New topology for spatial infinity?

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For spatial infinity we introduce the topology of a projective Lorentz sphere. This topology is indicated by the reduced variety of physically admissible solutions of both the electromagnetic and the gravitational field equations. In this topology past and future are fused, so that the notions of cause and effect lose their intuitive meanings.

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I. INTRODUCTION

In an asymptotically flat space-time both gravitational and electromagnetic fields at spatial infinity (SI) can be obtained as scalar solutions of linear wave equations on three-dimensional Lorentz spheres, with one timelike and two spacelike dimensions [1,2]. The appropriate wave equation for electromagnetic fields is

$$\nabla^2 u = 0 \ . \tag{1.1}$$

For gravitational fields it is

$$\nabla^2 u - 3u = 0 \ . \tag{1.2}$$

Both equations are manifestly invariant with respect to the six-parametric group of rotations and boosts, O(3,1). However, further examination reveals that in both cases one-half of the solutions do not lend themselves to a continuation from spatial infinity to finite regions of space-time, or to lightlike infinity.

When the wave equations (1.1) and (1.2) are solved by expansions into spherical harmonics with time-dependent coefficients, these coefficients satisfy ordinary second-order differential equations. Each of these equations admits a pair of solutions: one even in the time coordinate, the other odd. In the electromagnetic case the physically admissible solutions are even for odd values of l, the order of the spherical harmonic, and odd for even values of l. The reverse holds for the solutions of the gravitational wave equation (1.2).

The Lorentz-invariant character of these conditions is made evident if they are formulated in terms of *antipodal* points. Pairs of points are called antipodal to each other if, in the embedding pseudo-Euclidean space-time, they lie on a straight line passing through the center of symmetry of the Lorentz sphere. In terms of pseudo-Cartesian coordinates of the embedding Minkowski manifold antipodal points have coordinates that pairwise add up to zero. Compare Fig. 1.

Admissible solutions of the respective equations (1.1) and (1.2) can be characterized in terms of their properties at pairs of antipodal points: In electrodynamics the values at antipodal points must differ in sign only ($u' = -u$); such solutions will be called *odd*. Gravitational solutions must have the same values at antipodal pairs of points ($u' = u$); they will be called *even*. The pairing of antipodal points is invariant with respect to rotations and boosts. Hence the evenness or oddness of a scalar field is an invariant property.

II. ATTEMPT AT QUANTIZATION

In terms of Cauchy data on a spacelike hyperplane through the center of symmetry of a Lorentz sphere the restriction to even or to odd fields is equivalent to the imposition of constraints on the canonical (i.e., Hamiltonian) variables. In odd fields the configuration variables vanish on the chosen hyperplane, whereas in even fields the canonical momentum variables are zero.

If one proceeds strictly in formal analogy between Poisson brackets and quantum commutators, the surviving dynamical variables all commute with each other; hence, there are no Heisenberg uncertainty requirements. If instead one uses time-dependent annihilation and creation operators to construct Heisenberg analogues of

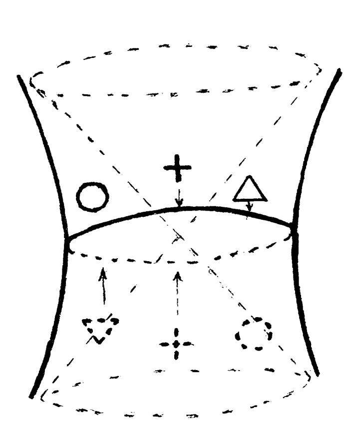

FIG. 1. Antipodal points on a (1,1) Lorentz sphere.

the physically acceptable solutions, one finds that the products of the minimal uncertainties of E and B of the Maxwell-Lorentz theory, and of the corresponding variables in Einstein gravitational theory, differ from the values at lightlike infinity ("scri"). In particular, as one approaches the light-cone boundary of SI, the uncertainties of the electromagnetic tend to zero, whereas on scri they are constant, and nonzero.

