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Plasma electrodynamics in the expanding Universe

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The 3+1 formalism of Thorne and Macdonald is used to formulate the electrodynamic equations for a plasma in a spatially flat Robertson-Walker metric. The conformal flatness of this space-time ensures that these equations closely mirror those of flat space-time. The linearized Vlasov-Maxwell equations are solved for the case of an unmagnetized plasma of ultrarelativistic particles and antiparticles, and the results are compared with those of classical kinetic theory and quantum field theory in special relativity. The Vlasov-Maxwell equations for a plasma of nonrelativistic particles are not conformally invariant, so a fluid approximation is used to obtain the linear modes of oscillation, again for an unmagnetized plasma. The plasma modes redshift at rates which depend on the rate of expansion of the Universe, and whether the electromagnetic fields or the particles dominate the dynamics.

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I. INTRODUCTION

Hubble's law has been observed for some time now, indicating that the Universe is expanding. Lemaitre, Friedmann, Robertson, and Walker have provided the simplest relativistic framework in which to model this expansion, allowing the effects of this expansion on a large array of physical processes to be studied from a theoretical viewpoint. One area that has attracted a large amount of interest over the last 50 years is the effect of density or metric perturbations on the matter distribution (for a recent example, see [1]).

Another outstanding problem, albeit one which has received much less attention, is the effect of the expansion on the electromagnetic interactions of the matter, including the longitudinal and transverse plasma modes. The first to tackle this problem appears to have been Holcomb and Tajima [2–4].

The first paper [2] treats plasmas at ultrarelativistic temperatures in a radiation-dominated Friedmann universe. The equations of motion are found for free photons, longitudinal and transverse oscillations, and Alfvén waves using the bulk properties of the plasma. It is found that all of the above modes of oscillation have the same form of equation and hence redshift at the same rate. Here we show how this arises from the conformal flatness of the metric and invariance of the equations, and extend the unmagnetized treatment to a kinetic theory approach, which exhibits the same form of solution as the flat space-time case.

The second paper [3] treats plasmas at nonrelativistic temperatures in a matter-dominated universe. The unmagnetized plasma is covered briefly, but the emphasis is on the modes in an external magnetic field. Here also it is claimed that the oscillations of the unmagnetized plasma redshift in the same manner as a photon, but that the shear Alfvén waves have a remarkably different dependence on time, namely, t^r, where r may take real or imaginary values, depending on the magnetic field and density of the plasma. We show that, for the unmagnetized plasma modes, the time dependences of the plasma oscillations and photon compete, as evidenced by the different rates at which the frequencies of each redshift, if not coupled to the other.

Reference [4] treats nonlinear aspects of primordial plasmas, which are of importance in the generation of large scale magnetic fields, but are beyond the scope of this paper.

Section II briefly reviews the 3+1 formalism of Thorne and Macdonald, which is used to put the general relativistic equations in a form similar to their special relativistic counterparts. The metric is introduced and the conformal flatness is utilized in choosing the most appropriate coordinates. Finally the kinetic theory is set forth.

Section III justifies the use of classical theory in the ultrarelativistic regime, and proceeds to solve for the longitudinal and transverse modes using a kinetic approach as well as a fluid approach.

Section IV finds solutions for the plasma modes in the nonrelativistic limit, for both radiation- and matter-dominated space-times. The asymptotic properties of these solutions demonstrate the balance that exists between the dynamics of the particles and the dynamics of the electromagnetic field.

II. FORMALISM

A. 3+1 split

This paper is concerned with plasma physics in curved space-time. In order to make use of the intuition obtained from nonrelativistic plasma physics it is preferable to split the electromagnetic field tensor $F^{\mu\nu}$ into electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ in terms of which the equations are more familiar. This requires choosing a par-

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ticular set of fiducial observers (FIDO's) at each point in space-time with respect to which **E**, **B**, and other physical quantities are measured. The familiar three-vectors and scalars are then given by the appropriate four-tensor projected perpendicular and parallel to the FIDO 4-velocity u^μ as given in Eqs. (2.4) to (2.10). If the motion of the FIDO's is simple with respect to an appropriate coordinate system the equations are simplified; it is generally possible (with the exception of structures of nontrivial topology such as wormholes) to choose FIDO's with world lines perpendicular to the hypersurfaces corresponding to a given universal time parameter — the "hypersurface" approach. This means that the directions the FIDO's associate with space mesh to form an absolute space given by one of these hypersurfaces. Often it is useful to require that the FIDO's remain at fixed coordinate positions (zero shift vector $\boldsymbol{\beta}$). This is not always convenient, for example, in the Kerr metric (rotating black hole), the dragging of inertial frames ensures that observers with world lines perpendicular to the hypersurfaces of Boyer-Lindquist time t move with respect to the fixed stars at a rate which is a function of the distance to the hole. In this case, if the spatial coordinates were to follow the observers, they would become increasingly twisted over time, and obscure any physical results obtained.

