Teleparallel theory of (2+1)-dimensional gravity

Toshiharu Kawai
Department of Physics, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka 558, Japan
(Received 18 March 1993)

A theory of (2+1)-dimensional gravity is developed on the basis of the Weitzenböck space-time characterized by the metricity condition and by the vanishing curvature tensor. The fundamental gravitational field variables are dreibein fields and the gravity is attributed to the torsion. The most general gravitational Lagrangian density quadratic in the torsion tensor is given by $L_G=\alpha t^{klm}t_{klm}+\beta v^k v_k+\gamma a^{klm}a_{klm}$. Here, t_{klm}, v_k, and a_{klm} are irreducible components of the torsion tensor, and α, β, and γ are real parameters. A condition is imposed on α and β by the requirement that the theory has a correct Newtonian limit. A static circularly symmetric exact solution of the gravitational field equation in the vacuum is given. It gives space-times quite different from each other, according to the signature of $\alpha\beta$. These space-times have event horizons, if and only if $\alpha(3\alpha+4\beta)<0$. Singularity structures of these space-times are also examined.

PACS number(s): 04.50.+h

I. INTRODUCTION

Recently, the Einstein theory of (2+1)-dimensional gravity has attracted considerable attention [1–6]. This theory is strange in various respects, among which are the absence of a Newtonian limit [1,2] and of a black-hole solution [1]. For the (1+1)-dimensional case, there is a theory [7] having a correct Newtonian limit and black-hole solution. Thus, it is natural to raise the question: Is there not a relativistic theory of (2+1)-dimensional gravity having a Newtonian limit and admitting black holes? For the (3+1)-dimensional case, a teleparallel theory of gravity, which can be alternative to the Einstein theory, has been proposed [8].

In view of the above, we give a teleparallel theory of (2+1)-dimensional gravity in the present paper, which has a Newtonian limit and black-hole solutions.

II. DREIBEINS, COVARIANT DERIVATIVE, AND TELEPARALLELISM

The three-dimensional space-time M is assumed to be a differentiable manifold endowed with the Lorentzian metric $g_{\mu\nu}dx^\mu\otimes dx^\nu$ ($\mu,\nu=0,1,2$) related to the fields $\mathbf{e}^k=e^k{}_\mu dx^\mu$ ($k=0,1,2$) through the relation $g_{\mu\nu}=e^k{}_\mu n_{kl}e^l{}_\nu$ with $(n_{kl})\stackrel{\rm def}{=}\mathrm{diag}(-1,1,1)$. Here, $\{x^\mu;\ \mu=0,1,2\}$ is a local coordinate of the space-time. The fields $\mathbf{e}_k=e^\mu{}_k\partial/\partial x^\mu$, which are dual to $\mathbf{e}^k$, are the dreibein fields. The field strength of $e^k{}_\mu$ is given by

$$T^k{}_{\mu\nu}=\partial_\mu e^k{}_\nu-\partial_\nu e^k{}_\mu\ . \tag{2.1}$$

We define the covariant derivative of the Lorentzian vector field V^k by

$$\nabla_l V^k=e^\mu{}_l\partial_\mu V^k\ . \tag{2.2}$$

For the world vector fields $\mathbf{V}=V^\mu\partial/\partial x^\mu$, the covariant derivative with respect to the affine connection $\Gamma^\mu_{\lambda\nu}$ is given by

$$\nabla_\nu V^\mu=\partial_\nu V^\mu+\Gamma^\mu_{\lambda\nu}V^\lambda\ . \tag{2.3}$$

The requirement

$$\nabla_l V^k=e^\nu{}_l e^k{}_\mu\nabla_\nu V^\mu \tag{2.4}$$

for $V^\mu\stackrel{\rm def}{=}e^\mu{}_k V^k$ leads to

$$\Gamma^\mu_{\lambda\nu}=e^\mu{}_k\partial_\nu e^k{}_\lambda\ , \tag{2.5}$$

and hence we have the relations

$$T^k{}_{\mu\nu}\equiv e^k{}_\lambda T^\lambda{}_{\mu\nu}\ , \tag{2.6}$$

$$R^\mu{}_{\nu\lambda\rho}\stackrel{\rm def}{=}\partial_\lambda\Gamma^\mu_{\nu\rho}-\partial_\rho\Gamma^\mu_{\nu\lambda}+\Gamma^\mu_{\tau\lambda}\Gamma^\tau_{\nu\rho}-\Gamma^\mu_{\tau\rho}\Gamma^\tau_{\nu\lambda}\equiv 0\ , \tag{2.7}$$

$$\nabla_\lambda g_{\mu\nu}\stackrel{\rm def}{=}\partial_\lambda g_{\mu\nu}-\Gamma^\rho_{\mu\lambda}g_{\rho\nu}-\Gamma^\rho_{\nu\lambda}g_{\mu\rho}\equiv 0\ , \tag{2.8}$$

where $T^\lambda{}_{\mu\nu}$ is defined by

$$T^\lambda{}_{\mu\nu}\stackrel{\rm def}{=}\Gamma^\lambda_{\nu\mu}-\Gamma^\lambda_{\mu\nu}\ . \tag{2.9}$$

The components $T^\lambda{}_{\mu\nu}$ and $R^\mu{}_{\nu\lambda\rho}$ are those of the torsion tensor and of the curvature tensor, respectively. Equation (2.7) implies the teleparallelism and it, together with (2.8), means that M is the Weitzenböck space-time. Also, from (2.8) we get

$$\Gamma^\lambda{}_{\mu\nu}=\{{}_\mu{}^\lambda{}_\nu\}+K^\lambda{}_{\mu\nu}\ , \tag{2.10}$$

where the first term denotes the Christoffel symbol,

$$\{{}_\mu{}^\lambda{}_\nu\}\stackrel{\rm def}{=}\tfrac{1}{2}g^{\lambda\xi}(\partial_\mu g_{\xi\nu}+\partial_\nu g_{\xi\mu}-\partial_\xi g_{\mu\nu})\ , \tag{2.11}$$

and the second stands for the contorsion tensor,

$$K^\lambda{}_{\mu\nu}\stackrel{\rm def}{=}-\tfrac{1}{2}(T^\lambda{}_{\mu\nu}-T_\mu{}^\lambda{}_\nu-T_\nu{}^\lambda{}_\mu)=-K_\mu{}^\lambda{}_\nu\ . \tag{2.12}$$

In what follows, the field components $e^k{}_\mu$ and $e^\mu{}_k$ are used to convert Latin and Greek indices, similarly to the case of V^μ and V^k. Also, raising and lowering the indices $k,l,m,\ldots$ are accomplished with the aid of $(\eta^{kl})\stackrel{\text{def}}{=}(\eta_{kl})^{-1}$ and (η_{kl}), respectively.

