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Dirty black holes: Entropy as a surface term

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It is by now clear that the naive rule for the entropy of a black hole, (entropy) = $\frac{1}{4}$ (area of event horizon), is violated in many interesting cases. Indeed, several authors have recently conjectured that in general the entropy of a dirty black hole might be given purely in terms of some surface integral over the event horizon of that black hole. A formal proof of this conjecture, using Lorentzian signature techniques, has recently been provided by Wald. This paper performs two functions. First, by extending a previous analysis due to the present author [Phys. Rev. D **48**, 583 (1993)] it is possible to provide a rather different proof of this result—a proof based on Euclidean signature techniques. The proof applies both to arbitrary static (aspheric) black holes, and also to arbitrary stationary axisymmetric black holes. The total entropy is

$$S = \frac{kA_H}{4\ell_P^2} + \int_H \mathcal{S}\ \sqrt{{}_2g}\ d^2x.$$

The integration runs over a spacelike cross section of the event horizon H. The surface entropy density $\mathcal{S}$ is related to the behavior of the matter Lagrangian under time dilations. Second, I shall consider the specific case of Einstein-Hilbert gravity coupled to an effective Lagrangian that is an arbitrary function of the Riemann tensor (though not of its derivatives). In this case a more explicit result is obtained:

$$S = \frac{kA_H}{4\ell_P^2} + 4\pi\frac{k}{\hbar}\int_H \frac{\partial\mathcal{L}}{\partial R_{\mu\nu\lambda\rho}}\ g^{\perp}_{\mu\lambda}g^{\perp}_{\nu\rho}\ \sqrt{{}_2g}d^2x.$$

The symbol $g^{\perp}_{\mu\nu}$ denotes the projection onto the two-dimensional subspace orthogonal to the event horizon. Although the derivation exhibited in this paper proceeds via Euclidean signature techniques the result can be checked against certain special cases previously obtained by other techniques, e.g., (Ricci)^n gravity, R^n gravity, and Lovelock gravity.

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I. INTRODUCTION

The entropy versus area relationship for generic "dirty" black holes has recently engendered considerable interest [1–7]. Generically a dirty black hole [8] is a black hole distorted by either (1) various classical matter fields, (2) higher curvature terms in the gravity Lagrangian [e.g., (Riemann)^n], or (3) infestation with some version of quantum hair.

The present paper addresses two main points.

First, it has recently been conjectured that the entropy of a dirty black hole can *always* be cast into the form of an integral of some quantity over the event horizon [3,9,10]. A formal proof of this result, based on Lorentzian signature Lagrangian techniques, has recently been announced [5]. Details and applications may be found in [6,7]. In this paper I present an alternative proof of this result. The present proof is obtained by utilizing Euclidean space techniques in the manner of [1], and is ultimately an extension of the original Gibbons-Hawking Euclidean signature technology [11]. The proof applies both to arbitrary static (aspheric) black holes, and also (with additional technical complications) to arbitrary stationary axisymmetric black holes. The total entropy is

$$S = \frac{kA_H}{4\ell_P^2} + \int_H \mathcal{S}\ \sqrt{{}_2g}\ d^2x. \tag{1}$$

The integration runs over a spacelike cross section of the event horizon H. The surface entropy density $\mathcal{S}$ is related (in a particular manner involving time dilations) to the surface term arising in the integration by parts that connects the stress-energy tensor with the variation of the action under a variation of the spacetime metric. For definiteness, the calculations are carried out in four-dimensional spacetime, but the generalization to arbitrary dimensionality is immediate.

Second, as a specific example, this paper will focus on the case of black holes in Einstein-Hilbert gravity coupled to an effective Lagrangian that is any arbitrary function of the Riemann tensor (though not of its derivatives). Interest in such a toy model is justified by noting that whatever the underlying quantum theory of gravity is, one would expect on general grounds that the low-energy theory should be describable by an effective Lagrangian that contains at least the class of terms indicated above.

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Applying the general formalism developed in this paper to this particular case yields

$$S = \frac{kA_H}{4\ell_P^2} + 4\pi\frac{k}{\hbar}\int_H J^{\mu\nu\lambda\rho}\, g^{\perp}_{\mu\lambda} g^{\perp}_{\nu\rho}\, \sqrt{{}_2g}\, d^2x. \qquad (2)$$

The tensor $J^{\mu\nu\lambda\rho} \equiv \partial\mathcal{L}/\partial R_{\mu\nu\lambda\rho}$ has the same symmetries at the Riemann tensor. The symbol $g^{\perp}_{\mu\nu}$ denotes the projection onto the two-dimensional subspace orthogonal to the event horizon.

It is instructive to check this formula against several special cases that have been derived by rather different methods. For instance, Jacobson and Myers [2] have recently evaluated the entropy for black holes in Lovelock gravity using Hamiltonian methods. The present analysis reproduces their result with no difficulty. More recently, Jacobson, Kang, and Myers [3,4] have extended their analysis to the case where the Lagrangian is an arbitrary function of the Ricci scalar. The entropy for black holes of this type was extracted by using a combination of field redefinition and Hamiltonian techniques. (Consider the behavior of the black hole under conformal deformations.) Again, this result can be shown to be a special case of the general formula given above. Furthermore, Jacobson, Kang, and Myers [3,4] have also considered the case where the Lagrangian is the Einstein-Hilbert Lagrangian augmented by the square of the Ricci tensor. The present techniques allow a simple extension of that result to the case of an arbitrary function of the Ricci tensor. The fact that different techniques give the same answer where they overlap is encouraging.

Notation. Adopt units where $c \equiv 1$, but all other quantities retain their usual dimensionalities, so that in particular $G \equiv \ell_P/m_P \equiv \hbar/m_P^2 \equiv \ell_P^2/\hbar$. The metric signature is $(+,+,+,+)$. The symbol T will always denote a temperature. The stress-energy tensor will be denoted by $t^{\mu\nu}$, and its trace by t.

II. GENERAL THEOREM

A. Reprise

In a previous paper [1], I have derived a general formula for the entropy of a dirty black hole in terms of (1) the area of the event horizon A_H, (2) the energy density in the classical fields surrounding the black hole ϱ, (3) the Euclideanized Lagrangian describing those fields $\mathcal{L}$, (4) the Hawking temperature T_H, (5) the entropy density s associated with the fluctuations (quantum hair, statistical hair), and finally (6) the metric. The total entropy is

$$S = \frac{kA_H}{4\ell_P^2} + \frac{1}{T_H}\int_\Sigma \{\varrho - \mathcal{L}\} K^\mu d\Sigma_\mu + \int_\Sigma s V^\mu d\Sigma_\mu. \quad (3)$$

This formula applies to all static black holes (not necessarily spherically symmetric), and to stationary nonstatic (axisymmetric) black holes. K^μ is the timelike Killing vector. V^μ is the four-velocity of a comoving observer. For a static black hole, this is just the four-velocity of a FIDO (fiducial observer). For a rotating black hole this is the four-velocity of a corotating observer. Σ denotes a constant time hypersurface. The first term in this formula agrees with Bekenstein's original suggestion [13], with the normalization constant fixed by Hawking's calculation [14]. For the time being fluctuations are ignored ($s = 0$, no quantum hair, no statistical mechanics effects).

