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Is it possible to assign physical meaning to field theory with higher derivatives?

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To overcome the difficulties with the energy indefiniteness in field theories with higher derivatives, we propose to use a mechanical analogy—the Timoshenko theory of the transverse flexural vibrations of beams or rods well known in mechanical engineering. This enables one to introduce the notion of "mechanical" energy in such field models that is wittingly positive definite. This approach can be applied at least to the higher derivative models which effectively describe the extended localized solutions in the usual first order field theories (vortex solutions in Higgs models and so on). Any problems with negative norm ghost states and unitarity violation do not arise here.

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I. INTRODUCTION

Field theories with higher derivatives acquire the stable reputation of nonphysical theories. Nevertheless, because they frequently arise in different areas of theoretical physics, interest in this issue is periodically revived [1–10].

A principal shortcoming of higher derivative theories, both classical and quantum, is the lack of a lower-energy bound. Here the energy is implied as a conserved Noether quantity corresponding to the translation invariance of the theory with respect to time or, what is the same, as a value of the Hamiltonian constructed according to Ostrogradsky's rules on the solution of equations of motion [11].

The attractive properties of quantum field theories with higher derivatives are also worth mentioning. In particular, the convergence of Feynman diagrams is improved owing to the higher derivative terms in the Lagrangian. For example, conformal gravity is found to be renormalizable whereas Einstein gravity is not [4,12]. Just this property of the theories in question is used to construct the gauge-invariant renormalization of Yang-Mills fields by adding higher derivative terms to the standard Lagrangian [13].

It should be noted that the lack of a lower-energy bound for a completely isolated system is admissible in principle if the energy is an integral of motion.[1] But, unfortunately, such isolated systems are not realized practically. A nonremovable interaction with an external environment inevitably results in pumping out an arbitrary amount of energy from the system, lowering its energy without limits.

Obviously, higher derivatives in time in the Lagrangian lead to additional degrees of freedom, since there is one-to-one correspondence between the dynamical degrees of freedom and the initial data for the relevant Euler-Lagrange equations. In the following, for the sake of definiteness we shall discuss field theories with Lagrangian functions depending, at most, on the second derivatives in time. Here there arises a very typical picture for higher derivative theories: In addition to the basic mode of oscillations which takes place even in the absence of the second derivatives in the Lagrangian there emerge additional, as a rule, higher-frequency modes. The contribution to the energy of the second mode has the opposite sign as compared with the basic one. Therefore, even at the classical level it turns out to be more profitable energetically to excite the oscillations from the second mode. The more oscillations of that sort are excited and the larger their amplitudes are, the lower the total energy of a system turns out to be. From this it follows that field theories with higher derivatives are unacceptable physically at least in making use of their standard interpretation.

All these arguments are applied exactly to the quantum level as well. Here the oscillations of both positive- and negative-energy modes are associated with the cor-

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[1] It is usually believed that the energy, being indefinite in sign, entails the instability of the classical dynamics for theories with higher derivatives, although a very special counterexample is known [14]. More exactly, if the energy of a system is not definite in sign, the problem of stability cannot be solved using the Lagrange-Dirichlet theorem [15] and, in general, it is not reduced to searching for the Lyapunov function as in the case of the usual theories with Lagrangian functions, containing, at most, first derivative in time of dynamical variables.

responding quanta of excitations. In virtue of the impossibility of removing the external perturbations, as has been noted previously, an unlimited number of negative-energy quanta will be created. As a result, in field theories with higher derivatives a problem such as the infrared catastrophe in quantum electrodynamics arises, but for all frequencies of the second mode now. This problem was successfully overcome in electrodynamics, but it still remains unsolved in higher derivative theories.

Some time ago, it was popular to use the formalism of an indefinite metric in the Fock space of the states. This metric can be introduced by a mutual interchange of the creation and annihilation operators of quanta of the second mode. As a result, the quantum states with excitations from the second mode acquire a negative norm, but the energy calculated as an expectation value of the Ostrogradsky Hamiltonian over these states turns out to be a positive definite quantity [1,2]. Thereby, the problem of negative energy is reduced to searching for a physical interpretation of theories with implicitly-violated unitarity. So far there has been no acceptable solution of the problem along this way [9]. Therefore in the following we shall only deal with the difficulty of the energy being indefinite in sign in theories with higher derivatives.

As far as we know, attempts to attach physical meaning to higher derivative theories are based on the conjecture forbidding the excitations with negative energy. This constraint should appear as the boundary condition following from cosmology [7] or as a by-product of the nonperturbative quantum solutions [5], or it has been introduced from the outset in formulating these models [10].