III. LAGRANGIAN DENSITIES AND GRAVITATIONAL FIELD EQUATION

For the matter field φ belonging to a representation of the three-dimensional Lorentz group, $L_M(\varphi,\nabla_k\varphi)$ with $\nabla_k\varphi\stackrel{\text{def}}{=}e^\mu{}_k\partial_\mu\varphi$ is a Lagrangian [9] invariant under global Lorentz transformation and under general coordinate transformation, if $L_M(\varphi,\partial_k\varphi)$ is an invariant Lagrangian on the three-dimensional Minkowski space-time.

For the dreibein fields $\mathbf{e}_k$, the most general Lagrangian, which is invariant under the transformations stated above and is quadratic in the torsion tensor, is given by

$$L_G=\alpha t^{klm}t_{klm}+\beta v^k v_k+\gamma a^{klm}a_{klm} \ . \tag{3.1}$$

Here, t_{klm}, v_k, and a_{klm} are the irreducible components of T_{klm}, which are defined by

$$t_{klm}\stackrel{\text{def}}{=}\tfrac{1}{2}(T_{klm}+T_{lkm})+\tfrac{1}{4}(\eta_{mk}v_l+\eta_{ml}v_k)-\tfrac{1}{2}\eta_{kl}v_m \ , \tag{3.2}$$

$$v_k\stackrel{\text{def}}{=}T^l{}_{lk} \ , \tag{3.3}$$

and

$$a_{klm}\stackrel{\text{def}}{=}\tfrac{1}{3}(T_{klm}+T_{mkl}+T_{lmk}) \ , \tag{3.4}$$

respectively, and α, β, and γ are real constant parameters. Then,

$$\mathbb{I}\stackrel{\text{def}}{=}\frac{1}{c}\int \mathbf{L}\, d^3x \tag{3.5}$$

is the total action of the system, where c is the light velocity in the vacuum and $\mathbf{L}$ is defined by

$$\mathbf{L}\stackrel{\text{def}}{=}\sqrt{-g}\,[L_G+L_M(\varphi,\nabla_k\varphi)] \tag{3.6}$$

with $g\stackrel{\text{def}}{=}\det(g_{\mu\nu})$.

The gravitational field equation following from the action $\mathbb{I}$ is

$$-2\nabla^k F_{ijk}+2v^k F_{ijk}+2H_{ij}-\eta_{ij}L_G=T_{ij} \tag{3.7}$$

with

$$F_{ijk}\stackrel{\text{def}}{=}\alpha(t_{ijk}-t_{ikj})+\beta(\eta_{ij}v_k-\eta_{ik}v_j)+2\gamma a_{ijk}=-F_{ikj} \ , \tag{3.8}$$

$$H_{ij}\stackrel{\text{def}}{=}T_{mni}F^{mn}{}_j-\tfrac{1}{2}T_{jmn}F_i{}^{mn}=H_{ji} \ , \tag{3.9}$$

$$\nabla^k F_{ijk}\stackrel{\text{def}}{=}e^{\mu k}\partial_\mu F_{ijk} \ . \tag{3.10}$$

Also, T_{ij} is the energy-momentum density defined by

$$\sqrt{-g}\,T_{ij}\stackrel{\text{def}}{=}e_{j\mu}\frac{\delta(\sqrt{-g}\,L_M)}{\delta e^i{}_\mu} \stackrel{\text{def}}{=}e_{j\mu}\left[\frac{\partial(\sqrt{-g}\,L_M)}{\partial e^i{}_\mu}-\partial_\nu\left[\frac{\partial(\sqrt{-g}\,L_M)}{\partial(\partial_\nu e^i{}_\mu)}\right]\right] . \tag{3.11}$$

Equation (3.7) can be rewritten as

$$\frac{1}{\kappa}G_{ij}(\{\ \})-2\nabla^k\hat{F}_{ijk}+2v^k\hat{F}_{ijk}+2\hat{H}_{ij}-\eta_{ij}\hat{L}_G=T_{ij} \ , \tag{3.7$'$}$$

where

$$\hat{F}_{ijk}\stackrel{\text{def}}{=}\left[\alpha+\frac{1}{3\kappa}\right](t_{ijk}-t_{ikj})+\left[\beta-\frac{1}{4\kappa}\right](\eta_{ij}v_k-\eta_{ik}v_j) +2\left[\gamma-\frac{1}{8\kappa}\right]a_{ijk}=-\hat{F}_{ikj} \ , \tag{3.12}$$

$$\hat{H}_{ij}=T_{mni}\hat{F}^{mn}{}_j-\tfrac{1}{2}T_{jmn}\hat{F}_i{}^{mn}=\hat{H}_{ji} \ , \tag{3.13}$$

$$\hat{L}_G\stackrel{\text{def}}{=}\left[\alpha+\frac{1}{3\kappa}\right]t^{klm}t_{klm}+\left[\beta-\frac{1}{4\kappa}\right]v^k v_k +\left[\gamma-\frac{1}{8\kappa}\right]a^{klm}a_{klm} \ , \tag{3.14}$$

and $G_{ij}(\{\ \})$ is the three-dimensional Einstein tensor defined by

$$G_{ij}(\{\ \})\stackrel{\text{def}}{=}e^\mu{}_i e^\nu{}_j G_{\mu\nu}(\{\ \}) \stackrel{\text{def}}{=}e^\mu{}_i e^\nu{}_j[R_{\mu\nu}(\{\ \})-\tfrac{1}{2}g_{\mu\nu}R(\{\ \})] \tag{3.15}$$

with

$$R_{\mu\nu}(\{\ \})\stackrel{\text{def}}{=}\partial_\rho\{{}_\mu{}^\rho{}_\nu\}-\partial_\nu\{{}_\mu{}^\rho{}_\rho\}+\{{}_\tau{}^\rho{}_\rho\}\{{}_\mu{}^\tau{}_\nu\} -\{{}_\tau{}^\rho{}_\nu\}\{{}_\mu{}^\tau{}_\rho\} , \tag{3.16}$$

$$R(\{\ \})\stackrel{\text{def}}{=}g^{\mu\nu}R_{\mu\nu}(\{\ \}) \ . \tag{3.17}$$

Here, κ denotes the "Einstein gravitational constant" $\kappa=8\pi G/c^4$ with G being the "Newton gravitational constant."