These negative outcomes have convinced us that before attempting to quantize fields at spatial infinity one should understand the nonquantum theory. In the sections that follow we shall confine ourselves to this less ambitious task.

III. HARMONIC FUNCTIONS ON THE TWO-DIMENSIONAL LORENTZ SPHERE

Functions on Lorentz spheres which assume either identical or opposite values on pairs of antipodal points can be defined, without loss of information, as single-valued or as double-valued functions on *projective Lorentz spheres.* A projective Lorentz sphere is defined as a manifold whose points represent pairs of antipodal points of (ordinary) Lorentz spheres. As they are multiply connected, functions can have more than a single value at the same point without necessitating discontinuities. Projective Lorentz spheres cannot be oriented; in some respects they resemble Möbius bands. Compare Fig. 2.

Though primary physical interest rests with three-dimensional projective Lorentz spheres, we shall deal with the two-dimensional projective Lorentz sphere in some detail, as it lends itself readily to a complete treatment. A spacelike Lorentz sphere embedded in a three-dimensional Minkowski manifold can be identified by its equation in standard Lorentz coordinates as

$$x^2+y^2-t^2=1 \ . \tag{3.1}$$

It is a single hyperboloid of rotation, ruled by two sets of

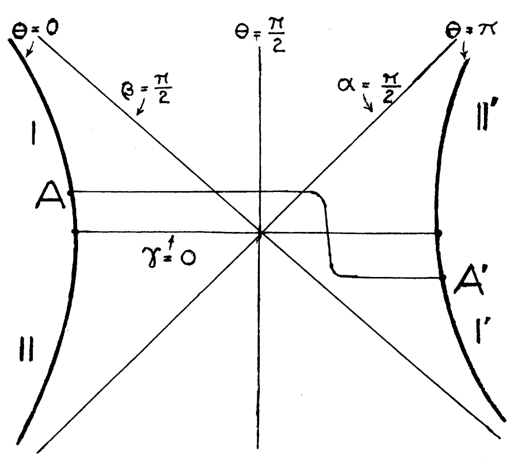

FIG. 2. Projective (1,1) Lorentz sphere, flattened out. The arcs I and I′ are identical with each other, and so are II and II′. The curve (A,A') is closed. It intersects the "equator" ($\gamma=0$) but once, demonstrating that the sphere cannot be time oriented.

straight null lines, each of which covers the whole Lorentz sphere once. That is to say, through each point of the Lorentz sphere pass exactly two of these null lines. If we identify a particular null line by its intersection with the circle $t=0$, $x^2+y^2=1$, denoting the point of intersection by α for one set and β for the other, then their equations, in terms of the pseudo-Cartesian coordinates of the embedding (2,1) Minkowski manifold, are, respectively,

$$x=\cos\alpha-t\sin\alpha, \quad y=\sin\alpha+t\cos\alpha \ , \tag{3.2}$$

and

$$x=\cos\beta+t\sin\beta, \quad y=\sin\beta-t\cos\beta \ . \tag{3.3}$$

Each point on the Lorentz sphere is identified unambiguously by the two null lines passing through it. Hence α and β, each ranging from 0 to 2π, will serve as a coordinate system on the (1,1) sphere. The line element is

$$d\sigma^2=\cos^{-2}\gamma \, d\alpha \, d\beta \ , \quad \gamma=\tfrac{1}{2}(\alpha-\beta) \ . \tag{3.4}$$

An alternative coordinate system, useful for some calculations, is

$$\gamma=\tfrac{1}{2}(\alpha-\beta), \quad \theta=\tfrac{1}{2}(\alpha+\beta) \ . \tag{3.5}$$