We would like to suggest another solution to this problem. Namely, we will show that the energy in the theory with higher derivatives can be redefined using a mechanical analogy. Here we have in mind the special class of higher derivative theories arising when the effective Lagrangians are constructed in extended object models (strings, in particular). Even at the classical level an extended object requires a field description. We shall suppose that the original field theory does not contain higher derivative terms in the Lagrangian so that its energy is bounded from below. The neglect of the details of the internal structure of the extended object along one or several of its internal dimensions results, as a rule, in higher derivative terms in the effective Lagrangian. Now the energy of the effective theory turns out to be unbounded from below.

As a specific model, we shall treat a relativistic rigid string with the action functional depending on the second derivatives of string coordinates [16,17]. Here the rigidity term takes effectively into account the thickness of the string. It may be imagined clearly that this system simulates, for example, the gluon tube of finite radius that connects the quarks inside the hadrons. Such a simple picture arises in certain approximations to QCD [18,19]. Taking the finite thickness of the cosmic strings into account one arrives at the model of the rigid string as well [20–22].

To solve the equations of motion in the model of the relativistic string with rigidity, we confine ourselves to the harmonic approximation in a timelike gauge.[2] Then we shall elucidate an analogy between the rigid string and the most simple mechanical system that takes into account the stiffness of an extended vibrating body. To this end we shall consider Timoshenko's theory of the flexural vibrations of beams and rods, well known in mechanical engineering [24]. This theory takes effectively into account the finite thickness of the beam via the second derivatives in time and in longitudinal coordinates in the Lagrange function. It is important that in this mechanical system there is no problem with the energy which is positive definite. Thus, this analogy points out in what way the definition of the energy in the model of the rigid string should be changed to get a positive definite energy.

The outline of the paper is as follows. In Sec. II the problem of the energy unbounded from below, typical of field theories with higher derivatives, is discussed in the framework of the relativistic string with rigidity by making use of the harmonic approximation. The theory of flexural vibrations of the Timoshenko beam is given in Sec. III. The Hamiltonian constructed here by applying the Ostrogradsky rules leads to an energy unbounded from below. Nevertheless, in this case there exists the notion of mechanical energy which is positive definite. In Sec. IV the analogy between the relativistic string with rigidity and the beam or rod is used for constructing the positive definite energy. In Sec. V the proposed method is compared with other attempts to overcome the drawback related to the energy unbounded from below in field theories with higher derivatives.

II. HARMONIC APPROXIMATION IN THE RIGID STRING MODEL

The localized vortex solutions to the classical equations of motion having the form of a flux tube or a string are well known in gauge fields models with the Higgs Lagrangian [25–27]. The behavior of these solutions can be described by some effective Lagrangians [22,28]. In the zeroth order approximation in the flux tube width one obtains here the Nambu-Goto action for the relativistic string [18]. The first order correction in the tube width leads to a rigid string model with the action depending on the second derivatives of the string coordinates [20,21]:

$$W = -\rho_0 c \int\int d^2u\sqrt{-g}\left(1 - \frac{\alpha}{2}r_s^2\Delta x^\mu \Delta x_\mu\right). \qquad (2.1)$$

Here $x^\mu(u^0, u^1)$, $\mu = 0, 1, \ldots, D-1$, are the string coordinates in D-dimensional space-time whose metric has the signature $(+, -, \ldots, -)$, ρ_0 is the linear mass density of the flux tube (or of the string), r_s is the transverse size of this tube, and c is the velocity of light. The in-

[2] As is known, the total theory of the relativistic string with rigidity owing to the reparametrization invariance of its action is a dynamical system with constraints in phase space [23]. However, the number of these constraints is not enough to remove all the quanta of negative energy.

ternal geometry on the string world surface is defined by the induced metric $g_{ij}(u) = \partial_i x^\mu \partial_j x_\mu$, $i,j = 0,1$, $g = \det(g_{ij})$, $g < 0$. The Laplace-Beltrami operator with respect to this metric reads explicitly

$$\Delta = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial u^i}\left(\sqrt{-g}g^{ij}\frac{\partial}{\partial u^j}\right), \quad g_{ij}g^{jk} = \delta_i^k. \qquad (2.2)$$