For the case such that the conditions

$$\alpha=-\frac{1}{3\kappa},\quad \beta=\frac{1}{4\kappa},\quad \gamma=\frac{1}{8\kappa} \ , \tag{3.18}$$

and

$$T_{[ij]}\stackrel{\text{def}}{=}\frac{1}{2}(T_{ij}-T_{ji})=0 \ , \tag{3.19}$$

are both satisfied, (3.7′) reduces to the three-dimensional Einstein equation. Even for this case, however, our theory does not reduce to the three-dimensional Einstein theory as a whole, because our connection and hence the covariant derivative are different from those of the Einstein theory.

The Lagrangian $L_D(\bar\psi,\psi,\nabla_k\bar\psi,\nabla_k\psi)$ of the three-dimensional two-component Dirac field ψ, which is an example of the matter field Lagrangian $L_M(\varphi,\nabla_k\varphi)$, is given by [10]

$$L_D(\bar\psi,\psi,\nabla_k\bar\psi,\nabla_k\psi)=\frac{i}{2}(\bar\psi\gamma^k\nabla_k\psi-\nabla_k\bar\psi\gamma^k\psi)-m\bar\psi\psi\ . \tag{3.20}$$

This can be rewritten as

$$L_D(\bar\psi,\psi,\nabla_k\bar\psi,\nabla_k\psi)=L_D(\bar\psi,\psi,\tilde\nabla_k\bar\psi,\tilde\nabla_k\psi)+\frac{i}{8}\varepsilon_{klm}a^{klm}\bar\psi\gamma^0\gamma^1\gamma^2\psi\ , \tag{3.20'}$$

where ε_{klm} stands for the totally antisymmetric Lorentzian tensor with $\varepsilon_{012}=-1$ and

$$L_D(\bar\psi,\psi,\tilde\nabla_k\bar\psi,\tilde\nabla_k\psi)\overset{\text{def}}{=}\frac{i}{2}(\bar\psi\gamma^k\tilde\nabla_k\psi-\tilde\nabla_k\bar\psi\gamma^k\psi)-m\bar\psi\psi \tag{3.21}$$

with

$$\begin{aligned}\tilde\nabla_k\psi&\overset{\text{def}}{=}e^\mu{}_k\left[\partial_\mu\psi+\frac{i}{2}\Delta^{lm}{}_\mu S_{lm}\psi\right]\ ,\\ \tilde\nabla_k\bar\psi&\overset{\text{def}}{=}e^\mu{}_k\left[\partial_\mu\bar\psi-\frac{i}{2}\Delta^{lm}{}_\mu\bar\psi S_{lm}\right]\ ,\end{aligned} \tag{3.22}$$

$$\Delta^{lm}{}_\mu\overset{\text{def}}{=}\tfrac{1}{2}e^n{}_\mu(T^{lm}{}_n-T^{ml}{}_n-T_n{}^{lm})\ , \tag{3.23}$$

$$S_{lm}\overset{\text{def}}{=}\frac{i}{4}[\gamma_l,\gamma_m]\ . \tag{3.24}$$

The Lagrangian $L_D(\bar\psi,\psi,\tilde\nabla_k\bar\psi,\tilde\nabla_k\psi)$ is the Dirac Lagrangian in the three-dimensional Einstein theory. Equation (3.20′) gives the relation between the Dirac Lagrangians in the two theories.

IV. WORLD LINE OF CLASSICAL PARTICLE, PATH OF LIGHT RAY, AND MACROSCOPIC EQUIVALENCE PRINCIPLE

From the fact that the gravitational action integral is invariant under general coordinate transformations, it follows that

$$\tilde\nabla_\nu B^{\mu\nu}-K^{\nu\lambda\mu}B_{\nu\lambda}\equiv0\ , \tag{4.1}$$

where we have defined

$$\sqrt{-g}\,B^{\mu\nu}\overset{\text{def}}{=}\sqrt{-g}\,e^{\mu k}B_k{}^\nu\overset{\text{def}}{=}-e^{\mu k}\frac{\delta(\sqrt{-g}\,L_G)}{\delta e^k{}_\nu}\ , \tag{4.2}$$

and $\tilde\nabla_\nu$ stands for the covariant derivative with respect to the Levi-Civita connection:

$$\tilde\nabla_\nu B^{\mu\nu}\overset{\text{def}}{=}\partial_\nu B^{\mu\nu}+\{{}_\lambda{}^\mu{}_\nu\}B^{\lambda\nu}+\{{}_\lambda{}^\nu{}_\nu\}B^{\mu\lambda}\ . \tag{4.3}$$

The gravitational field equation takes the form

$$B^{\mu\nu}=T^{\mu\nu}\ , \tag{4.4}$$

and hence we get

$$\tilde\nabla_\nu T^{\mu\nu}-K^{\nu\lambda\mu}T_{\nu\lambda}=0\ , \tag{4.5}$$

from (4.1).

