Massless minimally coupled fields in de Sitter space: O(4)-symmetric states versus de Sitter–invariant vacuum

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The issue of de Sitter invariance for a massless minimally coupled scalar field is reexamined. Formally, it is possible to construct a de Sitter–invariant state for this case provided that the zero mode of the field is quantized properly. Here we take the point of view that this state is physically acceptable, in the sense that physical observables can be computed and have a reasonable interpretation. In particular, we use this vacuum to derive a new result: that the squared difference between the field at two points along a geodesic observer's spacetime path grows linearly with the observer's proper time for a quantum state that does not break de Sitter invariance. Also, we use the Hadamard formalism to compute the renormalized expectation value of the energy-momentum tensor, both in the O(4)-invariant states introduced by Allen and Follaci, and in the de Sitter–invariant vacuum. We find that the vacuum energy density in the O(4)-invariant case is larger than in the de Sitter–invariant case.

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I. INTRODUCTION

Quantum field theory in curved spacetimes has been extensively studied during the past two decades or so (see, e.g., Ref. [1] for a review) with the purpose of understanding quantum effects in the presence of strong gravitational fields. In particular, a lot of attention has been devoted to de Sitter space, mainly because it has a high degree of symmetry and the wave equation can be exactly solved in this background. A four-dimensional de Sitter space can be conveniently defined as a hyperboloid embedded in a five-dimensional Minkowski space:

$$\xi^{\mu}(x)\xi_{\mu}(x) = H^{-2}, \tag{1}$$

where $\xi^{\mu}(x)$ denotes the position vector of the point x in the embedding space ($\mu = 0, ..., 4$). The manifest invariance of Eq. (1) under five-dimensional Lorentz transformations implies that de Sitter space has a ten-parameter group of isometries, the de Sitter group O(4,1).

Scalar fields of mass m with an arbitrary coupling ξ to the Ricci curvature scalar [see Eq. (4) below] can be easily quantized in de Sitter space, and quantum states that respect the O(4,1) invariance of the background can be constructed for these fields [2]. Physical quantities such as the two-point function and the renormalized expectation value of the energy-momentum tensor in the de Sitter–invariant states were computed exactly in early work [3].

Later, interest in this subject was motivated by the inflationary cosmology scenario [4] (since the geometry of spacetime during inflation is that of de Sitter space). In this context, it was realized that the mean-squared fluctuations of a massless minimally coupled field (i.e., $m = \xi = 0$) grow linearly with time during inflation [5]:

$$\langle\phi^2\rangle \approx \frac{H^3}{4\pi^2}t. \tag{2}$$

Note that this expression is not de Sitter invariant, essentially because the quantum state that was used in its derivation breaks the O(4,1) invariance explicitly.

The massless minimally coupled case is peculiar in that the de Sitter–invariant two-point function becomes infrared divergent in the limit $m \to 0$, $\xi \to 0$. This led some authors [6–8] to the definition of various other "vacua" with less symmetry than the full de Sitter group, but with a two-point function which is free from infrared divergences. In particular, here we will consider the two-parameter family of O(4)-invariant Fock "vacua" introduced by Allen and Folacci [7]. Note that the ξ^0 =const spatial sections of (1) are three-spheres. The O(4) vacua are not invariant under all de Sitter transformations, but only under spatial rotations of these three-spheres.

In this paper we would like to reconsider the possibility of constructing a de Sitter–invariant state for the massless minimally coupled field. (This state is not a Fock vacuum state.) That such state can be formally constructed was already implicit in Refs. [9, 10], where the quantization was studied in the functional Schrödinger

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picture. Our emphasis here will be in the physical interpretation. For $m = \xi = 0$ the action [Eq. (4) below] has a zero mode: It is invariant under constant shifts of the field

$$\phi \to \phi + \text{const.}$$

The two-point function is ill defined because all values of the spatially constant part of the field are equally probable in the de Sitter–invariant state (which is analogous to an eigenstate of momentum in the quantum mechanics of a free particle). However, such ambiguity does not prevent us from computing the expectation value of physical observables. To illustrate this point here we shall use this vacuum to derive a more powerful result than the one given in Eq. (2), namely, that one may have a de Sitter–invariant state $|0\rangle$, and in this state, any freely falling observer who picks a base point x in spacetime will see $\langle 0|[\phi(x) - \phi(y)]^2|0\rangle$ increasing with proper time along their path. We shall also compute the renormalized expectation value of the energy-momentum tensor, both in the one-parameter family of O(4)-invariant states and in the de Sitter–invariant vacuum. As we shall see, the vacuum energy density in the de Sitter–invariant case is lower than in the O(4)-symmetric case.

The rest of the paper is organized as follows. In Sec. II we briefly review the quantization of a scalar field in de Sitter space, with the purpose of fixing the notation. In Sec. III we compute the energy-momentum tensor in the two-parameter family of O(4)-invariant vacua. Section IV discusses the de Sitter–invariant vacuum for the massless minimally coupled field. In Sec. V we use this vacuum for the calculation of some observables. Finally, a discussion of the results is given in Sec. VI. The quantization of the scalar field in the functional Schrödinger picture is summarized in the Appendix.

II. SCALAR FIELD IN de SITTER SPACE

In this section we summarize the quantum theory of a scalar field of mass m and arbitrary coupling to the scalar curvature in de Sitter space, which was developed in Refs. [2, 11, 3, 6, 9].

The line element in de Sitter space reads

$$ds^2 = g_{ab}dx^a dx^b = H^{-2}\sin^{-2}\eta[-d\eta^2 + d\Omega^2], \qquad (3)$$

where we are using the closed coordinate system $x^a = (\eta, \Omega)$, $(a = 0, ..., 3)$ that covers the whole hyperboloid (1). Here $\eta \in (0, \pi)$ is the so-called conformal time, Ω is a set of angles on the three-sphere, and $d\Omega^2$ denotes the line element on the unit three-sphere.

The action for the scalar field is given by

$$S = \frac{-1}{2}\int \sqrt{-g}[\partial_a\phi\partial^a\phi + (m^2 + \xi R)\phi^2]d^4x, \qquad (4)$$

where g is the determinant of the metric, $R = 12H^2$ is the Ricci scalar, and ξ is an arbitrary coupling. It is convenient to expand the field as

$$\phi = \sum_{LM}\chi_{LM}(\eta)Y_{LM}(\Omega), \qquad (5)$$

where Y_{LM} are the usual spherical harmonics on the three-sphere, normalized as

$$\int Y_{LM}(\Omega)Y^*_{L'M'}(\Omega)d\Omega = \delta_{LL'}\delta_{MM'}. \qquad (6)$$

They are eigenfunctions of the Laplacian on the three-sphere:

$$\Delta^{(3)}Y_{LM} = -JY_{LM}, \qquad (7)$$

with $J = L(L + 2)$, $L = 0, ..., \infty$. The index M, $M = 0, ..., (L + 1)^2$, labels the degeneracy for given L.