The 3+1 approach has been used by a large number of authors, with the main applications being to formulate a canonical theory of gravity with a view to quantization, numerical solution of various problems in general relativity, and astrophysical problems where comparisons with nonrelativistic work are helpful, particularly black-hole accretion disks and cosmology. A 3+1 formulation of general relativistic magnetohydrodynamics is found in [5], and related numerical questions are addressed in [6]. These techniques are applied to black holes and their accretion disks in [7–9].

Our formalism is due to Thorne and Macdonald [10] and is general, only requiring that FIDO world lines be orthogonal to hypersurfaces of constant time (see above). This generality is what makes the numerical applications possible, however the real utility of the 3+1 approach is manifest in those systems with sufficient symmetry that a natural choice of FIDO presents itself. For the black-hole problem the "natural" FIDO's are the zero angular momentum observers (ZAMO's) which remain at a fixed distance from the hole. In our case the FIDO's are co-moving observers, fixed to the expanding motion of the galaxies. The equations in this section are quite general, specializing to our particular problem when the metric is introduced at the beginning of Sec. II B.

We use unrationalized units with $\hbar = c = k_b = 1$. The relativistic sign conventions are as in [11]. Space-time is split into an arbitrary (but carefully chosen) universal time η (t in [10]) and a three-space, the geometry of which is, in general, a function of η. World lines perpendicular to the hypersurfaces of constant η correspond to a family of observers (FIDO's) with respect to which all quantities are measured. FIDO proper time t (τ in [10]) is in general different from η as expressed by the lapse function α, with the relations being

$$\alpha = \frac{dt}{d\eta} = \frac{1}{\sqrt{-\eta^{;\mu}\eta_{;\mu}}} \ , \tag{2.1}$$

$$u^\mu = -\alpha\eta^{;\mu} \ , \tag{2.2}$$

$$\gamma^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \ , \tag{2.3}$$

where u^μ is the FIDO four-velocity and $\gamma^{\mu\nu}$ is the tensor which projects four-vectors onto the hypersurfaces of constant η, and plays the role of the metric in these spaces. The FIDO observed scalars and three-vectors are obtained by contracting the appropriate four-tensor with u^μ and $\gamma^{\mu\nu}$, respectively. A spatial vector is one which is orthogonal to u^μ; these are often written with latin indices, indicating only three components. Note however that if the FIDO's are moving with respect to the spatial coordinates (nonzero shift vector $\boldsymbol{\beta}$) then the lowered components with respect to the three- and four-dimensional bases may not coincide. We will always have $\boldsymbol{\beta} = 0$. The FIDO measured quantities we will use are

$$\rho = -J^\mu u_\mu \ , \tag{2.4}$$

$$j^\mu = \gamma^\mu{}_\nu J^\nu \ , \tag{2.5}$$

$$E^\mu = F^{\mu\nu}u_\nu \ , \tag{2.6}$$

$$B^\mu = -\tfrac{1}{2}\varepsilon^{\mu\nu\sigma\rho}u_\nu F_{\sigma\rho} \ , \tag{2.7}$$

$$\epsilon = T^{\mu\nu}u_\mu u_\nu \ , \tag{2.8}$$

$$S^\mu = -\gamma^\mu{}_\nu T^{\nu\sigma}u_\sigma \ , \tag{2.9}$$

$$W^{\mu\nu} = \gamma^\mu{}_\sigma T^{\sigma\rho}\gamma^\nu{}_\rho \ , \tag{2.10}$$

where the electromagnetic symbols take their usual meanings, ϵ, S^μ, and $W^{\mu\nu}$ are, respectively, energy density, momentum density and the pressure tensor, and $\varepsilon^{\mu\nu\sigma\rho}$ is the Levi-Civita tensor equal to $(-g)^{-1/2}$(antisymmetric symbol defined so that $\varepsilon^{0123} = 1$ in a right-handed future-directed orthonormal coordinate system).