From the Lorentz invariance of the action integral of the matter field φ which belongs to a representation σ of the three-dimensional Lorentz group, we can show

$$\sqrt{-g}\,T_{[kl]}-\partial_\mu(\sqrt{-g}\,S_{kl}{}^\mu)+\frac{i}{2}\frac{\delta\mathbb{L}}{\delta\varphi}M_{kl}\varphi\equiv0\ , \tag{4.6}$$

where $M_{kl}\overset{\text{def}}{=}-i\sigma_*(\bar M_{kl})$. Here, $\{\bar M_{kl},\ k,l=0,1,2\}$ is a basis of the Lie algebra of the three-dimensional Lorentz group satisfying the relation

$$[\bar M_{kl},\bar M_{mn}]=-\eta_{km}\bar M_{ln}-\eta_{ln}\bar M_{km}+\eta_{kn}\bar M_{lm}+\eta_{lm}\bar M_{kn}\ , \tag{4.7}$$

$$\bar M_{kl}=-\bar M_{lk}\ , \tag{4.8}$$

σ_* is the differential of σ, and $S_{kl}{}^\mu$ is the "spin" [11] density of the field φ defined by

$$\sqrt{-g}\,S_{kl}{}^\mu\overset{\text{def}}{=}\frac{i}{2}\frac{\partial(\sqrt{-g}L_M)}{\partial(\partial_\mu\varphi)}M_{kl}\varphi\ . \tag{4.9}$$

Thus, we have

$$\sqrt{-g}\,T_{[kl]}=\partial_\mu(\sqrt{-g}\,S_{kl}{}^\mu)\ , \tag{4.10}$$

when the field equation of the field φ is satisfied. The antisymmetric part $T_{[kl]}$ is due to the contribution of the intrinsic "spin" of φ. For macroscopic bodies such that the effects due to the intrinsic "spin" of the fundamental constituent particles can be ignored, the energy-momentum tensor can be regarded as symmetric and (4.5) reduces to

$$\tilde\nabla_\nu T^{\mu\nu}=0\ . \tag{4.11}$$

This is known by noting that $K^{\nu\lambda\mu}=-K^{\lambda\nu\mu}$. From (4.11), we can derive the equation of motion of a classical particle,

$$\frac{d^2x^\mu}{d\tau^2}+\{{}_\nu{}^\mu{}_\rho\}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}=0\ , \tag{4.12}$$

in a way quite similar to the case in the four-dimensional Einstein theory. Here, τ is the proper time of the particle, and (4.12) is the equation of the geodesic line of the metric $g_{\mu\nu}dx^\mu\otimes dx^\nu$.

We require, as in four-dimensional new general relativity, the U(1) gauge invariance of the electromagnetic interaction. Then, the electromagnetic Lagrangian L_{em} is given by

$$L_{\text{em}}\overset{\text{def}}{=}-\frac{1}{4}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{4.13}$$

with

$$F_{\mu\nu}\overset{\text{def}}{=}\partial_\mu A_\nu-\partial_\nu A_\mu\ , \tag{4.14}$$

and we can show that light rays propagate along the null geodesic line:

$$\frac{d^2x^\mu}{d\lambda^2}+\{{}^{\mu}_{\nu\rho}\}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda}=0\ ,$$
$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}=0\ , \tag{4.15}$$

where λ is a real parameter.

From (4.12) and (4.15) we see that equivalence principle holds for macroscopic body and light ray.

V. STATIC CIRCULARLY SYMMETRIC GRAVITATIONAL FIELD

We consider a static, circularly symmetric gravitational field produced by a static circularly symmetric body. As in the case of four-dimensional theory [8], we can assume without loss of generality that $(e^k{}_\mu)$ has a diagonal form:

$$(e^k{}_\mu)=\begin{bmatrix}A(r)&0&0\\0&B(r)&0\\0&0&B(r)\end{bmatrix} \tag{5.1}$$

with $r\overset{\text{def}}{=}\sqrt{(x^1)^2+(x^2)^2}$, which leads to $a_{klm}\equiv 0$.

In terms of $A(r)$ and $B(r)$, (3.7) can be expressed as

$$B^{-1}\left\{3\alpha\left[r\frac{d}{dr}\left[\frac{B^{-1}}{r}\frac{d}{dr}\ln\frac{A}{B}\right]+2B^{-1}\frac{1}{r}\frac{d}{dr}\ln\frac{A}{B}\right]+4\beta\left[r\frac{d}{dr}\left[\frac{B^{-1}}{r}\frac{d}{dr}\ln AB\right]+2B^{-1}\frac{1}{r}\frac{d}{dr}\ln AB\right]\right\}$$
$$+\frac{3}{2}\alpha B^{-2}\frac{d}{dr}\left[\ln\frac{A}{B}\right]\frac{d}{dr}(\ln AB^3)+2\beta B^{-2}\left[\frac{d}{dr}(\ln AB)\right]^2=-2T_{(0)(0)}\ , \tag{5.2}$$

$$\frac{x^ax^b}{r}B^{-1}\left[3\alpha\frac{d}{dr}\left[\frac{B^{-1}}{r}\frac{d}{dr}\ln\frac{A}{B}\right]-4\beta\frac{d}{dr}\left[\frac{B^{-1}}{r}\frac{d}{dr}\ln AB\right]+\frac{6\alpha B^{-1}}{r}\left[\frac{d}{dr}\ln A\right]\frac{d}{dr}\left[\ln\frac{A}{B}\right]\right]$$
$$-\frac{1}{2}\delta^{ab}\left\{3\alpha B^{-2}\left[\frac{d}{dr}\ln\frac{A}{B}\right]\left[\frac{d}{dr}\left[\ln\frac{A}{B}\right]-\frac{2}{r}\right]+4\beta B^{-2}\left[\frac{d}{dr}\ln AB\right]\left[\frac{d}{dr}\ln AB+\frac{2}{r}\right]\right\}$$
$$=\begin{cases}2T_{(1)(2)}=2T_{(2)(1)}, & a=1,b=2,\ \text{ or } a=2,b=1\ ,\\ -2T_{(2)(2)}, & a=1=b,\\ -2T_{(1)(1)}to, & a=2=b,\end{cases} \tag{5.3}$$

$$T_{(1)(0)}=0=T_{(0)(1)},\quad T_{(2)(0)}=0=T_{(0)(2)}\ , \tag{5.4}$$

where, to avoid confusion, Latin indices in T_{ij} are enclosed in parentheses.