Introducing (5) in (4) one finds

$$S = \frac{1}{2}\sum_{LM}\int (H\sin\eta)^{-2}[(\dot\chi_{LM})^2 - \omega_L^2(\eta)\chi^2_{LM}]d\eta, \qquad (8)$$

where

$$\omega_L^2(\eta) \equiv J + \frac{m^2 + \xi R}{(H\sin\eta)^2},$$

and the overdot indicates the derivative with respect to η. In going from (4) to (8) the term $\partial_i Y_{LM}\partial_j Y_{LM}$ has been integrated by parts and the relations (6) and (7) have been used. Equation (8) can be seen as the action for a collection of harmonic oscillators with time-dependent frequencies. The classical equations of motion for the modes $\chi_{LM}(\eta)$ read

$$\ddot\chi_{LM} - 2\cot\eta\dot\chi_{LM} + \omega_L^2(\eta)\chi_{LM} = 0. \qquad (9)$$

To quantize the theory, the field variables χ_{LM} and their canonically conjugate momenta

$$\pi_{LM} \equiv \frac{\partial L}{\partial\dot\chi_{LM}} = (H\sin\eta)^{-2}\dot\chi_{LM} \qquad (10)$$

are promoted to operators satisfying the canonical commutation relations

$$[\hat\chi_{LM}, \hat\pi_{L'M'}] = i\delta_{LL'}\delta_{MM'}. \qquad (11)$$

In the Heisenberg picture, these are time-dependent operators, and it is customary to expand them in terms of (time-independent) creation and annihilation operators a_{LM} and $a^\dagger_{LM}$:

$$\hat\chi_{LM} = U_{LM}a_{LM} + U^*_{LM}a^\dagger_{LM},$$
$$\qquad (12)$$
$$\hat\pi_{LM} = (H\sin\eta)^{-2}[\dot U_{LM}a_{LM} + \dot U^*_{LM}a^\dagger_{LM}].$$

Here $U_{LM}(\eta)$ are solutions of the field equation (9) (with $\chi_{LM} \leftrightarrow U_{LM}$) normalized according to the Wronskian condition

$$U_{LM}\dot U^*_{LM} - U^*_{LM}\dot U_{LM} = i(H\sin\eta)^2. \qquad (13)$$

The commutation relations (11) follow from (13) and the usual commutation relations for the creation and annihilation operators:

$$[a_{LM}, a^\dagger_{L'M'}] = \delta_{LL'}\delta_{MM'},$$

$$[a_{LM}, a_{L'M'}] = [a^\dagger_{LM}, a^\dagger_{L'M'}] = 0.$$

A "vacuum" state $|0\rangle$ can be defined by

$$a_{LM}|0\rangle = 0, \quad \forall L, M, \tag{14}$$

and the complete Hilbert space of states can be generated by repeated operation on $|0\rangle$ of the creation operators $a^\dagger_{LM}$. As is usual in curved space (see, e.g., [1]), the definition of this vacuum is somewhat arbitrary, since it depends on what particular choice we make for the set of modes $\{U_{LM}\}$. However, de Sitter space is a maximally symmetric space, invariant under a ten-parameter group of isometries [the de Sitter group O(4,1)], and it is natural to choose a vacuum state which also has the same symmetry. Actually, there exists a one-parameter family of de Sitter–invariant quantum states. Among them, we shall concentrate on the so-called Euclidean vacuum as the only one whose two-point function has Hadamard form and so the ultraviolet behavior is the same as for field theory in flat spacetime. The mode functions corresponding to the Euclidean vacuum are given by [2]

$$U_{LM} = A_L(\sin\eta)^{3/2}\left[P^\lambda_\nu(-\cos\eta) - \frac{2i}{\pi}Q^\lambda_\nu(-\cos\eta)\right], \tag{15}$$

where P^λ_ν and Q^λ_ν are Legendre functions on the cut, and

$$\lambda = \left[\frac{9}{4} - \frac{m^2+\xi R}{H^2}\right]^{1/2}, \quad \nu = L + \frac{1}{2}. \tag{16}$$

The normalization constants are given by

$$A_L = \frac{\sqrt{\pi}}{2}He^{i\lambda\pi/2}\left[\frac{\Gamma(L-\lambda+3/2)}{\Gamma(L+\lambda+3/2)}\right]^{1/2}. \tag{17}$$

The de Sitter invariance of this state is manifest in the symmetric two-point function

$$\begin{aligned} G^{(1)}(x,x') &= \langle 0|\phi(x)\phi(x') + \phi(x')\phi(x)|0\rangle \\ &= \sum_{LM}[U_{LM}(\eta)U^*_{LM}(\eta')Y_{LM}(\Omega)Y^*_{LM}(\Omega') \\ &\quad + U_{LM}(\eta')U^*_{LM}(\eta)Y_{LM}(\Omega')Y^*_{LM}(\Omega)], \end{aligned} \tag{18}$$

which can be evaluated to yield [2]

$$\begin{aligned} G^{(1)}(Z) = \frac{2H^2}{(4\pi)^2}&\Gamma\left(\frac{3}{2}-\lambda\right)\Gamma\left(\frac{3}{2}+\lambda\right) \\ &\times F\left(\frac{3}{2}-\lambda, \frac{3}{2}+\lambda, 2; \frac{1+Z}{2}\right). \end{aligned} \tag{19}$$

Here F is the hypergeometric function, and Z is given by [7]

$$Z(x,x') = H^2\xi^\mu(x)\xi_\mu(x') = \frac{\cos\gamma - \cos\eta\cos\eta'}{\sin\eta\sin\eta'}, \tag{20}$$

where γ is the angle between Ω and Ω'. Note that the two-point function depends only on Z, which is a Lorentz-invariant quantity in the embedding space, and therefore $G^{(1)}$ is de Sitter invariant.

The quantity $Z(x,x')$ can also be expressed as [7]

$$Z = \cos\sqrt{\frac{R\sigma}{6}}, \tag{21}$$

where $\sigma(x,x')$ is defined as one-half of the square of the geodesic distance between x and x'. If x and x' are timelike separated, then $\sigma < 0$ and $Z > 1$. On the other hand, if they are spacelike separated, then $Z < 1$. (However, a geodesic joining the two points exists only if $-1 \le Z$; hence, σ is undefined for $Z < -1$.)

III. MASSLESS MINIMALLY COUPLED CASE: O(4)-INVARIANT VACUUM

It should be noted that the two-point function (19) is ill-defined in the massless minimally coupled case ($m = \xi = 0$), since one of the γ functions has a pole at $\lambda = 3/2$. This divergence has led some authors [6–8] to the definition of other vacua with less symmetry than the full de Sitter group, but with a well-defined two-point function.

In the closed coordinate system that we are using, one such natural vacuum is the O(4)-invariant vacuum [7], which is symmetric under rotations of the η =const spatial sections (which are three-spheres). The set of modes that defines the O(4)-invariant quantum state is given by (15) for $L > 0$, but in order to avoid the infrared divergence, the $L = 0$ mode solution is chosen as [7]

$$U_0 = H\left[A\left(\eta - \frac{1}{2}\sin 2\eta - \frac{\pi}{2}\right) + B\right], \tag{22}$$

with

$$A = -i\alpha, \quad \alpha \in (0,\infty),$$

$$B = \frac{1}{\alpha}\left(\frac{1}{4} + i\beta\right), \quad \beta \in (-\infty,\infty).$$

The two complex parameters A and B have been reduced to two real parameters α and β because an overall phase is irrelevant and because (13) must be satisfied. In addition, requiring time-reversal invariance fixes $\beta = 0$ [7], which leaves us with just one parameter α. In what follows we take $\beta = 0$.