The 3+1 equations also contain the kinetic properties of the FIDO world lines:

$$\theta = u^\mu{}_{;\mu} \ , \tag{2.11}$$

$$a^\mu = u^\mu{}_{;\nu}u^\nu = (\ln\alpha)^{;\mu} \ , \tag{2.12}$$

$$\sigma_{\mu\nu} = \tfrac{1}{2}\gamma_\mu{}^\sigma\gamma_\nu{}^\rho(u_{\sigma;\rho} + u_{\rho;\sigma}) - \tfrac{1}{3}\theta\gamma_{\mu\nu} \ . \tag{2.13}$$

Here, θ is the expansion and is equal to three times the Hubble constant averaged over all directions, a^μ is the acceleration as measured by a FIDO accelerometer, and $\sigma_{\mu\nu}$ is the rate of shear of FIDO world lines. If the world lines were not perpendicular to the surfaces of constant η, another quantity would be necessary, the rotation $\omega_{\mu\nu}$.

Covariant spatial derivatives are defined equivalently as the four-dimensional covariant derivative projected onto the hypersurfaces of constant η,

$$(\nabla\mathbf{M})^\mu{}_\nu = \gamma^\mu{}_\sigma\gamma_\nu{}^\rho M^\sigma{}_{;\rho} \ , \tag{2.14}$$

or as a covariant derivative in these hypersurfaces,

$$(\nabla\mathbf{M})^i{}_j \equiv M^i{}_{|j} = M^i{}_{,j} + \Gamma^i_{jk}M^k \ , \tag{2.15}$$

where $\mathbf{M}$ is a spatial vector and Γ^i_{jk} is computed in the usual way from the spatial metric γ_{ij}. These derivatives do not commute if the spatial geometry is curved.

Now we are in a position to define the usual vector operations and time derivatives of vectors:

$$\mathbf{M}\cdot\mathbf{N} = M^i N^j \gamma_{ij} \ , \tag{2.16}$$

$$\nabla\cdot\mathbf{M} = M^i{}_{|i} \ , \tag{2.17}$$

$$(\mathbf{M}\times\mathbf{N})^i = \varepsilon^{ijk} M_j N_k \ , \tag{2.18}$$

$$(\nabla\times\mathbf{M})^i = \varepsilon^{ijk} M_{k|j} \ , \tag{2.19}$$

$$(D_t\mathbf{M})^\mu = \gamma^{\mu\nu} M_{\nu;\sigma} u^\sigma = M^\mu{}_{;\nu} u^\nu - u^\mu a_\nu M^\nu \ , \tag{2.20}$$

where $\varepsilon^{ijk} = \gamma^{-1/2}$(antisymmetric symbol). D_t is the derivative with respect to proper time which preserves scalar products; there are other time derivatives useful in the 3+1 formalism, but we will not need them here. D_t does not commute with spatial derivatives in general.

B. The metric

We will be solving the Vlasov equation coupled to the Maxwell equations, ignoring the effects of the fluctuation in particles and fields on the metric, in the spatially flat Robertson-Walker metric. The line element is often written as

$$ds^2 = -dt^2 + R^2(t)(dx^2 + dy^2 + dz^2) \ , \tag{2.21}$$

where x, y, and z are comoving coordinates and t is proper time as seen by observers at fixed x, y, and z, which we will use as our FIDO's. As Holcomb and Tajima [2] have noticed, a number of plasma equations in this metric have the same form as in flat space-time, with the usual $e^{-i\omega t}$ replaced by spherical Bessel functions multiplied by various powers of t. The reason for this becomes evident after making the coordinate transformation:

$$\eta = \int \frac{1}{R} dt \ , \tag{2.22}$$

$$ds^2 = R^2(\eta)(-d\eta^2 + dx^2 + dy^2 + dz^2) \ , \tag{2.23}$$

which differs from the Minkowski metric by a conformal transformation. A number of the equations of physics (more specifically, most equations describing only massless particles) are invariant under these transformations. For these systems the solution of the equations follow in the same manner as flat space-time. We will demonstrate this for the case of the free Maxwell equations, as well as the coupled Maxwell-Vlasov equations in the case of ultrarelativistic charged particles (Sec. III A).

The FIDO-related quantities for the metric above are found to be

$$\theta = \frac{3R'}{R^2} \ , \tag{2.24}$$

$$\alpha = R \ , \tag{2.25}$$

$$\sigma_{ij} = 0 \ , \tag{2.26}$$

where a prime indicates partial differentiation with respect to η.