A. Newtonian limit

First, we consider the case for which the conditions

$$T_{(0)(0)}\simeq\rho c^2>>|T_{(a)(b)}|\simeq 0,\quad a,b=1,2\ , \tag{5.5}$$

$$T_{(1)(0)}=0=T_{(0)(1)},\quad T_{(2)(0)}=0=T_{(0)(2)}\ , \tag{5.6}$$

$$A\simeq 1\simeq B,\quad A'\simeq 0\simeq B'\ , \tag{5.7}$$

are satisfied, where ρ is mass density of the gravitating body and a prime means differentiation with respect to r. For this case, (5.2) and (5.3) take the form

$$(3\alpha+4\beta)\Delta A-(3\alpha-4\beta)\Delta B\simeq -2\rho c^2\ , \tag{5.8a}$$

$$(3\alpha-4\beta)A'-(3\alpha+4\beta)B'\simeq 0\ , \tag{5.8b}$$

where terms quadratic in small quantities have been neglected. Here, Δ stands for the Laplacian of the two-dimensional Euclidean space. From (5.8a) and (5.8b), we can show that the potential U defined by

$$g_{00}=-A^2=-1-\frac{2U}{c^2} \tag{5.9}$$

satisfies the equation

$$\Delta U\simeq 4\pi G\rho\ , \tag{5.10}$$

if and only if

$$3\alpha+4\beta=-\frac{96\alpha\beta\pi G}{c^4},\quad \alpha\beta\neq 0\ . \tag{5.11}$$

Also, for the particle moving slowly, $|dx^\alpha/dt|<<c$ $(\alpha=1,2)$, (4.12) reduces to

$$\frac{d^2x^\alpha}{dt^2}\simeq-\frac{\partial U}{\partial x^\alpha}\ . \tag{5.12}$$

Thus, our theory has a Newtonian limit, if (5.11), which we shall assume in the text from now on, is satisfied.

B. Exact vacuum solution

The exact solution of (5.2)–(5.4) with $T_{ij}=0$ is given by

$$A(r)=\frac{\Lambda^2-1}{16\Lambda^2}(K_1 \ln r+K_3)(K_2 \ln r+K_4)\,, \tag{5.13a}$$

$$B(r)=\left[\frac{\Lambda-1}{4\Lambda}\right]^{(1+\Lambda)/(1-\Lambda)}\left[\frac{\Lambda+1}{4\Lambda}\right]^{(1-\Lambda)/(1+\Lambda)} \times(K_1 \ln r+K_3)^{(1+\Lambda)/(1-\Lambda)} \times(K_2 \ln r+K_4)^{(1-\Lambda)/(1+\Lambda)}\,, \tag{5.13b}$$

where $\Lambda \overset{\text{def}}{=} \sqrt{-4\beta/3\alpha}\neq 1$, and K_i ($i=1,2,3,4$) are complex constants satisfying

$$K_1K_2=\frac{\Lambda^2-1}{4\Lambda}(K_2K_3-K_1K_4)\,. \tag{5.14}$$

The derivation of this solution is given in Appendix B 1. [In Appendix B, solutions are also given for α and β violating (5.11).]

We require here that $A(r)$ and $B(r)$ are normalized as $A(r_0)=1=B(r_0)$ for some radius $r=r_0$; then K_3 and K_4 are expressed in terms of K_1 and K_2, and $A(r)$ and $B(r)$ take the form

$$A(r)=X(r)Y(r)\,, \quad B(r)=[X(r)]^{(1+\Lambda)/(1-\Lambda)}[Y(r)]^{(1-\Lambda)/(1+\Lambda)}\,, \tag{5.15}$$

where

$$X(r)\overset{\text{def}}{=}\left[1+\frac{\Lambda-1}{4\Lambda}K_1 \ln\frac{r}{r_0}\right], \quad Y(r)\overset{\text{def}}{=}\left[1+\frac{\Lambda+1}{4\Lambda}K_2 \ln\frac{r}{r_0}\right]. \tag{5.16}$$

The parameters K_1 and K_2 are now required to satisfy the relation

$$K_1K_2-(1-\Lambda)K_1-(1+\Lambda)K_2=0\,. \tag{5.17}$$

The equation of motion of a classical test particle moving slowly in the neighborhood of $r=r_0$ [12] agrees with that in the Newton theory, if and only if

$$\frac{c^2}{4\Lambda}[(\Lambda-1)K_1+(\Lambda+1)K_2]=2GM\,, \tag{5.18}$$

which we shall assume from now on. Here, M is the mass of the central gravitating body.

It is quite natural to make the requirement that the functions A and B are real values at least in a neighborhood of $r=r_0$. Then, certain restrictions are imposed on the parameters K_1 and K_2, which are classified into two cases by the signature of $\alpha\beta$.

1. The case with $\alpha\beta<0$

For this case, Λ is real and positive, and A and B are both real valued around $r=r_0$, if and only if K_1 and K_2 are both real. Then, the function A is real valued for an arbitrary positive r, but B can be complex valued in general for r which is far from the neighborhood of $r=r_0$ [13].

2. The case with $\alpha\beta>0$

For this case, Λ is pure imaginary, and the above requirements imposes the condition $K_1=K_2^*$, where the asterisk on the shoulder stands for the complex conjugation. Then, $X(r)=[Y(r)]^*$ and the solution (5.15) takes the form

$$A(r)=|X(r)|^2\,, \quad B(r)=[X(r)]^{(1+\Lambda)/(1-\Lambda)}\{[X(r)]^*\}^{(1-\Lambda)/(1+\Lambda)}\,. \tag{5.19}$$

These A and B are both real valued for *any* positive value of r, although we have not required it. The conditions (5.17) and (5.18) take the form

$$|K_1|^2-(1-\Lambda)K_1-(1+\Lambda)K_1^*=0\,, \tag{5.20}$$

$$\frac{c^2}{4\Lambda}[(\Lambda-1)K_1+(\Lambda+1)K_1^*]=2GM\,. \tag{5.21}$$

These two equations can be satisfied simultaneously, only if

$$1-\Lambda^2\geq|4\Lambda GM|/c^2\,, \tag{5.22}$$

which can be expressed, by the use of (5.11), as

$$48\pi^2\alpha\beta\geq c^4M^2\,. \tag{5.22$'$}$$

Thus, we see the following: *For a given positive value of $\alpha\beta$, the gravitational field produced by a body having (heavy) mass M violating* (5.22′) *does not have a naive correspondence to the Newton theory.*

VI. EVENT HORIZONS AND SINGULARITIES OF THE SPACE-TIMES GIVEN BY THE SOLUTION (5.15)

We shall examine the structures of the space-times given by the solution discussed in Sec. V.

