is certainly consistent and gives correct tree results, the issue is far from being completely understood. Related to this is the problem of constructing an effective tachyonic field theory ("collective field theory") starting from the Witten's gauge invariant formulation.

The last, but not least, is the problem of going off shell. Indeed, we have seen that scattering amplitudes of the class A_{DDDD} diverge. The same is true for A_{DDTT}^{deg} and $A_{D\bar{D}TT}$. To make these amplitudes sensible, one may have to regularize them. In the framework of the first quantization any SL(2,R) invariant regularization will do (Ref. [14])—there is no physical reason of choosing one instead of the other. In field theory, on the other hand, divergence is the consequence of degeneracy which, in turn, is the consequence of a peculiar two dimensional kinematics. One way to remove the degeneracy and, therefore, divergence is to go off shell. This is not, however, a trivial problem since while it is intuitively fairly clear what the off-shell generic tachyon is, it is not quite so for the states defined, at least on shell, only for some particular values of momenta. The issue of off-shell amplitudes has been discussed in Refs. [1,28] but it definitely needs more attention.

ACKNOWLEDGMENTS

I am grateful to A. Jevicki for his guidance and support throughout the work on this project and to M. Li for making valuable comments on the manuscript. This work was supported in part by the Department of Energy under Contract No. DE-FG02-91ER40688—Task A.

[1] B. Urošević, Phys. Rev. D **47**, 5460 (1993).
[2] M. Bershadsky and D. Kutasov, Phys. Lett. B **274**, 331 (1992); Nucl. Phys. **B382**, 213 (1992).
[3] I. R. Klebanov, in *String Theory and Quantum Gravity*, Proceedings of the Trieste Spring School, 1991, edited by J. Harvey *et al.* (World Scientific, Singapore, 1992).
[4] A. Jevicki, in Proceedings of the Spring School on String Theory, Trieste, Italy, 1993 (unpublished).
[5] B. Lian and G. Zuckerman Phys. Lett. B **254**, 417 (1991); **266**, 2 (1991).
[6] A. M. Polyakov, Mod. Phys. Lett. A **6**, 635 (1991); lectures given at 1991 Jerusalem Winter School (unpublished).
[7] D. Kutasov, in *String Theory and Quantum Gravity [3]*, p. 102.
[8] J. Avan and A. Jevicki, Phys. Lett. B **266**, 35 (1991); **272**, 17 (1992).
[9] E. Witten, Nucl. Phys. **B373**, 187 (1992); I. Klebanov and A. M. Polyakov, Mod. Phys. Lett. A **6**, 3273 (1991); N. Sakai and Y. Tanii, Prog. Theor. Phys. **86**, 547 (1991); Y. Matsumura, N. Sakai, and Y. Tanii, Report No. TIT (Tokyo)-HEEP 127, 1992 (unpublished); Nucl. Phys. **B395**, 354 (1993).
[10] E. Witten and B. Zwiebach, Nucl. Phys. **B377**, 55 (1992).
[11] P. Bouwknegt, J. McCarthy, and K. Pilch, Commun. Math. Phys. **145**, 541 (1992); Report No. CERN-TH.6279/91, 1991 (unpublished); K. Itoh and N. Ohta, Report No. OS-GE-22-91, 1991 (unpublished).
[12] C. Thorn, Phys. Rep. **174**, 1 (1989).
[13] N. Sakai and Y. Tanii, Mod. Phys. Lett. A **7**, 3486 (1992).
[14] M. Li, Nucl. Phys. **B382**, 242 (1992).
[15] I. Ya. Aref'eva, P. B. Medvedev, and A. P. Zubarev, Mod. Phys. Lett. A **8**, 2167 (1993).
[16] A. Schwarz, Commun. Math. Phys. **155**, 249 (1993).
[17] E. Witten, Phys. Rev. D **46**, 5467 (1992).
[18] H. Hata and B. Zwiebach, Ann. Phys. (to be published).
[19] S. Samuel, Nucl. Phys. **B296**, 187 (1988).