The two-point function in this state is

$$G^{(1)}_\alpha(x,x') = \hat{G}(x,x') + \frac{1}{2\pi^2}[U_0(\eta)U^*_0(\eta') + U_0(\eta')U^*_0(\eta)], \tag{23}$$

where $\hat{G}$ is defined as a sum over modes similar to (18) but without the $L = 0$ term. This sum is given (up to some irrelevant constant) in closed form by [7]

$$\begin{aligned} \hat{G}(x,x') = \frac{R}{48\pi^2}\Bigg[&\frac{1}{1-Z} - \ln(1-Z) - \ln(4\sin\eta\sin\eta') \\ &- \sin^2\eta - \sin^2\eta'\Bigg], \end{aligned} \tag{24}$$

with Z defined in (20).

We will be interested in constructing the energy-momentum tensor using the Hadamard formalism [12, 13]. For this we need to study the two-point function in the coincidence limit; that is, we have to bring $G^{(1)}$ into the Hadamard form [14, 15]

$$G^{(1)}(x,x') = \frac{1}{4\pi^2}\left[\frac{\Delta^{1/2}(x,x')}{\sigma} + V(x,x')\ln\sigma + W(x,x')\right], \quad (25)$$

where $\sigma(x,x')$ was defined in Eq. (21) and $\Delta(x,x')$ is the Van Vleck–Morette determinant. In de Sitter space it is given by [14]

$$\Delta(\sigma) = \left(\frac{R\sigma}{6}\right)^{3/2}\left[\sin\sqrt{\frac{R\sigma}{6}}\right]^{-3}. \quad (26)$$

In Eq. (25), $V(x,x')$ and $W(x,x')$ are symmetric functions of x and x', which are smooth in the coincidence limit.

Using (21) and (26), one can compare expressions (25) and (23) and (24) to find

$$V(x,x') = -\frac{R}{12}, \quad (27)$$

$$W(x,x') = F(\sigma) - \frac{R}{12}[\ln(4\sin\eta\sin\eta') + \sin^2\eta + \sin^2\eta'] + 2[U_0(\eta)U_0^*(\eta') + U_0^*(\eta)U_0(\eta')]. \quad (28)$$

Here

$$F(\sigma) \equiv \frac{R}{12}\left[\frac{1}{1-\cos X} - 2\frac{1}{(X\sin^3 X)^{1/2}} - \ln\left(\frac{R}{6X^2}[1-\cos X]\right)\right], \quad (29)$$

with

$$X = \sqrt{\frac{R\sigma}{6}}.$$

One can check that W is well behaved at $\sigma = 0$ (as expected from the general theory) by expanding each term in (29) in powers of σ. We find that the negative powers of σ cancel out, and we have

$$F(\sigma) = -\frac{R}{12}\left[\ln\left(\frac{R}{12}\right) + \frac{1}{3} - \frac{R\sigma}{480} + O(\sigma^2)\right]. \quad (30)$$

As usual, the singular part in (25) is purely geometrical, and all the dependence of $G^{(1)}$ on the quantum state is contained in the function $W(x,x')$.

The two-point function is now in a form ready for the computation of the renormalized expectation value of the energy-momentum tensor. Using the Hadamard formalism, this is given by [12, 13]

$$8\pi^2\langle T_{ab}\rangle_{\rm ren} = \tau_{ab}[W] - \tau_{ab}[V]\ln\mu^2 + 2v_1 g_{ab} - \frac{m^4}{16}g_{ab}, \quad (31)$$

where

$$\tau_{ab}[f] \equiv \lim_{x\to x'} \mathcal{D}_{ab'}(x,x')[f(x,x')].$$

Here $\mathcal{D}$ is the differential operator associated with the point-split expression of the formal energy-momentum operator. In the massless minimally coupled case,

$$\mathcal{D}_{ab'} \equiv \nabla_a\nabla_{b'} - \tfrac{1}{2}g_{ab'}g_{dd'}\nabla^d\nabla^{d'},$$

where $g_a^{b'}$ is the bivector of parallel transport [16]. In Eq. (31), μ^2 is a renormalization scale (arbitrary, in principle), and v_1 is the "trace anomaly" scalar, which in de Sitter space is equal to [13]

$$v_1 = \frac{29R^2}{8640}.$$

From (27) it is clear that, in our case,

$$\tau_{ab}[V] = 0,$$

and the dependence on the renormalization scale disappears. This is fortunate, since in the massles case there is no natural mass parameter in the problem. Also, the last term in (31) vanishes for $m = 0$.

All that we need to evaluate is $\tau_{ab}[W]$, with W given by (28). The term $\tau_{ab}[F(\sigma)]$ can be easily computed by noting that

$$\lim_{x\to x'}\nabla_a\nabla_{b'}F(\sigma) = \lim_{x\to x'}[F''(\sigma)\sigma_{,a}\sigma_{,b'} + F'(\sigma)\sigma_{;ab'}] = -F'(\sigma)|_{\sigma=0}g_{ab}, \quad (32)$$

where a prime indicates the derivative with respect to σ and we have used (see, e.g., [12])

$$\lim_{x\to x'}\sigma_{,a} = 0,$$

$$\lim_{x\to x'}\sigma_{ab'} = -g_{ab}.$$

The value of $F'(\sigma)$ at $\sigma = 0$ can be read off from (30), and using (32) we have

$$\tau_{ab}[F] = \frac{R^2}{5760}g_{ab}.$$

Also, it is clear that

$$\tau_{ab}[\ln(2\sin\eta) + \ln(2\sin\eta') + \sin^2\eta + \sin^2\eta'] = 0,$$

and one can check that

$$\tau_{ab}[U_0(\eta)U_0^*(\eta') + U_0^*(\eta)U_0(\eta')] = \tfrac{1}{36}R^2\alpha^2(1-2\delta_{a\eta})\sin^6\eta\, g_{ab}.$$

Substituting the previous expressions in (31), we have

$$\langle\alpha|T_{ab}|\alpha\rangle_{\rm ren} = \frac{119R^2}{138240\pi^2}g_{ab} + \frac{R^2}{144\pi^2}\alpha^2\sin^6\eta\, g_{ab}(1-2\delta_{a\eta}). \quad (33)$$

Therefore, the energy-momentum tensor is not de Sitter invariant, but only O(4) invariant, because of the explicit time dependence. Note also that the term which is not de Sitter invariant decays with the expansion of the Universe as a^{-6}, where a is the scale factor (compare with radiation, which behaves as a^{-4} or with the vacuum energy itself which behaves as a^0) and therefore it is unlikely to have any cosmological consequences. In the limit $\eta \to 0$ or $\eta \to \pi$, which corresponds to cosmological time going to $+\infty$ or $-\infty$, Eq. (33) reduces to the result (3.6) in Ref. [7] as corrected in Ref. [17].