In the present paper, the point at which $t^{klm}t_{klm}$ and/or v^kv_k do not have derivatives is called a singularity. Also, by an effective singularity, we mean the point at which the Riemann-Christoffel scalar curvature $R(\{\ \})$ is not differentiable, in view of the following fact: The Riemann-Christoffel curvature tensor is not the curvature tensor of the Weitzenböck space-time, but in our theory, classical test particles and light rays "feel" this curvature *effectively*, as is known from (4.12) and (4.15).

Let $x^0=ct$, x^α ($\alpha=1,2$) be the coordinate of a classical particle, and express x^α as

$$x^1=r\cos\theta,\quad x^2=r\sin\theta\,. \tag{6.1}$$

The equation of motion of the particle is derived from the action

$$I_p=-\int g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu d\tau = \int[A^2(\dot{x}^0)^2-B^2\dot{r}^2-B^2r^2\dot{\theta}^2]d\tau\,, \tag{6.2}$$

where τ is the proper time and $\dot{x}^0\overset{\text{def}}{=}dx^0/d\tau$, $\dot{r}\overset{\text{def}}{=}dr/d\tau$, and $\dot{\theta}\overset{\text{def}}{=}d\theta/d\tau$. From the variations of I_p with respect to x^0 and θ, we get

$$\frac{d}{d\tau}(A^2\dot{x}^0)=0 , \tag{6.3}$$

$$\frac{d}{d\tau}(B^2r^2\dot{\theta})=0 . \tag{6.4}$$

We also have [14]

$$c^2=-g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=A^2(\dot{x}^0)^2-B^2\dot{r}^2-B^2r^2\dot{\theta}^2 . \tag{6.5}$$

From (6.3) and (6.4), it follows that

$$A^2\dot{t}=k=\text{const} , \tag{6.6}$$

$$B^2r^2\dot{\theta}=h=\text{const} . \tag{6.7}$$

For the orbit with $\theta=\theta_0$=const, we have

$$\left[\frac{dr}{d\tau}\right]^2=\left[\frac{c}{AB}\right]^2(k^2-A^2) , \tag{6.8}$$

$$\left[\frac{dr}{dt}\right]^2=\left[\frac{cA}{kB}\right]^2(k^2-A^2) , \tag{6.9}$$

which follow from (6.5) and (6.6). Thus, the motion of the particle is restricted in the region $k^2\geq A^2$ [15]. We consider the time which the particle needs to go from the point (r_1,θ_0) to the point (r_2,θ_0) on this orbit [16]. It is given by

$$t_p=\frac{|k|}{c}\int_{r_1}^{r_2}\frac{|B|dr}{|A|\sqrt{k^2-A^2}} , \tag{6.10}$$

when it is measured by the coordinate time $t\stackrel{\text{def}}{=}x^0/c$. When it is measured by its own proper time, the particle needs the time

$$\tau_p=\frac{1}{c}\int_{r_1}^{r_2}\frac{|AB|dr}{\sqrt{k^2-A^2}} . \tag{6.11}$$

Let us describe the path of light ray with $x^\mu(\lambda)$ and use polar coordinate (r,θ) for $x^\alpha(\lambda)$ $(\alpha=1,2)$; we then have the equations

$$\frac{d}{d\lambda}(A^2\dot{x}^0)=0 , \tag{6.12}$$

$$\frac{d}{d\lambda}(B^2r^2\dot{\theta})=0 , \tag{6.13}$$

$$0=A^2(\dot{x}^0)^2-B^2\dot{r}^2-B^2r^2\dot{\theta}^2 , \tag{6.14}$$

where $\dot{x}^0\stackrel{\text{def}}{=}dx^0/d\lambda$, $\dot{\theta}\stackrel{\text{def}}{=}d\theta/d\lambda$, and $\dot{r}\stackrel{\text{def}}{=}dr/d\lambda$. This is shown in a way similar to the case of (6.3)–(6.5).

For the light ray with $\theta=\theta_0$=const, we have

$$\left[\frac{dr}{dt}\right]^2=c^2\left[\frac{A}{B}\right]^2 , \tag{6.15}$$

which follows from (6.14). To go from the point (r_1,θ_0) to the point (r_2,θ_0) on the path with $\theta=\theta_0$, a light ray needs the time t_l given by

$$t_l=\frac{1}{c}\int_{r_1}^{r_2}\left|\frac{B}{A}\right|dr , \tag{6.16}$$

when it is measured by the coordinate time.

The scalars $t^{klm}t_{klm}$, v^kv_k, and $R(\{\ \})$ have the expressions for the space-time given by (5.15):

$$t^{klm}t_{klm}=\frac{3}{16}\left[\frac{K_1}{X}+\frac{K_2}{Y}\right]^2X^{2(\Lambda+1)/(\Lambda-1)}\times Y^{2(\Lambda-1)/(\Lambda+1)}\left[\frac{1}{r^2}-\pi\delta(\mathbf{r})\right] , \tag{6.17}$$

$$v^kv_k=\frac{1}{4\Lambda^2}\left[\frac{K_1}{X}-\frac{K_2}{Y}\right]^2X^{2(\Lambda+1)/(\Lambda-1)}\times Y^{2(\Lambda-1)/(\Lambda+1)}\left[\frac{1}{r^2}-\pi\delta(\mathbf{r})\right] , \tag{6.18}$$

$$R(\{\ \})=\frac{2\pi}{\Lambda}(K_1Y-K_2X)X^{(\Lambda+3)/(\Lambda-1)}Y^{(\Lambda-3)/(\Lambda+1)}\delta(\mathbf{r})+\frac{1-\Lambda^2}{8\Lambda^2}(K_1Y+K_2X)^2X^{4/(\Lambda-1)}Y^{-4/(\Lambda+1)}\times\left[\frac{1}{r^2}-\pi\delta(\mathbf{r})\right] , \tag{6.19}$$

where $\mathbf{r}\stackrel{\text{def}}{=}(x^1,x^2)$. These are obtainable by using (2.1), (2.11), (3.2), (3.3), (3.16), (3.17), (5.1), (5.15), and (5.16).