The issue of interest is the evaluation of the term

$$\frac{1}{T_H}\int_\Sigma \{\varrho - \mathcal{L}\} K^\mu d\Sigma_\mu = \frac{k}{\hbar}\int_\Omega \{\varrho - \mathcal{L}\}\sqrt{g}\, d^4x. \qquad (4)$$

Here Ω denotes the entire Euclidean four-manifold. As is usual in the Euclidean formulation the time direction is compact with period $\hbar\beta = \hbar/(kT_H)$. The Hawking temperature T_H is related to the surface gravity κ by $kT_H = \hbar\kappa/2\pi$. By their very construction, Euclidean signature techniques are capable of addressing only the equilibrium thermodynamics of that class of black holes whose surface gravity is constant over the event horizon. Consequently, the "zeroth law" of black hole thermodynamics will be adopted *by fiat.*

By judicious use of several integrations by parts this integral will be transformed into a surface integral over the two-dimensional event horizon. To show this one must first introduce some extra technical machinery.

B. Metric

1. Static geometry

In the case of a static, possibly aspheric, black hole the Euclidean signature metric can be cast into the form

$$g = +N^2\, dt \otimes dt + g_{ij}\, dx^i \otimes dx^j. \qquad (5)$$

The quantity N is known as the lapse function. The event horizon occurs at $N = 0$. The timelike Killing vector is given by $K \equiv \partial/\partial t$. In coordinates $K^\mu = (1,0,0,0)$, $K_\mu = (N^2,0,0,0)$. FIDO's (fiducial observers) follow integral curves of the Killing vector; thus, the four-velocity of a FIDO is $V \equiv K/||K||$. In coordinates $V^\mu = (1/N,0,0,0)$, $V_\mu = (N,0,0,0)$.

Consider the one-form dt. Note that $||dt|| = 1/N$. In coordinates $(dt)_\mu = (1,0,0,0)$; $(dt)^\mu = (1/N^2,0,0,0)$. Consequently the one-form dt and Killing vector K are parallel, indeed $dt = K/N^2$.

The four-acceleration of a FIDO is given by $a \equiv (V \cdot \nabla)V$. In coordinates $a^\mu = V^\nu\nabla_\nu V^\mu = -(1/N)g^{\mu\nu}\nabla_\nu N$. Define the unit normal to the constant lapse hypersurface by n^μ; then by construction $a^\mu = -||a||n^\mu$. Using the fact that the Killing vector is hypersurface orthogonal, a brief computation shows

$$\begin{aligned}\nabla_\mu K_\nu &= -\frac{1}{N}\left(K_\mu\nabla_\nu N - K_\nu\nabla_\mu N\right)\\ &= (K_\mu a_\nu - a_\mu K_\nu)\\ &= \{||a||N\}\,(n_\mu V_\nu - n_\nu V_\mu)\ .\end{aligned} \qquad (6)$$

Furthermore, $\nabla_\mu V^\nu = V_\mu a^\nu = -||a||V_\mu n^\nu$. The surface gravity is defined by $\kappa = \lim_H\{N||a||\} = \lim_H\{||\nabla N||\} = \lim_H\{\partial N/\partial\eta\}$.

2. Stationary geometry

In the case of a stationary nonstatic black hole the Euclidean signature metric can be put into the form (see, e.g., [12])

$$g = +N^2\, dt \otimes dt + g_{ij}\, (dx^i - \beta^i dt) \otimes (dx^j - \beta^j dt). \quad (7)$$

In this more complicated situation it is possible to distinguish at least four interesting classes of fiducial observers — STATOR's, ZEVO's, ZAMO's, and ROTOR's.

As previously, N is known as the lapse function. The timelike Killing vector is still $K \equiv \partial/\partial t$. In coordinates $K^\mu = (1,0,0,0)$. A STATOR (stationary observer at rest) is one who follows the integral curves of the Killing vector. $V_S^\mu = K^\mu/||K||$. Note that $||K||^2 = N^2 + g_{ij}\beta^i\beta^j$. In Lorentzian signature the vanishing of $||K||$ defines the ergosphere, a concept that has no analogue in Euclidean signature. The notion of a STATOR will not prove particularly useful in what follows.

Consider the one-form dt. Even in a stationary (as opposed to static) geometry it is still true that $||dt|| = 1/N$. In either Lorentzian or Euclidean signature the vanishing of $N = 1/||dt||$ defines the event horizon. This one-form may be used to introduce the notion of a "minimally dragged " observer — a ZEVO (zero vorticity observer). A ZEVO is an observer whose (covariant) four-velocity is defined to be $V_Z = dt/||dt|| = N dt$. The appellation is justified by calculating the vorticity of such a system of observers: $\varpi = *(V \wedge dV) = 0$.

In coordinates $V_Z^\mu = (1; \beta^i)/N$. Define the relevant four-acceleration to be $a_Z \equiv (V \cdot \nabla)V$. A brief computation shows that the ZEVO's inherit much but not all of the structure of the FIDO's of a static geometry. For instance, $a_Z^\mu = -(1/N)(g^{\mu\nu} - V^\mu V^\nu)\nabla_\nu N$. The projection operator is needed because $(V \cdot \nabla N) = (\beta^i \partial_i N)/N \neq 0$, unless further assumptions are made. The surface gravity is defined by $\kappa = \lim_H\{N||a||\} = \lim_H\{||\nabla_\perp N||\}$. Note that if the stationary geometry is in fact static that the system of ZEVO's coincides with the system of STATOR's and one recovers the system of FIDO's.

It is believed that every black hole that is stationary but not static must be axially symmetric. Physically, the reason for this is that a rotating (i.e., nonstatic) black hole induces tidal dissipation in any system that is not axially symmetric. The final equilibrium state should thus be either static or axially symmetric. While some rigorous theorems along these lines can be proved for Einstein-Hilbert gravity the situation regarding more general theories is far from clear.