For the curvilinear coordinates u^0 and u^1 on the string world sheet we shall frequently use another, more ordinary, notation $u^0 = \tau$, $u^1 = \sigma$. The numerical parameter α in the action (2.1) is specified by the concrete mechanism generating the flux tube. In the Abelian gauge model with the simplest Higgs potential (the Nielsen-Olesen vortex model for the relativistic string) α proves to be about 20 [20]. The action (2.1) results in the nonlinear equations of motion containing the partial derivatives of the fourth order of the string coordinates x^μ [29]. To advance in their study, we employ the following parametrization including the time-like gauge on the string world surface:

$$x^\mu(u) = \left\{ct,\ \frac{l}{\pi}\sigma,\ \mathbf{x}(u)\right\}, \quad \tau = t, \qquad (2.3)$$

where $\mathbf{x}(u)$ are $(D-2)$ transverse string coordinates. Although the parametrization (2.3) holds true only for the limited string motions (so-called harmonic approximation [30]), it will be sufficient for our aims.

Inserting the ansatz (2.3) into (2.1) and expanding the integrand of (2.1) up to second order terms in powers of $\mathbf{x}(u)$ we obtain [30]

$$W = \frac{\rho_0}{2\pi}\int dt\int_0^\pi d\sigma\,[\dot{\mathbf{x}}^2 - a^2\mathbf{x}'^2 - \epsilon a^2(a^{-2}\ddot{\mathbf{x}} - \mathbf{x}'')^2], \qquad (2.4)$$

where $a = \pi c/l$, $\epsilon = \alpha(\pi r_s/l)^2$, and l is the string length. The dot means differentiation with respect to $t = \tau$ and the prime with respect to σ. Variation of the action (2.4) gives the equations of motion

$$(1 + \epsilon\Box)\Box\mathbf{x}(u) = 0, \qquad (2.5)$$

$$\Box \equiv a^{-2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial\sigma^2},$$

and the boundary conditions

$$(1 + \epsilon\Box)\,\mathbf{x}' = 0, \qquad (2.6)$$

$$\Box\mathbf{x} = 0, \quad \sigma = 0,\pi.$$

Owing to Eqs. (2.5) and (2.6) being linear, their general solution can be represented as the sum

$$\mathbf{x}(t,\sigma) = \mathbf{x}_1(t,\sigma) + \mathbf{x}_2(t,\sigma). \qquad (2.7)$$

Here $\mathbf{x}_1(u)$ are transverse degrees of freedom of the open Nambu-Goto string [18],

$$\Box\mathbf{x}_1(u) = 0, \qquad (2.8)$$

$$\mathbf{x}_1' = 0, \quad \sigma = 0,\pi.$$

The coordinates $\mathbf{x}_2(u)$ obey the equations

$$(1 + \epsilon\Box)\,\mathbf{x}_2(u) = 0, \qquad (2.9)$$

$$\mathbf{x}_2 = 0, \quad \sigma = 0,\pi.$$

As usual, the general solution of the boundary problems (2.8) and (2.9) is given by the expansions in the corresponding eigenfunctions [30];

$$\mathbf{x}_1(t,\sigma) = \mathbf{Q} + \frac{\mathbf{P}t}{\rho_0 l} + i\sqrt{\frac{\hbar}{\pi\rho_0 c}}\sum_{n\neq 0}\frac{\boldsymbol{\alpha}_n}{\overset{(1)}{\omega}_n}\cos n\sigma\, e^{-ia\overset{(1)}{\omega}_n t}, \qquad (2.10)$$

$$\mathbf{x}_2(t,\sigma) = \sqrt{\frac{\hbar}{\pi\rho_0 c}}\sum_{n\neq 0}\frac{\boldsymbol{\beta}_n}{\overset{(2)}{\omega}_n}\sin n\sigma\, e^{ia\overset{(2)}{\omega}_n t},$$

with two series of the eigenfrequencies:

$$\overset{(1)}{\omega}_n = -\overset{(1)}{\omega}_{-n} = n, \quad \overset{(2)}{\omega}_n = -\overset{(2)}{\omega}_{-n} = \sqrt{n^2 + \frac{1}{\epsilon}}, \quad n = 1,2,\ldots\ . \qquad (2.11)$$

Here $\mathbf{Q}$ and $\mathbf{P}$ are the coordinates of the center of mass and the total momentum of the string, respectively, and the amplitudes $\boldsymbol{\alpha}_n$ and $\boldsymbol{\beta}_n$, in virtue of the reality of the variables $\mathbf{x}_1$ and $\mathbf{x}_2$, obey the usual rules of complex conjugation:

$$\boldsymbol{\alpha}_n^* = \boldsymbol{\alpha}_{-n}, \quad \boldsymbol{\beta}_n^* = \boldsymbol{\beta}_{-n}, \quad n = 0,\pm 1,\pm 2,\ldots\ . \qquad (2.12)$$