IV. de SITTER–INVARIANT VACUUM FOR THE MASSLESS MINIMALLY COUPLED CASE

Sometimes it is said [6, 8] that the infrared divergence in $G^{(1)}$ indicates that de Sitter invariance is broken in the massless minimally coupled case. However, it is still possible to define a de Sitter–invariant vacuum for this case, and here we will take the point of view that this state is physically acceptable in the sense that physical quantities can be computed and have a reasonable interpretation. However, as we shall see, the space of states cannot be simply represented as a Fock space built by applying creation operators to this vacuum state. The quantization of ϕ in the case $m = \xi = 0$ is peculiar because the field contains a zero mode: The action is invariant under the transformation

$$\phi \to \phi + \text{const.} \tag{34}$$

It is well known that an expansion in terms of creation and annihilation operators, such as (12), is not adequate for the variables associated to the zero modes [18–21].

The situation is analogous to that of a quantum-mechanical harmonic oscillator: The expansion of the position and momentum operators x and p in terms of creation and annihilation operators breaks down in the limit when the frequency of the oscillator, ω, goes to zero (the free-particle case). In the Heisenberg picture we have

$$x(t) = (2M\omega)^{-1/2}(ae^{-i\omega t} + a^\dagger e^{+i\omega t}),$$

$$p(t) = -i(M\omega/2)^{1/2}(ae^{-i\omega t} - a^\dagger e^{+i\omega t}),$$

where M is the mass of the particle. Of course, these expressions are not valid in the limit $\omega \to 0$. The physical reason is that for a free particle the spectrum of the Hamiltonian becomes continuous and the number operator loses its meaning. Instead, we can consider the expansions

$$\begin{aligned} x(t) &= x_0 + p_0 t, \\ p(t) &= p_0, \end{aligned} \tag{35}$$

where the new operators satisfy the commutation relation $[x_0, p_0] = i$. At the classical level, x_0 and p_0 have the interpretation of the initial position and momentum (and are therefore constants), and so (35) can be seen as a Hamilton-Jacobi canonical transformation in which the new variables are constants of motion. The first equation in (35) is obviously the general solution of the equations of motion if we think of x_0 and p_0 as constants of integration. In this sense this equation is analogous to (22).

The simplest example of a field theory with zero modes is the massless scalar field in a flat compact space with finite volume V and topology of a torus $S^1 \times S^1 \times S^1$, discussed in Ref. [19]. In that case, a complete set of solutions of the wave equation is given by

$$\begin{aligned} f_{\mathbf{k}} &= (2V\omega)^{-1/2} \exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)] \quad (\mathbf{k} \neq 0), \\ f_0 &= At + B. \end{aligned} \tag{36}$$

Here $\omega = |\mathbf{k}|$ and the momenta $\mathbf{k}$ have the usual discrete spectrum due to finite volume. The Klein-Gordon normalization requires $A^*B - B^*A = i/V$. While the modes $f_{\mathbf{k}}$ ($\mathbf{k} \neq 0$) are the classical solutions for a harmonic oscillator of frequency ω, the mode f_0 is the classical solution for a free particle. Therefore, although it is formally possible to define creation and annihilation operators associated with f_0, in a manner analogous to the construction of the O(4)-invariant vacuum of the previous section, it is more natural to define position and momentum operators analogous to p_0 and x_0 above. With this the field expansion reads [19]

$$\phi = \frac{x_0 + p_0 t}{\sqrt{V}} + \sum_{\mathbf{k}\neq 0}(a_{\mathbf{k}} f_{\mathbf{k}} + \text{H.c.}).$$

It can be checked that the equal-time commutation relation for ϕ and its conjugate momentum are satisfied if $[x_0, p_0] = i$ and the usual commutation relations for the creation and annihilation operators are satisfied.

Note that in the limit of infinite volume the special treatment of the zero mode becomes irrelevant, as it makes a contribution of zero measure in the expansion of the field. An equivalent statement is that the set of modes with $\mathbf{k} \neq 0$ becomes complete in the limit of infinite volume. However, for finite volume the zero mode is important and makes a finite contribution to the energy. Indeed, it is straightforward to see that

$$E = \frac{p_0^2}{2} + \sum_{\mathbf{k}} |\mathbf{k}| \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2} \right).$$

One can define the ground state for this system through the equations $p_0|0\rangle = 0$, $a_{\mathbf{k}}|0\rangle = 0$. This ground state is not normalizable, in the same way that the ground state of a quantum-mechanical free particle is not (for a detailed discussion on this issue, see Ref. [18], Sec. 9). The field operator is seen to be equivalent to a collection of harmonic oscillators plus a free particle [whose position, in the Heisenberg picture, would be given by the operator $x(t) = x_0 + p_0 t$]. The space of states is equivalent to the direct product of a Fock space corresponding to the oscillators and an ordinary Hilbert space corresponding to the free particle. Since the energy is an observable, in addition to the usual Fock space operators, the momentum p_0 is also an observable.

The above construction can be generalized to arbitrary curved backgrounds [18]. Of course, in general, there is the usual caveat that for nonstationary backgrounds the energy is not conserved and the definition of a ground state is ambiguous. This is nothing new; it is the same problem that we encountered in Sec. II when discussing the massive field: The definition of a "vacuum" in nonstationary backgrounds is always a matter of choice. Here, as in Sec. II, we will be guided by considerations of symmetry in making this choice.

In the case of a massless minimally coupled field in de Sitter space, the zero mode associated with (34) is in the homogeneous sector ($L = 0$), and that is the reason why the coefficient A_0 [see Eq. (17)] becomes infinite for $\lambda \to 3/2$. Instead of defining creation and annihilation operators for $L = 0$ we replace the expansions (12) by [7]

$$\chi_0 = \frac{H}{\sqrt{2}}\left[Q + \left(\eta - \frac{1}{2}\sin 2\eta - \frac{\pi}{2}\right)P\right],$$
$$\pi_0 = \frac{\sqrt{2}}{H}P. \tag{37}$$

The coefficients of Q and P in the expansion of χ_0 are solutions of the field equation (9), and the expression for π_0 follows from (10). Moreover, the commutation relation between χ_0 and π_0 implies

$$[Q, P] = i,$$

and so, again, (37) can be seen as a Hamilton-Jacobi transformation in which the new canonical variables are constants of motion.

We define a vacuum state by

$$P|0\rangle = 0,$$
$$a_{LM}|0\rangle = 0, \quad L > 0, \tag{38}$$

where a_{LM} were defined in Sec. II.