Up to this point all of the equations have been independent of a choice of basis for spatial vectors. To solve the equations we need to specify such a basis. We choose the orthonormal basis, so that the components take on the numerical values as measured by a FIDO, and the components of $\overleftrightarrow{\gamma}$ are zero and one. Vectors and tensors measured with respect to an orthonormal basis will be denoted by carets; however all derivatives will be explicit with respect to the coordinates $(\eta, \mathbf{x})$. Using the above metric we obtain

$$(D_t\mathbf{M})^{\hat{\imath}} = \frac{1}{R}\frac{\partial}{\partial\eta} M^{\hat{\imath}} \equiv \frac{1}{R}(\hat{\mathbf{M}}')^{\hat{\imath}} \ , \tag{2.27}$$

$$\nabla\cdot\mathbf{M} = \frac{1}{R}\frac{\partial}{\partial x^i} M^{\hat{\imath}} \equiv \frac{1}{R}\boldsymbol{\partial}\cdot\hat{\mathbf{M}} \ , \tag{2.28}$$

$$(\nabla\times\mathbf{M})^{\hat{\imath}} = \frac{1}{R}\varepsilon^{\hat{\imath}\hat{\jmath}\hat{k}} \frac{\partial}{\partial x^j} M_{\hat{k}} \equiv \frac{1}{R}(\boldsymbol{\partial}\times\hat{\mathbf{M}})^{\hat{\imath}} \ , \tag{2.29}$$

where $\mathbf{M}$ is a spatial vector and $\boldsymbol{\partial}$ is vector notation for $\partial/\partial x$. Note that $R^{-1}\partial/\partial\eta$ is a derivative with respect to proper time and $R^{-1}\partial/\partial x$ is a derivative with respect to proper distance.

Maxwell's equations, the charge conservation and Lorentz force equations and energy and momentum conservation for continuous media have been written down in 3+1 form, in Sec. III of [10] and Sec. II of [2]. Substituting the FIDO related quantities and derivatives for our particular case we obtain

$$\tfrac{1}{R}\boldsymbol{\partial}\cdot\hat{\mathbf{E}} = 4\pi\rho_e \ , \tag{2.30}$$

$$\tfrac{1}{R}\boldsymbol{\partial}\cdot\hat{\mathbf{B}} = 0 \ , \tag{2.31}$$

$$\tfrac{1}{R}\hat{\mathbf{E}}' + \tfrac{2R'}{R^2}\hat{\mathbf{E}} = \tfrac{1}{R}\boldsymbol{\partial}\times\hat{\mathbf{B}} - 4\pi\hat{\boldsymbol{\jmath}} \ , \tag{2.32}$$

$$\tfrac{1}{R}\hat{\mathbf{B}}' + \tfrac{2R'}{R^2}\hat{\mathbf{B}} = -\tfrac{1}{R}\boldsymbol{\partial}\times\hat{\mathbf{E}} \ , \tag{2.33}$$

$$\tfrac{1}{R}\rho_e' + \tfrac{3R'}{R^2}\rho_e + \tfrac{1}{R}\boldsymbol{\partial}\cdot\hat{\boldsymbol{\jmath}} = 0 \ , \tag{2.34}$$

$$\left(\tfrac{1}{R}\tfrac{\partial}{\partial\eta} + \tfrac{1}{R}\hat{\mathbf{v}}\cdot\boldsymbol{\partial} + \tfrac{R'}{R^2}\right)\hat{\mathbf{p}} = e(\hat{\mathbf{E}} + \hat{\mathbf{v}}\times\hat{\mathbf{B}}) \ , \tag{2.35}$$

$$\tfrac{1}{R}\epsilon' + \tfrac{3R'}{R^2}\epsilon + \tfrac{1}{R}\boldsymbol{\partial}\cdot\hat{\mathbf{S}} + \tfrac{R'}{R^2}Tr(\overleftrightarrow{\hat{\mathbf{W}}}) = \hat{\boldsymbol{\jmath}}\cdot\hat{\mathbf{E}} \ , \tag{2.36}$$

$$\tfrac{1}{R}\hat{\mathbf{S}}' + \tfrac{4R'}{R^2}\hat{\mathbf{S}} + \tfrac{1}{R}\boldsymbol{\partial}\cdot\overleftrightarrow{\hat{\mathbf{W}}} = \rho_e\hat{\mathbf{E}} + \hat{\boldsymbol{\jmath}}\times\hat{\mathbf{B}} \ . \tag{2.37}$$