Nevertheless, if one adopts these physical arguments above to justify specializing to axial symmetry the metric may be further reduced to the form (see, e.g., [12])

$$\begin{aligned} g = &+ N^2\, dt \otimes dt + g_{\phi\phi}\, (d\phi - \omega dt) \otimes (d\phi - \omega dt) \\ &+ g_{AB}\, dx^A \otimes dx^B. \end{aligned} \quad (8)$$

There are now two Killing vectors, the timelike Killing vector $K \equiv \partial/\partial t$, and the axial Killing vector $\tilde{K} \equiv \partial/\partial\phi$. [$K^\mu = (1,0,0,0)$; $\tilde{K}^\mu = (0,0,0,1)$.] Because of the axial symmetry it is now possible to define the notion of angular momentum. The notion of the "minimally dragged" ZEVO system discussed above now particularizes to the notion of the ZAMO (zero angular momentum observer). For a ZAMO, $[V_\omega]^\mu = (1,0,0,\omega)/N$. This implies $V_\omega \propto (K + \omega\tilde{K})$. Note that $||K + \omega\tilde{K}|| = N$. Thus $V_\omega \equiv dt/||dt|| = (K + \omega\tilde{K})/||K + \omega\tilde{K}||$. Rearranging yields the useful result

$$dt = (K + \omega\tilde{K})/N^2. \quad (9)$$

Because $V \cdot \nabla N \propto (K + \omega\tilde{K}) \cdot \nabla N = 0$, the formulas for locally measured acceleration and surface gravity simplify from those appropriate to the "minimally dragged" ZEVO's, and one has results more closely related to those of the static FIDO's. For instance, one recovers $a^\mu = V^\nu\nabla_\nu V^\mu = -(1/N)g^{\mu\nu}\nabla_\nu N$, while for the surface gravity $\kappa = \lim_H\{N||a||\} = \lim_H\{||\nabla N||\}$.

Next, define the angular velocity of the event horizon by $\Omega_H \equiv \lim_H \omega$. For Kerr and Kerr-Newman black holes it is possible to show, as a mathematical theorem, that Ω_H is a constant everywhere on the event horizon. Indeed, for Kerr and Kerr-Newman black holes, as one approaches the horizon $\omega = \Omega_H + O(N^2)$. For arbitrary theories with arbitrary stress-energy tensors the truth or falsity of such results is far from clear. To obtain such results would require, at a minimum, the use of the field equations together with some form of the energy conditions. (This parallels the question of the constancy of the surface gravity over the event horizon.) As with the question of the surface gravity, Euclidean signature techniques cannot even be set up unless Ω_H is a constant. Physically, this is due to the fact that Euclidean signature techniques are intrinsically limited to the analysis of equilibrium thermodynamics. If Ω_H is not a constant then the implied differential rotation leads to shearing and dissipation so that the situation is decidedly not in equilibrium. Consequently the constancy of Ω_H will be adopted *by fiat.* (The assumed constancy of Ω_H is equivalent to assuming that the horizon of a stationary axisymmetric black hole is a Killing horizon, cf. [6].)

Having done this, it is now possible to introduce a fourth class of fiducial observers — the ROTOR's (corotating observers). Consider the Killing vector $K_\Omega = K + \Omega_H\tilde{K}$. In coordinates $K_\Omega^\mu = (1,0,0,\Omega_H)$. Consequently $||K_\Omega||^2 = N^2 + g_{\phi\phi}(\Omega_H - \omega)^2$. Thus K_Ω is that unique Killing vector that is null on the event horizon. The corotating observers are defined by $V_\Omega \equiv K_\Omega/||K_\Omega||$. Note that the ROTOR system of corotating observers and the ZAMO system have the same limit as one approaches the horizon. For convenience I shall sometimes write N_Ω for $||K_\Omega||$. Note that both N and N_Ω vanish on the event horizon.

In Lorentzian signature the system of corotating fiducial observers breaks down at sufficiently large distances. (K_Ω becomes spacelike for $\Omega_H r \geq c$.) There is no analogue of this behavior in Euclidean signature, and it can be safely ignored.

One has $a_\Omega \equiv (V_\Omega \cdot \nabla)V_\Omega = (K_\Omega \cdot \nabla)K_\Omega/||K_\Omega||^2 = -\nabla N_\Omega/N_\Omega$. The surface gravity is given by $\kappa = \lim_H\{N_\Omega||a_\Omega||\} = \lim_H\{||\nabla(N_\Omega)||\}$. That this definition in terms of ROTOR's coincides with the definition in terms of ZAMO's is yet another manifestation of the fact that these two systems tend to the same limit at the event horizon.

The necessity for this extended discussion of fiducial observers arises from the fact that these distinctions are both useful and necessary for the following discussion. For static black holes it suffices to use the simple system of FIDO's. For rotating black holes it is the ROTOR system of corotating observers that plays a primary role, first in defining the entropy, and second in performing the manipulations to be discussed below. The ZAMO system is also used, but is of secondary importance. It is to be emphasised that whatever stress-energy is surrounding the black hole it must, by the assumed internal equilibrium, be corotating with the hole. That is, the four-velocity of the ROTOR system of corotating observers must be an eigenvector of the stress-energy tensor.

C. Action

Take the Euclidean action to be

$$I_{\text{tot}}(g) = I_{\text{EH}}(g) + I_m(g). \qquad (10)$$

The Einstein-Hilbert action is

$$I_{\text{EH}}(g) = -\frac{1}{16\pi G}\int_\Omega R\sqrt{g}\, d^4x - \frac{1}{8\pi G}\int_{\partial\Omega} K\sqrt{_3g}\, d^3x, \qquad (11)$$

and consists of (1) the original Einstein-Hilbert Lagrangian, to be integrated over the entire Euclidean manifold, and (2) the Gibbons-Hawking surface term [11]. Here $_3g$ denotes the induced three-metric on the three-dimensional hypersurface $\partial\Omega$, while K denotes the trace of the second fundamental form. By the assumed asymptotic flatness of the black hole spacetime this term is to be integrated only over the three-surface at spatial infinity [11,15].

For an arbitrary variation of the metric

$$\delta I_{\text{EH}}(g) = \frac{1}{16\pi G}\int_\Omega G^{\mu\nu}\delta(g_{\mu\nu})\sqrt{g}\, d^4x - \int_{\partial\Omega}\Theta_{\text{EH}}(\delta g)\sqrt{_3g}\, d^3x. \qquad (12)$$

The surface term Θ_{EH} depends in a linear fashion on δg and its first derivative. For the augmented Einstein-Hilbert action, it is a special case result that these surface terms Θ_{EH} vanish provided that δg, though not necessarily its normal derivative, vanishes on the boundary.