The ambiguity in the choice of a vacuum corresponds to the freedom in the choice of the mode functions U_{LM} for $L \neq 0$ [which we take to be the same as for the O(4) vacuum], plus the freedom in choosing the mode solutions which appear as coefficients of Q and P in Eq. (37). In principle, we could have chosen any two homogeneous solutions of the wave equation, say, $f_1(\eta)$ and $f_2(\eta)$,

$$\chi_0 = f_1\tilde{Q} + f_2\tilde{P},$$
$$\pi_0 = (H\sin\eta)^{-2}(\dot{f}_1\tilde{Q} + \dot{f}_2\tilde{P}),$$

subject to the Wronskian condition $\dot{f}_2 f_1 - \dot{f}_1 f_2 = H^2\sin^2\eta$. With the choice (37) the equation $P|0\rangle = 0$ implies that the vacuum wave functional Ψ does not depend on χ_0:

$$P\Psi = \frac{H}{\sqrt{2}}\left(-i\frac{\partial}{\partial\chi_0}\right)\Psi = 0. \tag{39}$$

If we are interested in a de Sitter–invariant vacuum, this turns out to be the right choice.

In the Appendix we review the quantization of the scalar field in the Schrödinger picture. We show that in the limit $m \to 0$ and $\xi \to 0$, the de Sitter–invariant wave functional becomes independent of χ_0, and therefore it satisfies $P|0\rangle = 0$ [the other equations in (38) are also satisfied by construction]. Note that the solution of (39) is not normalizable, and that is the reason why $G^{(1)}$ is ill-defined in the de Sitter–invariant state. This should not be taken as an indication that the state is pathological: It simply means that all values of χ_0 are equally probable. The same problem would arise in the quantum mechanics of a free particle if we tried to compute $\langle p|x^2|p\rangle$, where $|p\rangle$ is an eigenstate of momentum.

Apart from considerations about de Sitter invariance (the group of isometries of the background spacetime), there is another (aesthetic) reason for choosing a state with $P|0\rangle = 0$, based on the symmetry of the Lagrangian under $\phi \to \phi + \text{const}$. The corresponding Noether current is $j_\mu = \partial_\mu\phi$. The generator of the symmetry is the "charge"

$$\hat{Q} = \int_{\eta=\text{const}} d\Sigma_\mu j^\mu,$$

and so the vacuum will be invariant under this symmetry if it is annihilated by the charge, $\hat{Q}|0\rangle = 0$. Introducing $j_\mu = \partial_\mu\phi$ in (4) we find $\hat{Q} = 2\pi H^{-1}P$, and so the condition becomes $P|0\rangle = 0$. Note that even though the current is linear in ϕ, the charge operator is nonvanishing and well defined precisely because the space has compact spatial sections.

As mentioned before, the vacuum state defined in this way is not simply a Fock-space vacuum (in fact, this would be in contradiction of the work of Allen [6]) but the direct product of a Fock space and an ordinary Hilbert space corresponding to the χ_0 variable. As we shall see, in order that the energy density T_{00} be a physical observable, in addition to the usual Fock-space observables the operator P is also a physical observable.

A basis for the space of states is the direct product of the basis for the Fock space times the basis for the Hilbert space of a particle in one dimension. The structure of the Fock space corresponding to the modes $L > 0$ is identical to the one corresponding to the O(4)-invariant vacuum, and we shall not discuss it further. The Hilbert space for one particle in one dimension is isomorphic to the usual space of square-integrable complex functions of a real variable, and a convenient basis is formed by the eigenstates of the momentum operator P with eigenvalue p:

$$|p\rangle \equiv e^{ipQ}|0\rangle.$$

Since they form a continuous basis, these states are not normalized in the discrete sense, but they have the continuous normalization $\langle p|p'\rangle = \delta(p - p')$. In the q representation they are the ordinary plane waves

$$\langle q|p\rangle = (2\pi)^{-1/2}e^{ipq}, \tag{40}$$

where $|q\rangle$ are eigenstates of Q with eigenvalue q, normalized as $\langle q|q'\rangle = \delta(q - q')$.

In this representation Q acts as a multiplicative operator and P as a derivative operator:

$$\langle q|Q|\psi\rangle = q\langle q|\psi\rangle,$$
$$\langle q|P|\psi\rangle = -i\frac{\partial}{\partial q}\langle q|\psi\rangle.$$

To make a connection with the previous section, one can see that the $L = 0$ sector of the O(4)-invariant vacuum $|\alpha\rangle$ discussed previously corresponds to the normalized Gaussian wave packet [7]

$$\langle q|\alpha\rangle \equiv \psi_\alpha(q) = \frac{\sqrt{2\alpha}}{\pi^{1/4}} e^{-2\alpha^2 q^2}. \tag{41}$$

Indeed, the operator a_0 of the O(4) vacuum can be expressed [using (37)] as $a_0 = i\sqrt{2}[B^* P - A^* Q]$, which clearly annihilates (41). Also, the "multiparticle" homogeneous ($L = 0$) excitations above $|\alpha\rangle$ are obtained by repeated operation of $a_0^\dagger$ on (41), which gives

$$\psi_\alpha^n \equiv \left\langle q \left| \frac{(a_0^\dagger)^n}{\sqrt{n!}} \right| \alpha \right\rangle = \left(\frac{2\alpha}{\pi^{1/2} 2^n n!} \right)^{1/2} H_n(2\alpha q) e^{-2\alpha^2 q^2}, \tag{42}$$

where H_n are the Hermite polynomials.

Throughout this section we have worked in the Heisenberg picture, and therefore the states (42) are time independent. To obtain the corresponding wave functions in the Schrödinger picture one can solve the Schrödinger equation with initial conditions (42). As we show in the Appendix, this Schrödinger equation is just the one for a free particle, and so the evolution of (41) is just that of a minimal wave packet which spreads in time.

V. DISPERSION OF THE FIELD AND ENERGY-MOMENTUM TENSOR

In order to gain intuition on the structure of the de Sitter–invariant vacuum defined in (38), let us consider the "dispersion" of the field, defined by

$$D^2(x, y) \equiv \langle 0|[\phi(x) - \phi(y)]^2|0\rangle, \tag{43}$$

which will give us an idea on how the value of the field fluctuates over space and time. Since D^2 contains terms of the form $\langle 0|\phi(x)\phi(x)|0\rangle$, we will encounter the usual ultraviolet divergences associated with the product of operators in the coincidence limit. A convenient way of getting around such divergences is to smear the field operator over a region of size s (see, e.g., [22, 23]):

$$\phi_s(x) \equiv \frac{1}{\mathrm{Vol}(s)} \int_{d(x,x')<s/2} \phi(x') d^3x',$$

where $d(x, y)$ is the geodesic distance between x and x' and Vol(s) is the volume of the smearing region. Here and for the rest of this section, d^3x stands for the three-dimensional invariant-volume element. The "diameter" s of the smearing region should be less than $2\pi H^{-1}$, since there are no spacelike geodesics longer than that [see comments after Eq. (21)], and we shall take $s \sim H^{-1}$. Also, in order to smear the field operator it is necessary to make a particular choice of the spacelike hypersurface at the point x. In what follows we shall always consider situations in which geodesic observers are involved, and so the smearing regions can be defined on the spacelike sections orthogonal to these geodesics. For instance, if x and y are timelike separated, we can consider the geodesic curve that links x with y, and take spacelike surfaces at x and y generated by the spacelike geodesics orthogonal to this curve. Later, we shall also consider the field measured by two observers moving along two different geodesics. Each observer can smear the field on the spacelike surface orthogonal to his or her geodesic. In any case, in the limit of large separation between x and y, the leading term in the dispersion will not depend on the details of how we smear the field.