Thus, the transverse coordinates of the relativistic string with rigidity $\mathbf{x}(u)$ are described in the harmonic approximation by the pair of independent variables $(\mathbf{x}_1,\mathbf{x}_2)$. This duplication of the number of dynamical degrees of freedom is general for higher derivative theories. It is also reflected explicitly in the canonical formalism worked out for higher derivative theories by Ostrogradsky more than a century ago [11]. In our case according to the Ostrogradsky method the independent generalized coordinates are $\mathbf{q}_1 = \mathbf{x}$ and $\mathbf{q}_2 = \dot{\mathbf{x}}$ and their conjugate momenta are defined by the expressions

$$\mathbf{p}_1 = \frac{\partial\mathcal{L}}{\partial\dot{\mathbf{x}}} - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\ddot{\mathbf{x}}}\right) = \frac{\rho_0 l}{\pi}(1 + \epsilon\Box)\,\dot{\mathbf{x}},$$

$$\mathbf{p}_2 = \frac{\partial\mathcal{L}}{\partial\ddot{\mathbf{x}}} = -\epsilon\frac{\rho_0 l}{\pi}\Box\mathbf{x}. \qquad (2.13)$$

With the use of (2.7), (2.8), and (2.9) from (2.13) we find $\mathbf{p}_1 = (\rho_0 l/\pi)\,\dot{\mathbf{x}}_1$, $\mathbf{p}_2 = (\rho_0 l/\pi)\,\mathbf{x}_2$. As a result, the canonical Ostrogradsky Hamiltonian

$$H = \frac{\rho_0 l}{2\pi}\int_0^\pi d\sigma\,(\mathbf{p}_1\dot{\mathbf{x}} + \mathbf{p}_2\ddot{\mathbf{x}} - \mathcal{L}), \qquad (2.14)$$

in terms of the variables $\mathbf{x}_1$ and $\mathbf{x}_2$, takes the form

$$H = \frac{\rho_0 l}{2\pi}\int_0^\pi d\sigma \left[(\dot{\mathbf{x}}_1^2 + a^2\mathbf{x}_1'^2) - \left(\dot{\mathbf{x}}_2^2 - a^2\mathbf{x}_2'^2 - \frac{a^2}{\epsilon}\mathbf{x}_2^2 - 2\mathbf{x}_2\ddot{\mathbf{x}}_2\right)\right]. \tag{2.15}$$

Hence it follows that already at the classical level the excitations of the degrees of freedom $\mathbf{x}_2$ may give a negative contribution to the total energy of the string. Indeed, inserting the general solution (2.10) into (2.15) we obtain

$$E = \frac{\mathbf{P}^2}{2M} + \frac{a\hbar}{2}\sum_{n=1}^{\infty}(\boldsymbol{\alpha}_n^*\boldsymbol{\alpha}_n + \boldsymbol{\alpha}_n\boldsymbol{\alpha}_n^*) - \frac{a\hbar}{2}\sum_{n=1}^{\infty}(\boldsymbol{\beta}_n^*\boldsymbol{\beta}_n + \boldsymbol{\beta}_n\boldsymbol{\beta}_n^*), \tag{2.16}$$

where $M = \rho_0 l$ is the total mass of the string.

Thus, in the rigid string model we arrive at the problem general for all higher derivative theories of the lack of a lower-energy bound [10,14]. In the quantum theory of this system the following annihilation and creation operators a_n^i and b_n^i are defined:

$$\alpha_n^i = \sqrt{\overset{(1)}{\omega}_n}\, a_n^i, \quad \alpha_{-n}^i = \alpha_n^{\dagger i} = \sqrt{\overset{(1)}{\omega}_n}\, a_n^{\dagger i},$$

$$\beta_n^i = \sqrt{\overset{(2)}{\omega}_n}\, b_n^i, \quad \beta_{-n}^i = \beta_n^{\dagger i} = \sqrt{\overset{(2)}{\omega}_n}\, b_n^{\dagger i},$$

with the standard commutation relations

$$[a_n^i, a_m^{\dagger j}] = [b_n^i, b_m^{\dagger j}] = \delta^{ij}\delta_{nm},$$

$$i,j = 1,2,\ldots,D-2, \quad n,m = 1,2,\ldots \ .$$

Therefore, taking account of the zero-point oscillations of the string we obtain the expression of the energy indefinite in sign:

$$E = \frac{\mathbf{P}^2}{2M} + a\hbar\sum_{n=1}^{\infty}\overset{(1)}{\omega}_n\left(\mathbf{a}_n^\dagger\mathbf{a}_n + \frac{D-2}{2}\right) - a\hbar\sum_{n=1}^{\infty}\overset{(2)}{\omega}_n\left(\mathbf{b}_n^\dagger\mathbf{b}_n + \frac{D-2}{2}\right). \tag{2.17}$$