The "matter" action is of the form

$$I_m(g) = \int_\Omega \mathcal{L}\sqrt{g}\, d^4x. \qquad (13)$$

Here $\mathcal{L}$ denotes the Euclideanized "matter" Lagrangian. (All higher order geometrical terms [e.g., $(\text{Riemann})^n$] are lumped into this "matter" Lagrangian.)

For an arbitrary variation of the metric,

$$\delta I_m(g) = \frac{1}{2}\int_\Omega t^{\mu\nu}\delta(g_{\mu\nu})\sqrt{g}\, d^4x - \frac{1}{2}\int_{\partial\Omega}\Theta(\delta g)\sqrt{_3g}\, d^3x. \qquad (14)$$

The surface term Θ depends in a linear fashion on δg and its first $n-1$ derivatives, where n denotes the highest order of the metric derivatives appearing in $\mathcal{L}$. In general, there is no particular reason to expect Θ to vanish unless δg and its first $n-1$ normal derivatives vanish on the boundary. The Einstein-Hilbert Lagrangian is special in this regard, as is the Lovelock Lagrangian [16].

D. Lemma: Volume term versus surface term

1. Static geometries

For clarity, I shall first discuss the case of a static, possibly aspheric, geometry. Consider the object $\int_\Omega \varrho\sqrt{g}d^4x$. For the time being, let Ω denote a four-volume that is bounded by hypersurfaces of constant lapse N. Let $\partial\Omega$ denote its three-boundary, whose normal is by construction orthogonal to the Killing flow. Note that by definition $\varrho = t^{\mu\nu}V_\mu V_\nu = t^{\mu\nu}\ \nabla_\mu t\ K_\nu$. Thus by considering $\delta(g_{\mu\nu}) \equiv \epsilon V_\mu V_\nu$, one has

$$\int_\Omega \varrho\sqrt{g}d^4x = \int_\Omega t^{\mu\nu}\ V_\mu V_\nu\ \sqrt{g}d^4x \qquad (15)$$

$$= \frac{d}{d\epsilon}\left[2I_m(g+\epsilon VV) + \int_{\partial\Omega}\Theta(\delta g = \epsilon V\otimes V)\sqrt{_3g}\, d^3x\right]. \qquad (16)$$

The derivative in the above equation is to be evaluated at $\epsilon = 0$. Introduce the notation $g_\epsilon \equiv g + \epsilon VV$. Differentiation yields

$$\frac{d}{d\epsilon}\left[I_m(g+\epsilon VV)\right] = \frac{d}{d\epsilon}\int_\Omega \mathcal{L}(g_\epsilon)\sqrt{g_\epsilon}\, d^4x = \frac{1}{2}\int_\Omega \mathcal{L}\sqrt{g}\, d^4x + \int_\Omega \frac{d\mathcal{L}}{d\epsilon}\sqrt{g}\, d^4x. \qquad (17)$$

Again, everything in the above equation is evaluated at $\epsilon = 0$. Now note that the substitution $g \mapsto g_\epsilon \equiv g + \epsilon V\otimes V$ merely corresponds to a coordinate change, a rescaling of the time direction by an amount $\sqrt{1+\epsilon}$. (One might also profitably think of this as a time dilation.) To be more explicit

$$\delta g_{\mu\nu} = \epsilon V_\mu V_\nu = \epsilon\nabla_{(\mu}t\ K_{\nu)} = \nabla_{(\mu}[\epsilon t\ K]_{\nu)}. \qquad (18)$$

Under a coordinate change $x^\mu \mapsto x^\mu + \xi^\mu(x)$, any arbitrary scalar transforms as $\delta\mathcal{L} = \xi^\mu(x)\partial_\mu\mathcal{L}$. Because the particular coordinate change under consideration is parallel to the Killing vector, the value of the Lagrangian is unaltered. That is, $d\mathcal{L}/d\epsilon = 0$. Introduce the notation $f_{\mu\nu} = V_\mu V_\nu = \nabla_{(\mu} t\ K_{\nu)}$. Consequently, for any four-volume Ω, bounded by constant lapse hypersurfaces, one has

$$\int_\Omega \{\varrho - \mathcal{L}\}\ \sqrt{g}\ d^4x = \int_{\partial\Omega} \Theta(\delta g = f)\sqrt{_3g}\ d^3x. \tag{19}$$

2. Stationary geometries

One must now repeat a minor variant of the above analysis, with additional technical complications to take care of the black hole's rotation. Recall that by the assumed internal equilibrium of the distribution one can show [1] that the stress-energy tensor has as one of its eigenvectors the four-velocity of the ROTOR system of corotating observers, V_Ω, with the associated eigenvalue being the energy density ϱ. Indeed

$$t^\mu{}_\nu (V_\Omega)^\nu = \varrho (V_\Omega)^\mu. \tag{20}$$

Now it is certainly true, but not useful, to observe that in the stationary case $\varrho = t^{\mu\nu}(V_\Omega)_\mu (V_\Omega)_\nu$. The reason that it is not useful is that explicit computation shows that it is not possible to interpret $\delta g = (V_\Omega) \otimes (V_\Omega)$ in terms of the effects of a coordinate transformation.

This is, fortunately, only a technical difficulty and not a fundamental problem. Introduce the notation $V^\perp$ to denote some arbitrary four-vector that is constrained only by the fact that it is assumed to be perpendicular to V_Ω. That is $V^\perp \cdot V_\Omega \equiv 0$. Then, because V_Ω is an eigenvector of the stress-energy tensor, for any such $V^\perp$ one has $\varrho = t^{\mu\nu}(V_\Omega)_\mu [(V_\Omega)_\nu + (V^\perp)_\nu]$. The trick is to pick $V^\perp$ in some appropriate manner. Without further ado, consider

$$V^\perp = ||K_\Omega|| dt - V_\Omega. \tag{21}$$

Note that $dt \cdot V_\Omega = dt \cdot K_\Omega / ||K_\Omega|| = 1/||K_\Omega||$, so that the perpendicularity requirement is indeed satisfied. Furthermore, by explicit construction, $V_\Omega \otimes [V_\Omega + V^\perp] = K_\Omega \otimes dt$. Consequently

$$\varrho = t^{\mu\nu}(K_\Omega)_\mu (dt)_\nu. \tag{22}$$

Now repeat the analysis used for the static case, this time considering

$$\delta g_{\mu\nu} = \epsilon \nabla_{(\mu} t\ [K_\Omega]_{\nu)} = \nabla_{(\mu}\{\epsilon t\ [K_\Omega]\}_{\nu)}. \tag{23}$$