Now we can consider the dispersion of the smeared field:

$$D_s^2(x, y) = \langle 0|[\phi_s(x) - \phi_s(y)]^2|0\rangle. \tag{44}$$

Note that this expression has no infrared divergences either, since the operator Q (which causes trouble in the two-point function because its expectation value is ill-defined in the de Sitter–invariant vacuum) cancels out when we consider the difference $\phi(x) - \phi(y)$.

As an intermediate step to compute (44) we "point-split" and symmetrize expression (43):

$$D_\epsilon^2(x', x''; y', y'') \equiv \tfrac{1}{2}\langle 0|\{[\phi(x') - \phi(y')], [\phi(x'') - \phi(y'')]\}|0\rangle.$$

Here x' and x'' are points within the smearing region surrounding x, separated by a geodesic distance ϵ ($\epsilon < s$) (similarly for y' and y''), and the curly brackets denote the anticommutator. Since $P|0\rangle = 0$, this expression reduces to

$$D_\epsilon^2 = \tfrac{1}{2}[\hat{G}(x', x'') + \hat{G}(y', y'') - \hat{G}(x', y'') - \hat{G}(y', x'')],$$

with $\hat{G}$ defined in (23) and (24). It is convenient to rewrite it as

$$D_\epsilon^2 = \frac{H^2}{8\pi^2}[g(x', x'') + g(y', y'') - g(x', y'') - g(y', x'')], \tag{45}$$

where

$$g(u, v) \equiv \frac{1}{1 - Z(u, v)} - \ln|1 - Z(u, v)|.$$

Note that (45) is a fully de Sitter–invariant expression (as it should be, since we are dealing with a de Sitter–invariant state). All the terms in (24) that depend explicitly on the conformal time η cancel out in the expression for D_ϵ.

Now we can easily estimate the smeared dispersion $D_s^2(x, y)$ for the case when the separation between x and y is much larger than H^{-1} (that is $|Z| \gg 1$). This can be done by smearing the expression (45) term by term. We note that when the two points x' and x'' lie within the same smearing region, the integrals

$$\frac{1}{[\mathrm{Vol}(s)]^2} \int_{d(x,x')<s/2} d^3x' \int_{d(x,x'')<s/2} d^3x'' g(x', x'')$$

give contributions of order 1, while if one of the points lies in the neighborhood of x and the other lies in the neighborhood of y, we have

$$\frac{1}{[\mathrm{Vol}(s)]^2}\int d^3x'd^3y''g(x',y'')\approx g(x,y)\approx -\ln|Z(x,y)|,$$

where we have used $|Z|\gg 1$. As a result,

$$D_s^2(x,y)\approx\frac{H^2}{4\pi^2}\ln|Z(x,y)|\quad(|Z|\gg 1). \tag{46}$$

If x and y are timelike separated, we can use (21) to write

$$\langle 0|[\phi_s(x)-\phi_s(y)]^2|0\rangle\approx\frac{H^3}{4\pi^2}\tau\quad(\tau\gg H^{-1}), \tag{47}$$

where τ is the proper time measured by a geodesic observer traveling from x to y.

Equation (47) embodies a familiar property of massless minimally coupled fields in de Sitter space, namely, that the mean-squared fluctuations in the field grow linearly with time [5, 23] [see Eq. (2)]. Here we have been able to derive this result in an invariant way, without the need of using a quantum state that breaks de Sitter invariance and without the need of introducing a cosmological time coordinate [τ in Eq. (47) is just the geodesic distance]. As noted by Vilenkin [22], the linear growth in time of the mean-squared fluctuation can be interpreted in terms of a random walk of the field ϕ. The magnitude of ϕ smeared over the interior of a Hubble-radius (H^{-1}) two-sphere changes by $\pm(H/2\pi)$ per expansion time H^{-1}. Then, the average displacement squared is $D_s^2\sim(H/2\pi)^2N$, where $N\sim H\tau$ is the number of steps. To support this interpretation, Vilenkin studied field correlations between points which were at large spacelike separations. We can repeat his arguments using the de Sitter–invariant formalism.

Since points separated by spacelike distances greater than πH^{-1} cannot be connected by geodesics (and we are interested in much larger separations), the discussion will require more work than in the case of timelike separations. Consider, to begin with, an arbitrary point x in de Sitter space and a timelike geodesic C_x passing through it. We can think of C_x as the trajectory of an inertial observer. Without loss of generality (by using de Sitter transformations) we can take x to have coordinates ($\eta=\pi/2,\Omega$) and C_x to be the curve Ω =const, while the metric still takes the form (3). Let x' be a second point on the spacelike hypersurface orthogonal to C_x at x, such that the geodesic distance between x and x' is much *smaller* than H^{-1}, and let $C_{x'}$ be a geodesic through the point x' which is initially parallel to C_x. In our coordinate system, x' has coordinates ($\eta=\pi/2,\Omega'$), $C_{x'}$ is the curve Ω' =const, and the distance between x and x' is γH^{-1} ($\gamma\ll 1$), where γ is the angle between Ω and Ω'. Parametrizing both geodesics by the proper time τ and taking $\tau=0$ at $\eta=\pi/2$, we can find what is the separation between points in C_x and C'_x at any given τ.

From (20) we have

$$Z(\tau,\gamma)=1+[\cos\gamma-1]\cosh^2 H\tau,$$

where $Z(\tau,\gamma)$ means the invariant function Z between the two points on the geodesics C_x and C'_x at proper time τ and we have used $(\sin\eta)^{-1}=\cosh H\tau$. Two observers at x and x', which were initially close and at rest relative to each other ($Z\approx 1$), are pulled apart by the expansion, so that eventually they reach large spacelike separation $Z\ll -1$. The distance between both observers will be equal to $(\pi/2)H^{-1}$ at the time τ_* when $Z(\tau_*,\gamma)=0$ [see Eq. (21)], and so we can write

$$Z(\tau,\gamma)=1-\frac{\cosh^2 H\tau}{\cosh^2 H\tau_*}.$$

For $\tau>>\tau_*$ we have [from (46)]

$$D_s^2\approx 2\frac{H^3}{4\pi^2}(\tau-\tau_*). \tag{48}$$

In the language of Ref. [23], this result can be phrased as follows. The field measured by each one of the two observers undergoes a random walk of step $\Delta\phi_s=\pm(H/2\pi)$. As long as both observers lie within the same Hubble volume their steps are correlated and the dispersion of the field does not grow. Approximately after time τ_*, the Hubble volumes around the two observers stop overlapping; this means the future light cones of the two observers fail to overlap, and so the respective random walks of the field become uncorrelated. Therefore, the dispersion is proportional to $(\tau-\tau_*)$. The factor of 2 in Eq. (48) arises because we have two independent random walks.