In the limit R =const (static universe) these equations reduce to their usual flat space-time counterparts. The electrodynamic equations (2.30)–(2.35) are simplified by the variable transformations

$$\bar{\mathbf{E}} = R^2\hat{\mathbf{E}} \ , \tag{2.38}$$

$$\bar{\mathbf{B}} = R^2\hat{\mathbf{B}} \ , \tag{2.39}$$

$$\bar{\rho}_e = R^3\rho_e \ , \tag{2.40}$$

$$\bar{\boldsymbol{\jmath}} = R^3\hat{\boldsymbol{\jmath}} \ , \tag{2.41}$$

$$\bar{\mathbf{p}} = R\hat{\mathbf{p}} \ , \tag{2.42}$$

leading to equations with the same form as the flat space-time equations in the barred quantities, with time replaced by η. Note that this simplification of the equations takes place independent of the functional form of R, and is a direct result of the conformal flatness of the space-time. The exception is the Lorentz force, which becomes

$$\left(\frac{\partial}{\partial\eta} + \hat{\mathbf{v}}\cdot\boldsymbol{\partial}\right)\bar{\mathbf{p}} = e(\bar{\mathbf{E}} + \hat{\mathbf{v}}\times\bar{\mathbf{B}}) \ . \tag{2.43}$$

$\bar{\mathbf{p}}$ is the conserved momentum if the fields are zero. The difference between this equation and that of flat space-time is that $\bar{\mathbf{p}}$ and $\hat{\mathbf{v}}$ are related differently. $\hat{\mathbf{v}}$ can be obtained from

$$\hat{\mathbf{v}} = \frac{\bar{\mathbf{p}}}{\sqrt{\bar{\mathbf{p}}^2 + m^2R^2}} \ . \tag{2.44}$$

C. Kinetic theory

In the Vlasov-Landau approximation, the charged particles are represented by a distribution function f over phase space and time $(\mathbf{x}, \bar{\mathbf{p}}, \eta)$. Interactions are treated by including an electromagnetic field which is self-consistently generated by the particles. Macroscopic quantities are obtained as integrals over the particle momentum. No detailed derivation will be given here; the reader is referred to [12] for how to do this. The kinetic equation derived from the above particle equation of motion is

$$\left(\frac{\partial}{\partial\eta} + \hat{\mathbf{v}}\cdot\boldsymbol{\partial} + e_s(\bar{\mathbf{E}} + \hat{\mathbf{v}}\times\bar{\mathbf{B}})\cdot\boldsymbol{\partial}_{\bar{\mathbf{p}}}\right) f_s(\mathbf{x}, \bar{\mathbf{p}}, \eta) = 0 \ , \tag{2.45}$$

where $\boldsymbol{\partial}_{\bar{\mathbf{p}}}$ is $\frac{\partial}{\partial\bar{\mathbf{p}}}$ and f_s is normalized so that it reduces to the usual special relativistic normalization in a local Lorentz frame. The subscript s allows for different species of particles. Thus the charge and current densities are given by

$$\rho_e = \sum_s e_s \int f_s d^3\hat{p} \ , \tag{2.46}$$

$$\hat{\jmath} = \sum_s e_s \int f_s \hat{\mathbf{v}} d^3\hat{p} \ , \tag{2.47}$$

$$\text{or } \bar{\rho}_e = \sum_s e_s \int f_s d^3\bar{p} \ , \tag{2.48}$$

$$\bar{\jmath} = \sum_s e_s \int f_s \hat{\mathbf{v}} d^3\bar{p} \ , \tag{2.49}$$

and similarly

$$\epsilon = \sum_s \int f_s \sqrt{\hat{\mathbf{p}}^2 + m_s^2}\, d^3\hat{p} \ , \tag{2.50}$$

$$\hat{\mathbf{S}} = \sum_s \int f_s \hat{\mathbf{p}} d^3\hat{p} \ , \tag{2.51}$$

$$\hat{\overleftrightarrow{\mathbf{W}}} = \sum_s \int f_s \frac{\hat{\mathbf{p}}\otimes\hat{\mathbf{p}}}{\sqrt{\hat{\mathbf{p}}^2 + m_s^2}} d^3\hat{p} \ . \tag{2.52}$$