As is well known [1,2,9], the negative-energy $(-a\hbar\overset{(2)}{\omega}_n)$ creation operators $\mathbf{b}_n^\dagger$ can be regarded as positive-energy $(+a\hbar\overset{(2)}{\omega}_n)$ annihilation ones. Thereby, in the Fock space of the states the positive norm negative-energy excitations are transformed into negative norm positive-energy ones. So the violation of unitarity in the quantum theory is really a reflection of the essentially classical problem of the lack of a lower-energy bound [see (2.16) and [9,31,32]]. In a recent papers (see [10] for review) it was proposed to apply the perturbative constraints to freeze out the excitations of those degrees of freedom which give rise to the negative contribution to the energy. In the present paper using the mechanical analogy we would like to show that there exists another solution of the problem in question.

III. FLEXURAL VIBRATIONS OF THE TIMOSHENKO BEAM

To elucidate the analogy between the rigid string and the mechanical vibrating systems we consider in this section the flexural vibrations of the so-called Timoshenko beam.

In principle, the flexural vibrations of three-dimensional extended objects such as rods or beams are described by the general equations of the three-dimensional theory of elasticity [33]. However, in virtue of their complication this description is not suitable for practical use. Therefore, one has to employ here some approximations.

If a rod or a beam is considered as an infinitely thin one (that is, if we fully neglect its transverse sizes), then we obtain the string described by the equation for the lateral deflection $y(x,t)$:

$$Ty'' - \mu\ddot{y} = 0. \tag{3.1}$$

Here T is the string tension and μ is the linear density of the string matter. As was to be expected, none of the characteristics of the transverse string sizes enter into (3.1). By taking into account the beam thickness effectively, Eq. (3.1) is modified as [24]

$$EIy'''' - Ty'' + F\rho\ddot{y} = 0, \tag{3.2}$$

where E is Young's modulus, I is the momentum of inertia of a cross section around the principal axis normal to the plane of motion, F is the cross section area, and ρ is the mass density. In applications the case of the absence of longitudinal strength ($T = 0$) is frequently considered. If it is really the case, then Eq. (3.2) is transformed into the Bernoulli-Euler equation

$$EIy'''' - F\rho\ddot{y} = 0. \tag{3.3}$$

The effect of the transverse sizes of the beam leads to the appearance, in Eqs. (3.2) and (3.3), of higher derivatives as compared with the string case (3.1). The corresponding Lagrange densities contain the $(y'')^2$ term, but the problem with the positive definiteness of the energy does not arise there. Only theories with higher derivatives in time suffer from the above problem. The model of flexural vibrations of beams proposed at the begining of our century by Timoshenko [24,34] belongs to such theories. In addition to the bending of the beam under the flexural vibrations the Timoshenko model takes into account the shear deformations of its elements.[3] Two degrees of freedom are associated with each cross section of the beam: the deflection due to bending and that due to shear. This duplication of the number of degrees of freedom in the Timoshenko model leads to the equation

[3]Apart from this, the inertia of the gyration of the beam cross sections is taken into account in the Timoshenko model (the Rayleigh correction [35]). However, this fact itself does not lead to the appearance of higher derivatives in time in the theory.

of the fourth order in time:

$$EIy'''' + F\rho\ddot{y} - \rho I\left(1 + \frac{E}{kG}\right)\ddot{y}'' + \rho I\frac{\rho}{kG}\ddddot{y} = 0. \tag{3.4}$$

Here G is the shear modulus and k is the shear coefficient (the phenomenological parameter depending on the geometry of the beam cross section).