This is nothing more than the effect of the coordinate change $x^\mu \mapsto x^\mu + \epsilon t [K_\Omega]^\mu$. Consequently the logic of the preceding case continues to hold, and the lemma is not disturbed by the black hole's rotation. For this stationary case introduce the notation $f_{\mu\nu} = \nabla_{(\mu} t\ [K_\Omega]_{\nu)}$. Again, for any four-volume Ω, bounded by constant lapse hypersurfaces, one has

$$\int_\Omega \{\varrho - \mathcal{L}\}\ \sqrt{g}\ d^4x = \int_{\partial\Omega} \Theta(\delta g = f)\sqrt{_3g}\ d^3x. \tag{24}$$

E. Entropy

The preceding lemma essentially solves the problem. Apply the lemma to the volume term in the entropy formula. As one pushes Ω outward to cover the whole Euclidean four-manifold two potential sources of surface terms should be considered: surface terms arising at spatial infinity, and surface terms arising at the horizon. The surface terms arising at spatial infinity should be quietly discarded by the assumed asymptotic flatness of spacetime. The only remaining piece is the boundary term at the horizon (cf. [17]). A suitably careful definition of the entropy is in terms of the limit

$$S = \frac{kA_H}{4\ell_P^2} + \frac{k}{\hbar} \lim_H \int_{\partial\Omega} \Theta(f)\sqrt{_3g} d^3x. \tag{25}$$

The limiting procedure $\partial\Omega \to H \times [0, \hbar\beta]$ may be handled in terms of surfaces of constant lapse function N. (One could just as easily work with hypersurfaces of constant $N_\Omega = ||K_\Omega||$. Nothing is gained or lost by such a choice.) Then $\sqrt{_3g}\ d^3x \mapsto N\sqrt{_2g}\ d^2x\ dt$. The t integration runs over the range $[0, \hbar\beta]$. Thus, as one approaches the event horizon,

$$S = \frac{kA_H}{4\ell_P^2} + k \int_H \beta \lim_H [N\Theta(f)]\ \sqrt{_2g}\ d^2x. \tag{26}$$

This can be interpreted in terms of a surface entropy density defined on the event horizon:

$$\mathcal{S} = k\beta \lim_H [N\Theta(f)]\ , \tag{27}$$

whence

$$S = \frac{kA_H}{4\ell_P^2} + \int_H \mathcal{S}\ \sqrt{_2g}\ d^2x. \tag{28}$$

An interesting nonzero result is obtained only if $\Theta(f)$ blows up as $N \to 0$. To see why and when this occurs requires a deeper understanding of the surface term. This is as far as I have currently been able to push the program in the general case. Further advances seem to require the choice of some specific class of Lagrangian as template.

Finally, let us reinstate the (volume) entropy density term associated with the statistical and quantum fluctuations occurring outside the black hole event horizon. Then

$$S = \frac{kA_H}{4\ell_P^2} + \int_H \mathcal{S}\ \sqrt{_2g}\ d^2x + \int_\Sigma s\ \sqrt{_3g}\ d^3x. \tag{29}$$

Insofar as the quantum fluctuations can be described by some effective Lagrangian $\mathcal{L}_{\text{eff}}$, they may be extracted from the volume density s, and pushed into the surface density term $\mathcal{S}$. This trade-off between volume and surface effects parallels the trade-off between integrating out fast modes (and describing them by an effective Lagrangian), and keeping the slow modes available for explicit computation.

III. SPECIFIC EXAMPLES

A. $\mathcal{L} = \mathcal{L}$ (Riemann)

Consider now the case where one takes $\mathcal{L}$ to be some arbitrary function of the Riemann tensor (though not of its derivatives). Interest in this class of Lagrangians is justified on the grounds that *any* quantum theory of gravity will induce terms of this type in the low-energy effective theory. Many of the examples considered in the literature are special cases of this reasonably large class. By the preceding general analysis, evaluation of the entropy is equivalent to the determination of the value of the surface term Θ at the horizon. This surface term is best evaluated by indirection. Define the object

$$J^{\mu\nu}{}_{\lambda\rho} \equiv \frac{\partial \mathcal{L}}{\partial R_{\mu\nu}{}^{\lambda\rho}}, \tag{30}$$

so that, in particular,

$$\delta(\mathcal{L}) = J^{\mu\nu}{}_{\lambda\rho}\, \delta(R_{\mu\nu}{}^{\lambda\rho}) \ . \tag{31}$$

Without loss of generality one may take $J^{\mu\nu}{}_{\lambda\rho}$ to inherit the symmetry structure of the Riemann tensor itself. Specifically $J^{\mu\nu}{}_{\lambda\rho} = J^{[\mu\nu]}{}_{[\lambda\rho]} = J_{\lambda\rho}{}^{\mu\nu}$. Consider a general variation of the metric. Define $\delta g^{\mu}{}_{\nu} = g^{\mu\sigma}\delta g_{\sigma\nu}$. One has

$$\delta(R_{\mu\nu}{}^{\lambda\rho}) = -2\nabla_{[\mu}\nabla^{[\lambda}(\delta g)_{\nu]}{}^{\rho]} + R_{\mu\nu}{}^{\sigma[\lambda}\delta g_{\sigma}{}^{\rho]}. \tag{32}$$

This allows us to write

$$\int_\Omega \delta(\mathcal{L})\sqrt{g}\; d^4x = \int_\Omega J^{\mu\nu}{}_{\lambda\rho}\left\{-2\nabla_\mu\nabla^\lambda(\delta g)_\nu{}^\rho + R_{\mu\nu}{}^{\sigma\lambda}(\delta g)_\sigma{}^\rho\right\}\sqrt{g}\; d^4x. \tag{33}$$

Here one has been able to drop the explicit antisymmetrization in view of the symmetry properties of $J^{\mu\nu}{}_{\lambda\rho}$ itself.