A. The space-time given by the solution (5.15) with negative $\alpha\beta$

Since K_1K_2 is not vanishing, each of the functions $X(r)$ and $Y(r)$ has a zero for positive r; we denote them by a_1 and a_2, $X(a_1)=0$, and $Y(a_2)=0$. We have $a_1\neq a_2$, as is easily known.

By substituting the expression (5.15) into (6.10), (6.11), and (6.16), we know that (1) t_p is infinite, if $\Lambda>1$ and at least one of a_1 and a_2 is in the interval $[r_1,r_2]$, but otherwise it is finite, (2) τ_p is infinite, if $3\geq\Lambda>1$ and $a_1\in[r_1,r_2]$, but otherwise it is finite, and (3) t_l is infinite, if (a) $r_2=\infty$ or (b) $\Lambda>1$ and at least one of a_1 and a_2 is in the interval $[r_1,r_2]$, but otherwise it is finite.

The following is seen from (1), (2), and (3).

(a) For the case with $\Lambda>3$, the circles $r=a_1$ and $r=a_2$ are both event horizons.

(b) For the case with $3\geq\Lambda>1$, the circle $r=a_2$ is an event horizon, and to reach (or to come from) the circle $r=a_1$, *classical particle needs infinitely long time, even when it is measured by its own proper time.* Thus, if $3\geq\Lambda>1$, $r=a_1$ is an "ultra" event horizon in this sense.

(c) When $\Lambda<1$, there is no event horizon at all. Also, we know the following from (6.17), (6.18), and (6.19).

(d) There are singularities and also effective singularities both at the origin $r=0$ and at $r=a_2$.

(e) For the case with $\Lambda>5$, the circle $r=a_1$ is a singularity.

(f) For the case with $\Lambda<1$, the circle $r=a_1$ is a singularity as well as an effective singularity.

B. The space-time given by the solution (5.15) with positive $\alpha\beta$

For this case, the function $X(r)=[Y(r)]^*$ does not have a zero point for real r, and t_p and τ_p are both finite for any points (r_1,θ_0) and (r_2,θ_0) on the orbit $\theta=\theta_0$ in the region $k^2\geq A^2$. The time t_l is infinite, if $r_2=\infty$, but otherwise it is finite. These can be known by substituting (5.19) into (6.10), (6.11), and (6.16).

The scalars $t^{klm}t_{klm}$, $v^k v_k$, and $R(\{\ \})$ for this case are singular only at $r=0$, as is seen from (6.17), (6.18), and (6.19) by noting that $X(r)$ does not have a zero point. There is no event horizon in this space-time, but it has a singularity and an effective singularity at $r=0$.

VII. SUMMARY AND COMMENTS

We have formulated a teleparallel theory of (2+1)-dimensional gravity and the results can be summarized as follows.

(1) The theory is underlain with the Weitzenböck space-time, and the gravity is attributed to the torsion. The most general gravitational Lagrangian quadratic in the torsion tensor is given by (3.1), and it has real parameters α, β, and γ.

(2) The gravitational field equation (3.7′) [or equivalently (3.7)] agrees with the Einstein equation, if and only if the conditions (3.18) and (3.19) are satisfied. Even in that case, our theory does not reduce to the Einstein theory as a whole.

(3) The requirement that a circularly symmetric gravitational field should have a Newtonian limit imposes the condition (5.11) on the parameters α and β. Equation (3.18) violates this condition, which is expected from the fact that the three-dimensional Einstein theory does not have a Newtonian limit.

(4) The static circularly symmetric exact solutions of the gravitational field equation in vacuum have been obtained.

(5) (a) For the case with negative $\alpha\beta$ satisfying (5.11), the space-time given by the solution (5.15) has various different structures classified by the value of $\Lambda=\sqrt{-4\beta/3\alpha}$, as is summarized in (a)–(f) of Sec. VI. (b) For a positive $\alpha\beta$ satisfying (5.11), however, the solution (5.15) gives a space-time having a rather simple structure; i.e., there is no event horizon and it has a singularity and an effective singularity at $r=0$. Remarkable in this case is the fact that a body having a mass M violating (5.22′) produces gravitational field which does not have a naive correspondence to the Newton gravity. The solution (5.15) gives black-hole space-times, if and only if $\alpha(3\alpha+4\beta)<0$. The circularly symmetric space-times given by (5.15) are markedly different from the Schwarzschild space-time in the Einstein theory of (3+1)-dimensional gravity.

The theory developed above seems to present us with a toy model useful to examine basic concepts in theories of gravity.

APPENDIX A: THE DIRAC FIELD ON THE THREE-DIMENSIONAL MINKOWSKI SPACE-TIME

We consider the two-component Dirac field ψ on the (2+1)-dimensional space-time. The adjoint $\bar{\psi}$ of ψ and the γ matrices γ^k $(k=0,1,2)$ are defined by

$$\bar{\psi}\overset{\text{def}}{=}\psi^\dagger\gamma^0 \tag{A1}$$

and

$$\gamma^0\overset{\text{def}}{=}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad \gamma^1\overset{\text{def}}{=}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},\quad \gamma^2\overset{\text{def}}{=}\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}, \tag{A2}$$

respectively. Here, $\psi^\dagger$ stands for the Hermitian conjugate of ψ. We have the relation $\gamma^k\gamma^l+\gamma^l\gamma^k=-2\eta^{kl}$.

The free Lagrangian of the field ψ on the three-dimensional Minkowski space-time is given by

$$L_D=\frac{i}{2}(\bar{\psi}\gamma^k\partial_k\psi-\partial_k\bar{\psi}\gamma^k\psi)-m\bar{\psi}\psi\ , \tag{A3}$$

where m is the mass of ψ.

APPENDIX B: EXACT SOLUTIONS OF (5.2)–(5.4) WITH $T_{ij}=0$

We now derive exact solutions of (5.2)–(5.4) with $T_{ij}=0$. In this Appendix, the parameters α and β are not necessarily required to satisfy (5.11).