Finally, we should say that although (47) and (48) have been derived using the de Sitter–invariant state, they would hold for any O(4)-invariant state (in the limit of large τ). This is because the contribution of $L=0$ to D_s^2 is

$$\frac{H^2}{2\pi}[\eta-\tfrac{1}{2}\sin 2\eta+(\eta\leftrightarrow\eta')^2]\langle P^2\rangle.$$

This term remains bounded in time and eventually becomes subdominant with respect to the vacuum terms (47) and (48). Similarly, because the modes U_{LM} are bounded in time, any finite number of particles in the modes $L>0$ will make a bounded contribution which will be irrelevant at late times.

Another physical quantity that we can compute using $|0\rangle$ is the expectation value of the energy-momentum tensor. Since the differential operator $\mathcal{D}_{ab'}$ acting on a constant is zero, the operator Q will not be present in the formal expression of $T_{ab'}$. Also, since $P|0\rangle=0$, it is clear that

$$\langle 0|\mathcal{D}_{ab'}\{\phi(x),\phi(x')\}|0\rangle=\mathcal{D}_{ab'}\hat{G}(x,x').$$

The computation of $\langle 0|T_{ab}|0\rangle$ now reduces to the one presented in Sec. IV, replacing $G^{(1)}_{A,B}$ by $\hat{G}$. Obviously, the result is given by Eq. (33) with $A=0$:

$$\langle 0|T_{ab}|0\rangle_{\mathrm{ren}}=\frac{119}{138240\pi^2}R^2g_{ab}, \tag{49}$$

which is de Sitter invariant as expected.

Since we chose a state with $P|0\rangle=0$, there is no contribution from the $L=0$ sector to $\langle T_{ab}\rangle$. The $L=0$ contribution to the energy-momentum tensor operator is

$$\hat{T}_{ab}^{(L=0)} = \frac{R^2}{144\pi^2}(1-2\delta_{a0})\sin^6\eta\, g_{ab}\frac{\hat{P}^2}{2}.$$

In a state with nonvanishing momentum, the expectation value of this operator has to be added to the right-hand side (RHS) of (49). In particular, for the O(4)-invariant states, $\langle P^2\rangle_\alpha = 2\alpha^2$ and we recover (33). Clearly, for the energy $\langle T_{00}\rangle$ to be an observable, P has to be observable.

It is interesting to compare Eq. (49) with the general result for a massive and nonminimally coupled field [3]:

$$\langle T_{ab}\rangle_{\rm ren} = \frac{-g_{ab}}{64\pi^2}\Bigg\{ m^2\left[m^2+\left(\xi-\frac{1}{6}\right)R\right]\left[\psi\left(\frac{3}{2}-\lambda\right)+\psi\left(\frac{3}{2}+\lambda\right)+\ln\frac{R}{12m^2}\right] -m^2\left(\xi-\frac{1}{6}\right)R-\frac{1}{18}m^2R-\frac{1}{2}\left(\xi-\frac{1}{6}\right)^2R^2+\frac{R^2}{2160}\Bigg\}.$$

Note that the limit of this expression as $m\to 0$ and $\xi\to 0$ is ambiguous, because the term

$$\frac{-g_{ab}}{64\pi^2}m^2\left[m^2+\left(\xi-\frac{1}{6}\right)R\right]\psi\left(\frac{3}{2}-\lambda\right) \longrightarrow \frac{-g_{ab}}{1536\pi^2}\frac{R^2}{1+\frac{\xi R}{m^2}} \qquad (50)$$

gives different answers by approaching the origin of the (ξ, m^2) plane in different ways. It is intriguing that in order to recover the result (49) the limit $m^2,\xi\to 0$ has to be taken along a path such that

$$\frac{\xi R}{m^2}\to -2. \qquad (51)$$

The origin of the ambiguity can be traced back to the contribution of the mode $L=0$ to $\langle T_{\mu\nu}\rangle$ in the Euclidean vacuum. It is easy to see that this contribution is given by

$$\frac{-R^2}{1536\pi^2}\frac{(\xi R+2m^2)}{(m^2+\xi R)}+O(m^2,\xi),$$

and therefore it will vanish only if the limit is taken according to the path (51). This is equivalent to taking the limit $m^2,\xi\to 0$ in the formal expression of the energy-momentum tensor operator before taking the vacuum expectation value.

VI. CONCLUSIONS AND DISCUSSION

We have used the Hadamard formalism to compute the renormalized expectation value of the energy-momentum tensor for a massless minimally coupled field in de Sitter space in the two-parameter family of O(4) Hadamard vacua. We find that this tensor is not de Sitter invariant but only O(4) invariant (in disagreement with the result of Ref. [7], which was subsequently corrected in Ref. [17]).

We have also studied the de Sitter–invariant state for the massless minimally coupled field. It is worth noting that such state is not a Fock vacuum (indeed, Allen [6] has shown that for $m=\xi=0$ there is no de Sitter–invariant Fock vacuum): The discrete zero mode is not quantized in terms of creation and annihilation operators, but rather using the canonical position and momentum operators. In particular, we have used it to derive a covariant version of Eq. (2). We find that the expectation value of the square of the difference $\phi(x)-\phi(y)$ grows linearly with the geodesic distance between x and y, for timelike separations which are large compared with H^{-1} [see Eq. (47)]. The linear growth $D(x,y)\propto H\tau$ has the same physical origin as the linear growth in time of Eq. (2), and it can be interpreted, along the same lines, as a "Brownian motion" of the field due to quantum fluctuations (see, e.g., Ref. [23]).

We have computed the renormalized expectation value of the energy-momentum tensor in the de Sitter–invariant vacuum. We find that the renormalized vacuum energy density $\langle T_{00}\rangle_{\rm ren}$ is lower in this state than in any of the O(4)-invariant states. In this sense, only the de Sitter–invariant state deserves to be called vacuum.

The O(4)-invariant $\langle T_{\mu\nu}\rangle_{\rm ren}$ [Eq. (33)] approaches the de Sitter–invariant value (49) at timelike infinity. Also, the dispersion $D(x,y)$ computed in Sec. IV using the de Sitter–invariant state coincides with the limit $\eta\to\pi$ of the dispersion computed in a O(4)-invariant state. Therefore, the de Sitter–invariant state can be seen as the limit into which the O(4)-invariant states evolve at sufficiently late times. This behavior is familiar from the massive case, and it corresponds to the fact that any excitations above the de Sitter–invariant vacuum are redshifted away by the exponential expansion.

After this paper was submitted, we became aware that the result of Ref. [7] had already been corrected in Ref. [17].

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APPENDIX

For completeness, in this appendix we summarize the field quantization in the Schrödinger picture (see, e.g., Ref. [9]).