Equation (3.4) should be supplemented with the boundary conditions at the ends $x_1 = 0$, $x_2 = l$ of the beam. In the following for the sake of simplicity we shall consider the hinged-hinged beam, when both the flexure and the bending moment of the beam are equal to zero:

$$y(t,0) = y''(t,0) = 0, \quad y(t,l) = y''(t,l) = 0. \tag{3.5}$$

The general solution of Eq. (3.4) and the boundary conditions (3.5) have the form

$$y(t,x) = \sum_{n\neq 0}^{\infty} \sin\lambda_n x\,[q_{n1}(t) + q_{n2}(t)], \tag{3.6}$$

where $\lambda_n = n\pi/l$ and the functions $q_{ns}(t) = A_{ns}\cos(\omega_{ns}t + \epsilon_{ns})$, $s = 1,2$, are the normal coordinates corresponding to two series of the eigenfrequencies $\omega_{ns} = \lambda_n\sqrt{E/\rho}\,\omega_{*ns}$, $s = 1,2$, respectively. The dimensionless frequencies ω_{*ns} are defined by the formula

$$\left.\begin{matrix}\omega^2_{*n1}\\ \omega^2_{*n2}\end{matrix}\right\} = \frac{1}{2}\left[1 + \xi + \frac{\xi}{\lambda_n^2 r^2} \mp\sqrt{\left(1 + \xi + \frac{\xi}{\lambda_n^2 r^2}\right)^2 - 4\xi}\,\right], \tag{3.7}$$

where $\xi = kG/E$ is the dimensionless parameter and r is the radius of gyration of the beam cross section around the principal axis normal to the plane of motion, $r^2 = I/F$.

When the shear modulus G tends formally to the infinity, the Timoshenko equation (3.4) is reduced to the Bernoulli-Euler one with the Rayleigh correction

$$EIy'''' + \rho F\ddot{y} - \rho I\ddot{y}'' = 0. \tag{3.8}$$

In this case the frequencies of the first series (3.7) in the Timoshenko theory tend to finite values,

$$\omega^2_{*n1} \to \frac{\lambda_n^2 r^2}{1 + \lambda_n^2 r^2}, \tag{3.9}$$

and those of the second mode of oscillation go to infinity.

The Timoshenko equation (3.4) and the corresponding boundary conditions (3.5) can be derived by varying the Lagrangian density [36]

$$\mathcal{L} = \tfrac{1}{2}\left(\dot{y}^2 - a_1 y''^2 - a_3\ddot{y}^2 + a_2\ddot{y}y''\right). \tag{3.10}$$

Here a_i, $i = 1,2,3$, are the coefficients of Eq. (3.4):

$$a_1 = \frac{EI}{\rho F}, \quad a_2 = \frac{I}{F}\left(1 + \frac{E}{kG}\right), \quad a_3 = \frac{\rho I}{FkG}. \tag{3.11}$$

Further, using (2.13) one can define the canonical variables

$$q_1 = y, \quad q_2 = \dot{y},$$

$$p_1 = \frac{\partial\mathcal{L}}{\partial\dot{y}} - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\ddot{y}}\right) = \dot{y} + a_3\dddot{y} - \frac{a_2}{2}\dot{y}',$$

$$p_2 = \frac{\partial\mathcal{L}}{\partial\ddot{y}} = -a_3\ddot{y} + \frac{a_2}{2}y'', \tag{3.12}$$

and construct the Ostrogradsky canonical Hamiltonian

$$H = \frac{1}{2}\int_0^l dx\left[2p_1q_2 - \frac{p_2^2}{a_3} - q_2^2 + \left(a_1 - \frac{a_2^2}{4a_3}\right)q_1''^2 + \frac{a_2}{a_3}p_2q_1''\right]$$
$$= \frac{1}{2}\int_0^l dx\left(\dot{y}^2 + 2a_3\dot{y}\dddot{y} - a_2\dot{y}\dot{y}'' - a_3\ddot{y}^2 + a_1y''^2\right). \tag{3.13}$$

This Hamiltonian is conserved in time and it generates the time translations $t \to t + \Delta t$. The value of H in the general solution (3.6) is the energy of the Timoshenko beam calculated according to Ostrogradsky:

$$E_O = \frac{l}{4}a_3\sum_{n=1}^{\infty}\left(\omega_{n2}^2 - \omega_{n1}^2\right)\left(\omega_{n1}^2A_{n1}^2 - \omega_{n2}^2A_{n2}^2\right). \tag{3.14}$$

Thus, the flexural vibrations with amplitudes A_{n2} give the negative contribution to E_O [36] because for all n's we have, from (3.7),

$$\omega_{n2}^2 - \omega_{n1}^2 > 0.$$

Formula (3.14) is completely equivalent to that (2.16) for the energy of the relativistic string with rigidity in the harmonic approximation. In spite of the principal difference of these objects, they suffer from the same lack of lower-energy bound. However, in the case of the flexural vibrations of beams there exists the well defined notion of mechanical energy which is always a positive quantity, of course.