From the above, one reads off

$$t^{\mu\nu} = -2\nabla_\alpha\nabla_\beta J^{\alpha\mu\beta\nu} - 2\nabla_\alpha\nabla_\beta J^{\alpha\nu\beta\mu} - J^{\alpha\beta\gamma\mu}R_{\alpha\beta\gamma}{}^{\nu} - J^{\alpha\beta\gamma\nu}R_{\alpha\beta\gamma}{}^{\mu} + \mathcal{L}g^{\mu\nu}. \tag{34}$$

Recall the notation

$$f_{\mu\nu} \equiv [K_\Omega]_{(\mu}\nabla_{\nu)}t. \tag{35}$$

For the static case interpret $\Omega_H = 0$, and discard the axial symmetry. Thus this definition is seen to make sense for both the static (aspheric) and stationary axisymmetric cases. In either case we have by construction $\varrho = t^{\mu\nu}f_{\mu\nu}$. Construct the integral

$$\mathcal{X} \equiv \int_\Omega \{\varrho - \mathcal{L}\}\sqrt{g}d^4x = \int_\Omega \{t^{\mu\nu}\, f_{\mu\nu} - \mathcal{L}\}\;\sqrt{g}\; d^4x. \tag{36}$$

Then

$$\mathcal{X} = \int_\Omega \{-4\nabla_\alpha\nabla_\beta J^{\alpha\mu\beta\nu} - 2J^{\alpha\beta\gamma\mu}R_{\alpha\beta\gamma}{}^{\nu}\}f_{\mu\nu}\;\sqrt{g}d^4x. \tag{37}$$

Integrate by parts once:

$$\mathcal{X} = \int_\Omega \left\{+4\nabla_\beta J^{\alpha\mu\beta\nu}\nabla_\alpha(f_{\mu\nu})\right\} + \left\{-2J^{\alpha\beta\gamma\mu}R_{\alpha\beta\gamma}{}^{\nu}f_{\mu\nu}\right\}\sqrt{g}\; d^4x - 4\int_{\partial\Omega}\left\{n_\alpha(\nabla_\beta J^{\alpha\mu\beta\nu})f_{\mu\nu}\right\}\sqrt{3g}\; d^3x. \tag{38}$$

Integrate by parts a second time:

$$\begin{aligned}\mathcal{X} = &\int_\Omega \left\{-4J^{\alpha\mu\beta\nu}\nabla_\beta\nabla_\alpha(f_{\mu\nu})\right\} + \left\{-2J^{\alpha\beta\gamma\mu}R_{\alpha\beta\gamma}{}^{\nu}f_{\mu\nu}\right\}\sqrt{g}\; d^4x \\ &-4\int_{\partial\Omega}\left\{n_\alpha(\nabla_\beta J^{\alpha\mu\beta\nu})f_{\mu\nu} - n_\beta J^{\alpha\mu\beta\nu}(\nabla_\alpha f_{\mu\nu})\right\}\sqrt{3g}\; d^3x.\end{aligned} \tag{39}$$

Rearrange

$$\begin{aligned}\mathcal{X} = &-2\int_\Omega J^{\mu\nu\lambda\rho}\left\{2\nabla_\mu\nabla_\lambda(f_{\nu\rho}) + R_{\mu\nu\lambda}{}^{\sigma}f_{\rho\sigma}\right\}\sqrt{g}\; d^4x \\ &-4\int_{\partial\Omega}\left\{n_\alpha(\nabla_\beta J^{\alpha\mu\beta\nu})(f_{\mu\nu}) - J^{\alpha\mu\beta\nu}(\nabla_\alpha f_{\mu\nu})n_\beta\right\}\sqrt{3g}\; d^3x.\end{aligned} \tag{40}$$

The volume integral above vanishes identically. To see this, note that after appropriate explicit antisymmetrization the volume term is just

$$\int_{\Omega} J^{\mu\nu\lambda\rho}\,\delta(R_{\mu\nu\lambda\rho})\,\sqrt{g}\,d^4x, \tag{41}$$

where $\delta(R_{\mu\nu\lambda\rho})$ is just that due to taking $\delta(g_{\mu\nu}) = f_{\mu\nu}$. But, as we have already seen $f_{\mu\nu} = \nabla_{(\mu}[tK_\Omega]_{\nu)}$, which corresponds to just the effect of a coordinate transformation.

The evaluation of the surface terms proceeds as follows. First note that the surface term at spatial infinity is automatically suppressed by the assumption of asymptotic flatness. Second, near the horizon $\sqrt{3g}\;d^3x \to N\sqrt{2g}\;d^2x dt$. Since the Riemann tensor and its derivatives are well behaved at the horizon, as are the limits of n_α and $f_{\mu\nu}$, it is easy to see that the first surface term vanishes, being suppressed by the factor of N in the metric determinant.

The only remaining term is

$$\mathcal{X} = 4\lim_H \int_{\partial\Omega} \left\{J^{\alpha\mu\beta\nu}\nabla_\alpha(f_{\mu\nu})n_\beta\right\}\sqrt{3g}\;d^3x. \tag{42}$$

This formula is, of course, nothing more nor less than the special case explicit evaluation of the surface term $\Theta(f)$ previously encountered in the general argument.

At this stage it proves useful to treat the stationary and static cases separately.

Static geometry. For a static geometry $f_{\mu\nu} = V_\mu V_\nu$. The gradient term includes pieces such as

$$\begin{aligned}\nabla_\alpha(V_\mu V_\nu) &= (\nabla_\alpha V_\mu)V_\nu + V_\mu(\nabla_\alpha V\nu)\\ &= -||a||\,(V_\alpha n_\mu V_\nu + V_\mu V_\alpha n_\nu)\,.\end{aligned} \tag{43}$$

Now, take the limit as one approaches the horizon. The only surviving term in the surface integral comes from the cancellation between the N arising from the metric determinant and $||a||$. Note that

$$\lim_H \int_0^{\hbar\beta} dtN||a|| = \lim_H \hbar\beta N||a|| = 2\pi. \tag{44}$$

This yields

$$\int_\Omega \{\varrho - \mathcal{L}\}\sqrt{g}\;d^4x = 8\pi\int_H \left\{J^{\alpha\mu\beta\nu}V_\alpha V_\beta n_\mu n_\nu\right\}\sqrt{2g}\;d^2x. \tag{45}$$

Stationary geometry. The limiting procedure is now a little more delicate, and requires some tedious technical steps. Recall that $f_{\mu\nu} = [K_\Omega]_{(\mu}[dt]_{\nu)}$. The gradient term includes pieces such as

$$\nabla_\alpha([K_\Omega]_\mu[dt]_\nu) = (\nabla_\alpha[K_\Omega]_\mu)[dt]_\nu + [K_\Omega]_\mu(\nabla_\alpha\nabla_\nu t). \tag{46}$$

Because the Killing vector K_Ω is not hypersurface orthogonal, in general, the best we can say is that the covariant derivative of the Killing vector satisfies

$$\nabla_\mu[K_\Omega]_\nu = 2(a_\Omega)_{[\mu}(K_\Omega)_{\nu]} + \pi_{\mu\nu}. \tag{47}$$

Here $\pi_{\mu\nu}$ is an antisymmetric tensor orthogonal to K_Ω. On the other hand, it is known that K_Ω is hypersurface orthogonal on the event horizon, so that $\pi_{\mu\nu}$ vanishes in that limit. Consequently

$$\nabla_\mu[K_\Omega]_\nu = -2N_\Omega||a_\Omega||n_{[\mu}(V_\Omega)_{\nu]} + O(N_\Omega). \tag{48}$$