In the Schrödinger picture, $\hat{\chi}_{LM}$ and $\hat{\pi}_{LM}$ are time-independent operators satisfying the commutation relations

$$[\hat{\chi}_{LM}, \hat{\pi}_{L'M'}] = i\delta_{LL'}\delta_{MM'} \tag{A1}$$

and acting on a Hilbert space of time-dependent physical states Ψ. In the q representation, such states are described by wave functionals $\Psi(\{\chi_{LM}\}, \eta)$ and the action of the operators is given by

$$\hat{\chi}_{LM}\Psi = \chi_{LM}\Psi,$$

$$\hat{\pi}_{LM}\Psi = -i\frac{\partial}{\partial\chi_{LM}}\Psi.$$

The time evolution is governed by the Schrödinger equation

$$\hat{H}\Psi = -i\frac{\partial}{\partial\eta}\Psi, \tag{A2}$$

where $\hat{H}$ is the Hamiltonian derived from the action (8), with χ_{LM} and π_{LM} replaced by its operator counterparts:

$$\hat{H} = \sum_{LM}\frac{1}{2}\left[\frac{\hat{\pi}^2_{LM}}{(H\sin\eta)^{-2}} + (H\sin\eta)^{-2}\omega_L^2\chi^2_{LM}\right]. \tag{A3}$$

Note that throughout this appendix χ_{LM} are not functions of η (as they were in Sec. II), but they are the time-independent position operators of the Schrödinger picture (see, e.g., [24]).

Factorizing the wave functional as

$$\Psi = \prod_{LM}\Psi_{LM}(\chi_{LM}, \eta),$$

Eq. (A2) separates into a set of Schrödinger equations, one for each individual mode:

$$\frac{1}{2}\left[\frac{-1}{(H\sin\eta)^{-2}}\frac{\partial^2}{\partial\chi^2_{LM}} + (H\sin\eta)^{-2}\omega_L^2\chi^2_{LM}\right]\Psi_{LM} = -i\frac{\partial}{\partial\eta}\Psi_{LM}. \tag{A4}$$

These can be solved by using the ansatz

$$\Psi_{LM} = g_{LM}\exp\left[\frac{i}{2}(H\sin\eta)^{-2}\frac{\dot{V}_{LM}}{V_{LM}}\chi^2_{LM}\right], \tag{A5}$$

where $g_{LM}(\eta)$ and $V_{LM}(\eta)$ are unspecified functions. Substituting (A5) into (A4) and collecting the terms proportional to χ^2_{LM} one finds

$$\ddot{V}_{LM} - 2\cot\eta\dot{V}_{LM} + \omega_L^2(\eta)V_{LM} = 0, \tag{A6}$$

and so V_{LM} must be a solution of the field equation (9). Collecting the terms which are independent of χ_{LM}, one finds a differential equation for g_{LM} which can be solved immediately to yield

$$g_{LM} = C_{LM}V_{LM}^{-1/2},$$

where C_{LM} is just an overall normalization constant. Choosing one solution of (A6) for each L and M specifies a particular quantum state. In order to know what set of solutions $\{V_{LM}\}$ corresponds to the de Sitter–invariant quantum state defined in Sec. II, one has to impose that the wave functional be annihilated by the operators a_{LM} associated with the set of modes that defines such vacua [Eq. (15)]:

$$a_{LM}\Psi = \left[U^*_{LM}\frac{\partial}{\partial\chi_{LM}} - i\frac{\dot{U}^*_{LM}}{(H\sin\eta)^2}\chi_{LM}\right]\Psi = 0,$$

where we have inverted (12) to express a_{LM} in terms of $\hat{\chi}_{LM}$ and $\hat{\pi}_{LM}$. Clearly, these conditions are satisfied if and only if

$$V_{LM} = U^*_{LM}.$$

In summary, the de Sitter–invariant wave functional is given by

$$\Psi = \prod_{LM}(2\pi)^{-1/4}U_{LM}^{-1/2}\exp\left[\frac{i}{2}(H\sin\eta)^{-2}\frac{\dot{U}^*_{LM}}{U^*_{LM}}\chi^2_{LM}\right], \tag{A7}$$

with U_{LM} given by (15). It can be checked that this wave functional is annihilated by the operator generators of the de Sitter group [24] and is thus de Sitter invariant. Note also that this wave functional is properly normalized, in the sense that

$$\int_{\infty}^{\infty}\prod_{LM}d\chi_{LM}|\Psi(\{\chi_{LM}\}, \eta)|^2 = 1.$$

Note that the case $m^2 = \xi = 0$ is special. From (15) we find that U_0 becomes constant in the massless minimally coupled limit,

$$U_0 = A_0\left(-\sqrt{\frac{2}{\pi}}\right),$$

and so the Wronskian condition cannot be satisfied and the normalization constant A_0 becomes infinite [see Eq. (17)]. Such infinity can be understood by noting that, since

$$\lim_{m^2,\xi\to 0}\frac{\dot{U}_0}{U_0} = 0,$$

the wave functional becomes independent of χ_0,

$$\pi_0\Psi = -i\frac{\partial}{\partial\chi_0}\Psi = 0 \quad (m^2 = \xi = 0),$$

and therefore Ψ is not normalizable in the discrete sense (which is natural for an eigenstate of momentum).

To conclude, let us study in more detail the $L = 0$ term of the Schrödinger equation (A2). Using the notation $\psi \equiv \Psi_{L=0}$ we have

$$-\frac{1}{2}\frac{\partial}{\partial\chi_0^2}\psi = \frac{-i}{H^2}\frac{\partial}{\partial\tilde{t}}\psi,$$

where we have introduced the new time variable

$$\tilde{t} \equiv \frac{1}{2}\left(\eta - \frac{1}{2}\sin 2\eta - \frac{\pi}{2}\right).$$

In this notation the basic solutions are the eigenstates of momentum that we discussed in Sec. IV:

$$\psi_p(\chi_0) \propto e^{i(pq - p^2\tilde{t})},$$

with $q \equiv \sqrt{2}H^{-1}\chi_0$ [see Eq. (37)]. For $\tilde{t} = 0$ these are the Heisenberg wave functions (40).

The wave packet (41) is just a superposition of these modes, and its time evolution can be found in any elementary textbook. It represents a Gaussian wave packet that spreads in time. Noting that $\langle\alpha|P^2|\alpha\rangle = 2\alpha^2$ and $\langle\alpha|Q^2|\alpha\rangle = (2\alpha^2)^{-1}$ we have, from (37),

$$\langle\chi_0^2\rangle_\alpha = H^2\left[\frac{1}{4\alpha^2} + 4\alpha^2\tilde{t}^{\,2}\right].$$

Since the range of $\tilde{t}$ is finite, $\tilde{t} \in [-\pi/4, \pi/4]$, the expectation value of χ_0^2 does not grow unbounded, but reaches a constant in the asymptotic past and future. Therefore the asymptotic growth in time of $\langle\phi^2\rangle$ in de Sitter space is due to the $L > 0$ modes.

This behavior is somewhat different from that of the theory of a massless field on a compact toroidal flat spacetime which we briefly discussed in Sec. IV. There, the contribution of the $L = 0$ mode to $\langle\phi^2\rangle$ also has a term proportional to $\langle p_0^2\rangle t^2$. However, in that case, t is the Minkowski time. If we choose a state with $\langle p_0^2\rangle \neq 0$, then $\langle\phi^2\rangle \propto t^2$ grows unbounded as time increases due to the $L = 0$ contribution alone. On the other hand, for the ground state $\langle p_0^2\rangle = 0$, but $\langle x_0^2\rangle = \infty$, and therefore $\langle\phi^2\rangle$ is infinite, just like in the de Sitter–invariant state studied in this paper.

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