The mechanical energy of a rod or a beam is a sum of the kinetic and potential ones of their elements. Let $y_1(t,x)$ be a lateral deflection of the beam due to bending only and $y_2(t,x)$ be that due to shear and $y(t,x) = y_1(t,x) + y_2(t,x)$ is the total lateral deflection of the beam. In the Timoshenko model the kinetic energy contains the contribution from the transverse motion of beam elements,

$$T_{\rm tr} = \frac{\rho FI}{2}\int_0^l dx\,\dot{y}^2, \tag{3.15}$$

and that from the gyration of the beam cross section,

$$T_{\rm gyr} = \frac{\rho I}{2}\int_0^l dx\,\dot{y}_1'^2. \tag{3.16}$$

According to the Hooke law one can easily find the potential energy of the flexural vibrations of the beam. This energy consists of the elastic energy of the bending deformations,

$$V_{\text{ben}} = \frac{EI}{2}\int_0^l dx\, y_1''^2, \tag{3.17}$$

and that of the shear deformations,

$$V_{\text{sh}} = \frac{kFG}{2}\int_0^l dx\, y_2'^2. \tag{3.18}$$

Joining together formulas (3.15)–(3.18) we obtain the action functional of the Timoshenko model:

$$W = \frac{\rho F}{2}\int_0^l dx\,\left(\dot{y}^2 + r^2\dot{y}_1'^2\right) - \frac{EI}{2}\int_0^l dx\, y_1''^2 - \frac{kFG}{2}\int_0^l dx\, y_2'^2. \tag{3.19}$$

Variation of the action (3.19) gives the following equations for $y_1(t,x)$ and $y_2(t,x)$:

$$\frac{\rho}{E}\ddot{y}_1 - y_1'' = \frac{kG}{r^2E}y_2, \tag{3.20}$$

$$\ddot{y}_2 - \frac{kG}{\rho}y_2'' = -\ddot{y}_1, \tag{3.21}$$

and the boundary conditions which take for the hinged-hinged beam the form

$$y(t,0) = y(t,l) = 0, \quad y''(t,0) = y''(t,l) = 0,$$

$$\psi'(t,0) = \psi'(t,l) = 0. \tag{3.22}$$

Here $\psi(t,x) \equiv y_1'(t,x)$. Combining Eqs. (3.20) and (3.21) one may obtain the Timoshenko equation (3.4) for the total lateral deflection $y = y_1 + y_2$.

The sum of (3.15)–(3.18) is the total mechanical energy of flexural vibrations of the Timoshenko beam:

$$E = \frac{\rho F}{2}\int_0^l dx\,\left(\dot{y}^2 + r^2\dot{\psi}^2\right) + \frac{EI}{2}\int_0^l dx\,\psi'^2 + \frac{kFG}{2}\int_0^l dx\,(y' - \psi)^2. \tag{3.23}$$

In the case of the hinged-hinged beam we have the general solution (3.6) for $y(t, x)$ and the analogous expansion for $\psi(t, x)$:

$$\psi(t,x) = \sum_{n=1}^{\infty}\cos\lambda_n x\left[\frac{k_{n1}}{l}q_{n1}(t) + \frac{k_{n2}}{l}q_{n2}(t)\right], \tag{3.24}$$

where k_{ns}/l are the amplitude ratios in the expansions (3.6) and (3.24)

$$k_{ns} = n\pi\left(1 - \xi^{-1}\omega_{*ns}^2\right), \quad s = 1,2, \quad n = 1,2,\dots\ . \tag{3.25}$$

Substituting (3.6) and (3.24) into (3.23) we obtain the expression for the mechanical energy in terms of the amplitudes A_{ns}, $s = 1,2$:

$$E_M = \frac{l}{4}\sum_{n=1}^{\infty}\left[\left(1 + \frac{r^2k_{n1}^2}{l^2}\right)\omega_{n1}^2A_{n1}^2 + \left(1 + \frac{r^2k_{n2}^2}{l^2}\right)\omega_{n2}^2A_{n2}^2\right]. \tag{3.26}$$

As was expected, the energy (3.26) is positive definite in sign because of the positive definiteness of the original functional (3.23).