Expressed in these coordinates, the line element is

$$d\sigma^2=\cos^{-2}\gamma(d\theta^2-d\gamma^2) \ . \tag{3.6}$$

The angle θ, which is the spacelike coordinate, represents the geographical longitude, whereas γ, the timelike coordinate, is related to the pseudo-Cartesian coordinate t by

$$t=\tan\gamma \ . \tag{3.7}$$

Next we shall examine d'Alembert's equation

$$\nabla^2 u=0 \tag{3.8}$$

on the (1,1) sphere. In terms of α and β, that equation has the form

$$\frac{\partial^2 u}{\partial\alpha\,\partial\beta}=0 \ , \tag{3.9}$$

with the general solution

$$u(\alpha,\beta)=f(\alpha)+g(\beta) \ . \tag{3.10}$$

f and g are arbitrary (continuous, differentiable) functions of their respective arguments, periodic with the period 2π if their arguments are to be extended beyond their original range.

Next we shall restrict ourselves to even and to odd solutions. These are

$$u(\alpha,\beta)=f(\alpha)+f(\beta+\pi) \tag{3.11}$$

and

$$u(\alpha,\beta)=f(\alpha)-f(\beta+\pi) \ , \tag{3.12}$$

respectively. Thus, by requiring either evenness or oddness we have reduced the two arbitrary functions to one. Moreover, though f is a function of but one argument, it determines the solutions in a two-dimensional domain.

Can these relationships be converted into the dependence of $u(\alpha,\beta)$ on Cauchy data on a spacelike one-dimensional domain, such as half the circle $t=0$, corresponding in our coordinate system to $\alpha=\beta$? (Obviously, only the data on half the circle will do, as the other half of the circle is antipodal to the first.) This is indeed the case.

Denote the value of $u(\theta,\theta)$ on that half-circle by $v(\theta)$, and its normal derivative by $\dot{v}(\theta)$:

$$\dot{v}(\theta)=\frac{\partial u}{\partial \alpha}-\frac{\partial u}{\partial \beta}\,, \quad \theta=\alpha=\beta\,. \tag{3.13}$$

As antipodal points are related to each other by the equalities

$$\alpha'=\beta+\pi,\quad \beta'=\alpha+\pi\,, \tag{3.14}$$

the values of the Cauchy data on the entire circle must satisfy the conditions

$$v(\theta+\pi)=v(\theta),\quad \dot{v}(\theta+\pi)=-\dot{v}(\theta) \tag{3.15}$$

for even fields, and

$$v(\theta+\pi)=-v(\theta),\quad \dot{v}(\theta+\pi)=\dot{v}(\theta) \tag{3.16}$$

for odd fields.

The derivative of the function $v(\theta)$ along the circle, $v'\equiv dv/d\theta$, is related to those of $u(\alpha,\beta)$ at the same locations by the formula

$$v'(\theta)=\frac{\partial u}{\partial \alpha}+\frac{\partial u}{\partial \beta}\,. \tag{3.17}$$

Combining this expression with Eq. (3.13), as well as with (3.11) or with (3.12), we obtain for $df/d\theta$ an expression that depends only on Cauchy data:

$$\frac{df}{d\theta}\equiv f'(\theta)=\frac{1}{2}(\dot{v}+v')\,. \tag{3.18}$$

f itself is thereby determined up to a constant of integration.

For odd fields that constant of integration is irrelevant. Substituting into Eq. (3.12) one obtains immediately:

$$u(\alpha,\beta)=\frac{1}{2}\left[v(\alpha)-v(\beta+\pi)+\int_{\theta=\beta+\pi}^{\alpha}\dot{v}\,d\theta\right]$$
$$=\frac{1}{2}\left[v(\alpha)+v(\beta)+\int_{\theta=\beta+\pi}^{\alpha}\dot{v}(\theta)d\theta\right]\,. \tag{3.19}$$

For even fields we can make use of Eqs. (3.15) in conjunction with (3.18). The result is

$$f(\theta)=\frac{1}{2}v(\theta)+\frac{1}{2}\int_{\theta'=0}^{\theta}\dot{v}\,d\theta'-\frac{1}{4}\int_{\theta'=0}^{\pi}\dot{v}\,d\theta' \tag{3.20}$$

and

$$u(\alpha,\beta)=\frac{1}{2}\left[v(\alpha)+v(\beta)+\int_{\theta=\beta}^{\alpha}\dot{v}(\theta)d\theta\right]\,. \tag{3.21}$$

A few examples will illustrate the role of the Cauchy data.