Now consider

$$\begin{aligned}(\nabla_\mu\nabla_\nu t) &= \nabla_\mu[(K_\nu + \omega\tilde{K}_\nu)/N^2]\\ &= -2N^{-3}\nabla_{(\mu}N\;[K+\omega\tilde{K}]_{\nu)} + N^{-2}\nabla_{(\mu}\omega\;\tilde{K}_{\nu)}\\ &= -2N^{-1}[a_\omega]_{(\mu}\;[V_\omega]_{\nu)} + N^{-2}\nabla_{(\mu}\omega\;\tilde{K}_{\nu)}\\ &= +2\frac{||a_\omega||}{N}\;[n_\omega]_{(\mu}\;[V_\omega]_{\nu)} + N^{-2}\nabla_{(\mu}\omega\;\tilde{K}_{\nu)}.\end{aligned} \tag{49}$$

Now because $\omega = \Omega_H + O(N^2)$ one has $\nabla\omega = O(N\nabla N) = O(N^2 a)$. So finally

$$(\nabla_\mu\nabla_\nu t) = +2\frac{||a_\omega||}{N}\;\{[n_\omega]_{(\mu}\;[V_\omega]_{\nu)} + O(N)\}\;. \tag{50}$$

Now, take the limit as one approaches the horizon. The ZAMO's V_ω and the ROTOR's V_Ω approach the same limit. The same occurs for the relevant normals. Again, the only surviving term in the surface integral comes from the cancellation between the N arising from the metric determinant, $||a_\omega||$ and $||a_\Omega||$. Various subdominant pieces vanish in the limit. As before, this yields

$$\int_\Omega \{\varrho - \mathcal{L}\}\sqrt{g}\;d^4x = 8\pi\int_H \left\{J^{\alpha\mu\beta\nu}V_\alpha V_\beta n_\mu n_\nu\right\}\sqrt{2g}\;d^2x. \tag{51}$$

Returning to the general case (static or stationary), and backtracking to the general entropy formula, one now obtains

$$S = \frac{kA_H}{4\ell_P^2} + 8\pi\frac{k}{\hbar}\int_H J^{\mu\nu\lambda\rho}\;V_\mu n_\nu V_\lambda n_\rho\;\sqrt{2g}\;d^2x. \tag{52}$$

A further refinement is to define $g^\perp_{\mu\nu} \equiv V_\mu V_\nu + n_\mu n_\nu$. This is essentially the metric in the two directions perpendicular to the event horizon, in terms of which the symmetries of $J^{\mu\nu\lambda\rho}$ imply

$$S = \frac{kA_H}{4\ell_P^2} + 4\pi\frac{k}{\hbar}\int_H J^{\mu\nu\lambda\rho}\;g^\perp_{\mu\lambda}g^\perp_{\nu\rho}\;\sqrt{2g}\;d^2x. \tag{53}$$

This is our final form for the entropy. Note that it exhibits all of the properties expected from the general analysis.

B. $\mathcal{L} = \mathcal{L}$ (Ricci)

A further specialization of the above is to consider the case where the Lagrangian is an arbitrary function of the Ricci tensor, rather than the full Riemann tensor. The analysis is straightforward.

Define

$$\tilde{J}^{\mu\nu} \equiv \frac{\partial\mathcal{L}}{\partial R_{\mu\nu}}. \tag{54}$$

Then

$$J^{\mu\nu}{}_{\lambda\rho} \equiv \frac{\partial \mathcal{L}}{\partial R_{\mu\nu}{}^{\lambda\rho}} = \tilde{J}^{[\mu}{}_{[\lambda} g^{\nu]}{}_{\rho]} = \frac{1}{4}\left[\tilde{J}^{\mu}{}_{\lambda} g^{\nu}{}_{\rho} - \tilde{J}^{\mu}{}_{\rho} g^{\nu}{}_{\lambda} + \tilde{J}^{\nu}{}_{\rho} g^{\mu}{}_{\lambda} - \tilde{J}^{\nu}{}_{\lambda} g^{\mu}{}_{\rho}\right] . \tag{55}$$

Inserting into the previous formula, one extracts

$$S = \frac{kA_H}{4\ell_P^2} + 2\pi \frac{k}{\hbar}\int_H \frac{\partial \mathcal{L}}{\partial R_{\mu\nu}}\, g^{\perp}_{\mu\nu}\, \sqrt{{}_2 g}\, d^2x. \tag{56}$$

This formula is instructively similar to that obtained by Jacobson, Kang, and Myers [3,4]. Using field redefinition techniques under conformal rescalings they were limited to the case $\mathcal{L} = R_{\mu\nu}R^{\mu\nu} + \mathcal{L}_{\text{matter}}$. (The extra matter was required to enforce a nontrivial solution to the field equations; it was assumed that the extra matter was sufficiently well behaved not to contribute to the entropy in its own right.) Under these assumptions, Jacobson, Kang, and Myers showed that

$$S = \frac{kA_H}{4\ell_P^2} + 4\pi \frac{k}{\hbar}\int_H R^{\mu\nu}\, g^{\perp}_{\mu\nu}\, \sqrt{{}_2 g}\, d^2x. \tag{57}$$

This is exactly the result enunciated by Jacobson, Kang, and Myers [3,4].

A simple consistency check is to lump the Einstein-Hilbert action in with $\mathcal{L}$. Taking $\mathcal{L} = \frac{1}{16\pi G}R = \frac{\hbar}{16\pi \ell_P^2}R$ reproduces the ordinary area term.

C. $\mathcal{L} = \mathcal{L}$ (Tr[Ricci])

A completely analogous analysis can be applied in the case that the Lagrangian is an arbitrary function of the scalar Ricci curvature.