So in the Timoshenko model there exists a mechanical energy positive definite in sign [formulas (3.23), (3.26)] and an Ostrogradsky energy unbounded from below [formulas (3.13), (3.14)]. Both these quantities are integrals of motion and they are mutually related:

$$E_M = E_O + \frac{l}{4}\left(\frac{a_3}{r}\right)^2\sum_{n=1}^{\infty}\frac{\left(\omega_{n2}^2 - \omega_{n1}^2\right)}{\lambda_n^2} \times \left(\omega_{n2}^4A_{n2}^2 - \omega_{n1}^4A_{n1}^2\right). \tag{3.27}$$

However, the mechanical energy (3.23) in contrast with the Ostrogradsky energy (3.14) has quite a clear physical meaning.

IV. "MECHANICAL ENERGY" OF THE RIGID STRING

The description of the rigid string dynamics [Eqs. (2.5), (2.7), (2.8), and (2.9)] is in many respects analogous to that of the flexural vibrations of the Timoshenko beam [Eqs. (3.4), (3.20), (3.21), and (3.22)]. Indeed, both objects can be described either by one equation of the fourth order [Eqs. (2.5) and (3.4), respectively] or by two equations of the second order [Eqs. (2.8), (2.9), (3.20), and (3.21) for the "partial" deflections]. The "material" of the gluon tube in comparison with that of a beam has very distinct mechanical properties, of course. Therefore, in these models there is no complete identity between the corresponding equations. But it is important that starting from Eqs. (2.8) and (2.9) in the rigid string model one may identify according to the usual rules the energy corresponding to the mechanical one in the Timoshenko model.

For Eqs. (2.8) and (2.9) we have the standard Lagrangian densities

$$\mathcal{L}_1 = \frac{1}{2}\left(\dot{\mathbf{x}}_1^2 - \mathbf{x}_1'^2\right), \quad \mathcal{L}_2 = \frac{\epsilon}{2}\left(\dot{\mathbf{x}}_2^2 - \mathbf{x}_2'^2\right) - \frac{\mathbf{x}_2^2}{2}. \tag{4.1}$$

The total energy is defined by the formula

$$E_M = \frac{1}{2}\int_0^\pi d\sigma\,\left(\dot{\mathbf{x}}_1^2 + \mathbf{x}_1'^2\right) + \frac{1}{2}\int_0^\pi d\sigma\,\left(\dot{\mathbf{x}}_2^2 + \mathbf{x}_2'^2 + \mathbf{x}_2^2\right). \tag{4.2}$$

Substituting the general solution (2.10) into (4.2) one finds

$$E_M = \frac{\mathbf{P}^2}{2M} + \frac{a\hbar}{2}\sum_{n=1}^{\infty}(\boldsymbol{\alpha}_n^*\boldsymbol{\alpha}_n + \boldsymbol{\alpha}_n\boldsymbol{\alpha}_n^*) + \frac{a\hbar}{2}\sum_{n=1}^{\infty}(\boldsymbol{\beta}_n^*\boldsymbol{\beta}_n + \boldsymbol{\beta}_n\boldsymbol{\beta}_n^*)\,. \tag{4.3}$$

As was expected, the mechanical energy (4.3) in the rigid string model is the quantity positive definite in sign. Obviously, this property of the energy also holds at the quantum level. Taking account of zero-point oscillations one may write the mechanical energy of the rigid string as

$$E_M = \frac{\mathbf{P}^2}{2M} + a\hbar\sum_{n=1}^{\infty}\omega_{n1}\left(\mathbf{a}_n^\dagger\mathbf{a}_n + \frac{D-2}{2}\right) + a\hbar\sum_{n=1}^{\infty}\omega_{n2}\left(\mathbf{b}_n^\dagger\mathbf{b}_n + \frac{D-2}{2}\right). \tag{4.4}$$

In this case all string states in Fock space are positive in norm; hence, the above mentioned problem with the violation of unitarity does not arise here.

V. CONCLUSION

Thus in the framework of the rigid string model we have shown that one can construct, for this object, a positive definite "mechanical" energy instead of the Ostrogradsky energy unbounded from below. Obviously, the same can be done for any field model describing extended objects at the classical level. An appealing aspect of our approach is the absence of any constraints on the physical degrees of freedom introduced "by hand" in some other papers on this subject. This enables one to construct a complete quantum theory instead of a truncated one. Further, at the quantum level the problems with negative norm states and the loss of unitarity do not arise.

On the other hand, the energy constructed according to Ostrogradsky generates time translations, but the mechanical one does not. Therefore, the sole difficulty which can occur here is to prove the relativistic invariance of such theories by making use of the notion of mechanical energy.

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