Equations (5.2)–(5.4) with $T_{ij}=0$ are equivalent to

$$(3\alpha+4\beta)\left[P'+\frac{1}{r}P+\frac{1}{2}P^2\right]-(3\alpha-4\beta)\left[Q'+\frac{1}{r}Q\right]-\frac{1}{2}(3\alpha+4\beta)Q^2=0\ , \tag{B1}$$

$$(3\alpha-4\beta)\left[P'-\frac{1}{r}P\right]-(3\alpha+4\beta)\left[Q'-\frac{1}{r}Q-Q^2\right]+6\alpha P^2-(9\alpha-4\beta)PQ=0\ , \tag{B2}$$

$$(3\alpha+4\beta)(P^2+Q^2)-2(3\alpha-4\beta)PQ-\frac{2}{r}(3\alpha-4\beta)P+\frac{2}{r}(3\alpha+4\beta)Q=0 \tag{B3}$$

with

$$P\overset{\text{def}}{=}\frac{A'}{A},\quad Q\overset{\text{def}}{=}\frac{B'}{B}\ . \tag{B4}$$

1. The case with $3\alpha\neq-4\beta$ and $\alpha\beta\neq0$ [17]

By considering Eq. (B1)$+L\times$ Eq. (B2)$-L\times$ Eq. (B3) for $L=L_+$ and for $L=L_-$ with

$$L_+\overset{\text{def}}{=}\frac{\Lambda-1}{\Lambda+1},\quad L_-\overset{\text{def}}{=}\frac{\Lambda+1}{\Lambda-1}\ , \tag{B5}$$

where $\Lambda\overset{\text{def}}{=}\sqrt{-4\beta/3\alpha}\neq1$, we get

$$\frac{d}{dr}\{[r(1-\Lambda)P-(1+\Lambda)Q]\}$$
$$-\frac{1-\Lambda}{4\Lambda}[(1-\Lambda)P-(1+\Lambda)Q]$$
$$\times r[(1-\Lambda)P-(1+\Lambda)Q]=0 \quad \text{for } L=L_+ , \quad \text{(B6a)}$$

$$\frac{d}{dr}\{[r(1+\Lambda)P-(1-\Lambda)Q]\}$$
$$+\frac{1+\Lambda}{4\Lambda}[(1+\Lambda)P-(1-\Lambda)Q]$$
$$\times r[(1+\Lambda)P-(1-\Lambda)Q]=0 \quad \text{for } L=L_- . \quad \text{(B6b)}$$

Equations (B6a) and (B6b) are integrated to give

$$r\left[(1-\Lambda)\frac{A'}{A}-(1+\Lambda)\frac{B'}{B}\right] =K_1 A^{(1-\Lambda)^2/4\Lambda}B^{-(1-\Lambda^2)/4\Lambda} , \quad \text{(B7a)}$$

$$r\left[(1+\Lambda)\frac{A'}{A}-(1-\Lambda)\frac{B'}{B}\right] =K_2 A^{-(1+\Lambda)^2/4\Lambda}B^{(1-\Lambda^2)/4\Lambda} , \quad \text{(B7b)}$$

respectively, where $K_i(i=1,2)$ are complex constants. Integrating these equations further, we get the expressions (5.13a) and (5.13b). Substituting these into (B3), the restriction (5.14) is obtained. Thus, the solution of (B1), (B2), and (B3) as given by (5.13a) and (5.13b) with (5.14) has been derived.

2. The case with $3\alpha=-4\beta\neq 0$

For this case, there are two solutions, which are easily obtained and are given by

$$A(r)=1+a\ln(r/r_0), \quad B(r)=r_0/r , \quad \text{(B8)}$$

and

$$A(r)=1, \quad B(r)=(r/r_0)^b , \quad \text{(B9)}$$

where a and b are constants and we have normalized as $A(r_0)=1=B(r_0)$.

Event horizons and singularity structures of the space-times given by these solutions can be discussed as in Sec. VI and we know the following.

(1) The space-time given by the solution (B8) has an event horizon at $r=l$, where l is the zero point of $A(r)$; $A(l)=0$. It has a singularity at $r=0$, but there is no effective singularity at all.

(2) The space-time given by the solution (B9) has a singularity at $r=0$, if and only if $b>-3/2$. The point $r=0$ is an effective singularity also, if $b>0$. Light ray can reach the point $r=0$, but it needs infinitely long time, if $b\leq -1$. A classical particle also can reach the point $r=0$, if it has a sufficiently high energy. Even in that case, it needs infinitely long time to reach there, if $b\leq -1$. This is so, even if the time is measured by its own proper time.

3. The cases with $\alpha\beta=0$

When $\alpha\beta=0$, solutions are obtained quite easily, but they are trivial: (a) For the case with $\alpha=0=\beta$, $A(r)$ and $B(r)$ are both arbitrary; (b) for the case with $\alpha=0\neq\beta$ ($\alpha\neq 0=\beta$), the product $A(r)B(r)$ [ratio $A(r)/B(r)$] is required to be constant. Otherwise, $A(r)$ and $B(r)$ are arbitrary.

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[9] From now on, Lagrangian density is simply called Lagrangian.

[10] See Appendix A for the definitions of γ^k and of $\bar{\psi}$.

[11] By "spin" we mean here the quantum number associated with the three-dimensional Lorentz group.

[12] It is worth mentioning that the space-time metric is Minkowskian at $r=r_0$.

[13] A similar situation occurs also in the solution (5.23) of Ref. [8]; i.e., the functions A and B given there are not necessarily real valued, in general, for an arbitrary positive r.

[14] (a) We employ (6.5) instead of the Euler equation for r which is not independent from the former. (b) The coordinate time $t=x^0/c$ is the proper time of the observer staying on the circle $r=r_0$, as is known from (6.5).

[15] From a physical point of view, we here understand the variable r appearing in discussions on motions of particles and on light rays to be in a region where A and B are both real valued.

[16] We choose r_1 and r_2 satisfying the conditions (a) $r_1\leq r_2$ and (b) the interval $[r_1,r_2]$ is in a connected part of the region $k^2\geq A^2$.

[17] Note that the conditions $3\alpha\neq -4\beta$ and $\alpha\beta\neq 0$ are satisfied, if (5.11) holds.