Consider

$$J^{\mu\nu}{}_{\lambda\rho} \equiv \frac{\partial \mathcal{L}}{\partial R_{\mu\nu}{}^{\lambda\rho}} = \frac{\partial \mathcal{L}}{\partial R}\, g^{[\mu}{}_{[\lambda} g^{\nu]}{}_{\rho]}. \tag{58}$$

Inserting into the general result one obtains

$$S = \frac{kA_H}{4\ell_P^2} + 4\pi \frac{k}{\hbar}\int_H \frac{\partial \mathcal{L}}{\partial R}\, \sqrt{{}_2 g}\, d^2x. \tag{59}$$

D. Lovelock gravity

As a final example, I shall discuss Lovelock gravity. While the analysis presented so far has, for definiteness, been presented in four dimensions there is nothing essentially four-dimensional about these techniques. In D dimensions the Lovelock Lagrangian is given by (see, e.g., [2])

$$\mathcal{L} = \sum_{m=0}^{[D/2]} c_m\, \mathcal{L}_m. \tag{60}$$

In this sum, $[D/2]$ indicates the integer part of $D/2$. The individual terms are given by

$$\mathcal{L}_m(g) = \frac{1}{2^m}\, \delta^{\mu_1\nu_1\cdots\mu_m\nu_m}_{\lambda_1\rho_1\cdots\lambda_m\rho_m}\; R_{\mu_1\nu_1}{}^{\lambda_1\rho_1}\cdots R_{\mu_m\nu_m}{}^{\lambda_m\rho_m}. \tag{61}$$

The δ symbol is a totally antisymmetric product of $2m$ Kronecker deltas, suitably normalized to take values 0 and ± 1. It is convenient to define $\mathcal{L}_0 = 1$, this term corresponding to a cosmological constant. Furthermore $\mathcal{L}_1 = R$ is the Einstein-Hilbert Lagrangian. In general, $\mathcal{L}_m$ is the Euler density for a $2m$-dimensional manifold. Because of the antisymmetrization, no derivative appears at higher than second order in the equations of motion.

For the purposes currently at hand, consider the object

$$[J_m]^{\mu\nu}{}_{\lambda\rho} \equiv \frac{\partial \mathcal{L}_m}{\partial R_{\mu\nu}{}^{\lambda\rho}} = \frac{m}{2^m}\, \delta^{\mu_1\nu_1\cdots\mu_{m-1}\nu_{m-1}\mu\nu}_{\lambda_1\rho_1\cdots\lambda_{m-1}\rho_{m-1}\lambda\rho}\; R_{\mu_1\nu_1}{}^{\lambda_1\rho_1}\cdots R_{\mu_{m-1}\nu_{m-1}}{}^{\lambda_{m-1}\rho_{m-1}} . \tag{62}$$

Applying the general formula, the contractions with $g^{\perp}$, together with the total antisymmetrization of the indices, imply that the only components of Riemann tensor that contribute to the entropy density are those that are tangential to the $(D-2)$-dimensional event horizon. Specifically, we note that

$$\delta^{\lambda_1\rho_1\cdots\mu_{m-1}\nu_{m-1}\mu\nu}_{\lambda_1\rho_1\cdots\lambda_{m-1}\rho_{m-1}\lambda\rho}\; (g^{\perp})^{[\lambda}{}_{[\mu}(g^{\perp})^{\rho]}{}_{\nu]} = \tilde{\delta}^{\mu_1\nu_1\cdots\mu_{m-1}\nu_{m-1}}_{\lambda_1\rho_1\cdots\lambda_{m-1}\rho_{m-1}}. \tag{63}$$

Here $\tilde{\delta}$ is the totally antisymmetric product of $2(m-1)$ Kronecker deltas, restricted to the subspace orthogonal to $g^{\perp}$.

The rest of the derivation now parallels that due to Jacobson and Myers [2]. The entropy is

$$S = 4\pi \frac{k}{\hbar}\int_H \sum_{m=1}^{[D/2]} m\, c_m\, \mathcal{L}_{m-1}(h)\, \sqrt{h}\, d^{D-2}x. \tag{64}$$

In this particular case the entropy is given solely in terms of the intrinsic geometry (h) of the event horizon — this result is special to this particular type of Lagrangian and does not generalize.

IV. DISCUSSION

The computation of black hole entropies in various model theories is an issue of great current interest. Based on the work of several authors the situation has by now become significantly clarified. Several points are worth making.

First, the naive area law for black hole entropies is in general false:

$$S \neq \frac{kA_H}{4\ell_P^2}. \tag{65}$$

The naive law certainly holds for Einstein-Hilbert gravity coupled to matter whose kinetic energy is quadratic [1,18]. Once one moves beyond quadratic kinetic energies the naive law fails in general.

Second, for a general higher derivative Lagrangian the entropy of a black hole is given by an integral of some suitable density over (a fixed time spacelike cross section of) the event horizon

$$S = \frac{kA_H}{4\ell_P^2} + \int_H \mathcal{S}\ \sqrt{{}_2g}\ d^2x. \tag{66}$$

The entropy surface density is a simple function of the surface term that connects the stress-energy tensor with the variation of the action under a variation of the spacetime metric.

Third, in the specific case of a Lagrangian that is solely a function of the Riemann tensor

$$S = \frac{kA_H}{4\ell_P^2} + 4\pi\frac{k}{\hbar}\int_H \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\lambda\rho}}\ g^{\perp}_{\mu\lambda} g^{\perp}_{\nu\rho}\ \sqrt{{}_2g}\ d^2x. \tag{67}$$

This relatively general formula can be checked against a number of more specific examples where the entropy is known by other means. The fact that different types of calculation give the same answer where they overlap is certainly encouraging.

Fourth, the present paper has obtained its results via extensive use of Euclidean signature techniques. The underlying physics is perhaps somewhat obscured by this formalism. It is encouraging to note that similar results have by now been presented using a number of different techniques [3–6]. The overall agreement between these various different techniques is a further useful consistency check.

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[1] M. Visser, Phys. Rev. D **48**, 583 (1993).
[2] T. Jacobson and R. Myers, Phys. Rev. Lett. **70**, 3684 (1993).
[3] T. Jacobson, G. Kang, and R. Myers, talk delivered by T. Jacobson at the Conference on Quantum aspects of Black Holes, Santa Barbara, 1993 (unpublished).
[4] T. Jacobson, G. Kang, and R. Myers (unpublished).
[5] R. Wald, talk delivered at the conference on Quantum aspects of Black Holes [3].
[6] R. Wald, Phys. Rev. D **48**, 3427 (1993).
[7] V. Iyer and R. Wald (unpublished).
[8] M. Visser, Phys. Rev. D **46**, 2445 (1992).
[9] I. Moss and R. Myers (private communication).
[10] J. Simon and B. Whiting (unpublished).
[11] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2752 (1977).
[12] K. S. Thorne, R. H. Price, and D. A. Macdonald, *Black Holes: The Membrane Paradigm* (Yale University, New Haven, CT, 1986).
[13] J. Bekenstein, Phys. Rev. D **7**, 2333 (1973).
[14] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[15] S. W. Hawking, in *General Relativity: An Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).
[16] R. Myers, Phys. Rev. D **36**, 392 (1987).
[17] R. Kallosh, T. Ortín, and A. Peet, Phys. Rev. D **47**, 5400 (1993).
[18] I. Moss, Phys. Rev. Lett. **69**, 1852 (1992).