(i) Even field–$v(\theta)=\cos 2n\theta$, $\dot{v}(\theta)=0$:

$$u(\alpha,\beta)=\cos 2n\gamma\,\cos 2n\theta,\quad \gamma=\tfrac{1}{2}(\alpha-\beta),\quad \theta=\tfrac{1}{2}(\alpha+\beta)\,.$$

(ii) Even field–$v(\theta)=0$, $\dot{v}(\theta)=\cos(2n+1)\theta$:

$$u(\alpha,\beta)=(2n+1)^{-1}\sin(2n+1)\gamma\,\cos(2n+1)\theta\,.$$

(iii) Odd field–$v(\theta)=\cos(2n+1)\theta$, $\dot{v}(\theta)=0$:

$$u(\alpha,\beta)=\cos(2n+1)\gamma\,\cos(2n+1)\theta\,.$$

(iv) Odd field–$v(\theta)=0$, $\dot{v}(\theta)=\cos 2n\theta$:

$$u(\alpha,\beta)=(2n)^{-1}\sin 2n\gamma\,\cos 2n\theta\,.$$

In all these examples n is to be a nonzero integer. As the field $u(\alpha,\beta)$ depends on the Cauchy data linearly, these examples also serve as guides to the Fourier decomposition of the data and of the resulting field.

IV. THE PROJECTIVE LORENTZ SPHERE

When pairs of antipodal points are mapped into a single point of a target manifold, the result is a projective Lorentz sphere. Conversely, the (ordinary) Lorentz sphere is a covering (though not the universal covering) manifold of the projective Lorentz sphere. The contraction preserves the two sets of ruling null lines as separate sets, but it cuts each set in half. Projective Lorentz spheres are not orientable, nor are they time orientable [3]: One cannot tell the future and past apart. Though the Cauchy data on the half-circle determine the field $u(\alpha,\beta)$ everywhere, that half-circle does not cut the projective Lorentz sphere in two. Thus the intuitive distinction between cause and effect loses all meaning.

On the projective Lorentz sphere *even* fields are simply those fields which at each point of the manifold have but a single value. *Odd* fields have two values at every point, which differ only by their signs. This change of sign occurs if the point in question is connected with itself by a closed curve that passes an odd number of times through the Cauchy surface. The Cauchy surface itself in the case of the (1,1) projective Lorentz sphere is a closed curve; it has no end points.

V. THE THREE-DIMENSIONAL PROJECTIVE LORENTZ SPHERE

The notion of antipodal points permit the construction of (2,1) projective Lorentz spheres, in perfect analogy to the (1,1) construction. Three-dimensional projective Lorentz spheres are also not orientable, nor are they time orientable. Their Cauchy surfaces are projective spheres. Even and odd fields are defined the same as in the two-dimensional case, and the Cauchy data are again the values of the fields, and of their normal derivatives, on the Cauchy surface.

In this paper we have primarily dealt with the two-dimensional structure because of its relative ease of treatment in terms of the arbitrary functions $f(\theta)$. We have been able to study the properties of solutions in three dimensions only by expanding them with respect to spherical harmonics on the Cauchy surface. The expansion coefficients, which are functions of the third dimension and which obey ordinary differential equations, with initial values on the Cauchy surface itself, play the same role in three dimensions as the functions of γ do in the Fourier expansions of two-dimensional fields, indicated at the end of Sec. III.

The proposed topology of SI differs significantly from the topology of scri, and that difference is related somehow to the *prima facie* conflict between the irreducible quantum uncertainties on SI and on scri. Conceivably, this conflict may be mitigated by an appropriate quantum theory addressing itself to the altered causality on an unoriented manifold. It remains to be seen whether this hope can be satisfied.

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