Applying $\int d^3\bar{p}$ to the kinetic equation (2.45), we obtain the equation of charge conservation. Similarly the equations of energy and momentum conservation (2.36) and (2.37) can be obtained. The condition for equilibrium distributions is found by setting the fields and spatial derivatives to zero in the kinetic equation. The solution is

$$f_s = \text{arbitrary function of } \bar{\mathbf{p}} \ . \tag{2.53}$$

Comparing this with a Boltzmann distribution at rest with respect to a FIDO,

$$f_B = \text{const} \times e^{-\beta\sqrt{\hat{\mathbf{p}}^2 + m^2}} \ , \tag{2.54}$$

$$\beta = \frac{1}{T}, \tag{2.55}$$

where T is the temperature, it is seen that f_B is only a solution of (2.45) if either

$$\hat{\mathbf{p}} \gg m \text{ and } T \sim R^{-1} \ , \tag{2.56}$$

or

$$\hat{\mathbf{p}} \ll m \text{ and } T \sim R^{-2} \ . \tag{2.57}$$

The first possibility corresponds to the ultrarelativistic limit, and the second to the nonrelativistic limit. In the intermediate region the distributions are nonequilibrium and collisions cannot be ignored, as has been well documented [13]. Of course, for ultrarelativistic particles, antiparticles should be included, and the correct quantum distribution used. This limit is the subject of the next section.

III. ULTRARELATIVISTIC LIMIT

A. Kinetic treatment

A natural question that arises in connection with plasmas at ultrarelativistic temperatures is whether classical kinetic theory gives a sufficiently accurate description or whether the more involved subject of quantum field theory in curved space-time is necessary to treat particle creation and/or annihilation and other quantum processes. We assume that the curvature of the space-time has a much greater length scale than the Compton wavelength of the particles, so that no particles are created out of the curvature itself. In flat space-time it is found that, to terms of first order in the fine structure constant, the classical and quantum formulations of the plasma give the same plasma frequency, provided that the correct distributions are used, for both spin-0 and spin-$\frac{1}{2}$ particles. This result will be discussed further later in this section. In addition, the factors of R that appear in expanding universe calculations conspire to ensure that no overall creation or annihilation takes place, whether or not there is an imbalance between particles and antiparticles. These considerations indicate that, under the approximations mentioned above, a classical treatment should be satisfactory, although a quantum treatment would be preferred, and as such would be a natural generalization of this work.

We consider only linear oscillations, that is, first order perturbations about an equilibrium solution $f_{s0}(\bar{\mathbf{p}})$ given by the Fermi-Dirac or Bose-Einstein distribution in the ultrarelativistic limit

$$f_{s0}(\bar{\mathbf{p}}) = \frac{2S+1}{(2\pi)^3}\frac{1}{e^{\bar{\beta}(\bar{\mathbf{p}}-\bar{\mu}_s)} \pm 1} \ , \tag{3.1}$$

$$\bar{\beta} = R^{-1}\beta \ , \tag{3.2}$$

$$\bar{\mu}_s = R\mu_s \ , \tag{3.3}$$

where $2S+1$ is the spin degeneracy factor, μ_s is the chemical potential of species s, and the + (−) sign applies to fermions (bosons). $\bar{\beta}$ and $\bar{\mu}_s$ must be independent of time for f_{s0} to be a solution of (2.45). Using a subscript + (−) to apply to particles (antiparticles), the constraint that they are produced in pairs is

$$\bar{\mu}_+ + \bar{\mu}_- = 0 \ . \tag{3.4}$$

Henceforth, $\bar{\mu}_+$ will be simply written as $\bar{\mu}$.

The other condition placed on the chemical potentials arises from the FIDO observed difference in number density between particles and antiparticles. When multiplied by R^3 this quantity is independent of time, that is,

$$\Delta\bar{n} = R^3\Delta n = R^3(n_+ - n_-) = \int (f_{+0} - f_{-0}) d^3\bar{p} \ . \tag{3.5}$$

The fact that both sides of the above equation are independent of time indicates that the assumption that there is no overall creation or annihilation of particles in the ultrarelativistic regime is correct.

The full distribution function will be written as

$$f_s(\mathbf{x}, \bar{\mathbf{p}}, \eta) = f_{s0}(\bar{\mathbf{p}}) + f_{s1}(\mathbf{x}, \bar{\mathbf{p}}, \eta) \ , \tag{3.6}$$

where f_{s1} and also $\bar{\mathbf{E}}$ and $\bar{\mathbf{B}}$ are assumed to be small, so that second order terms are negligible. A possible extension of this work would be to include an external magnetic field which is not assumed to be small.

The coupled Maxwell-Vlasov equations become

$$\boldsymbol{\partial} \cdot \bar{\mathbf{E}} = 4\pi \textstyle\sum_s e_s \int f_{s1} d^3\bar{p} \ , \tag{3.7}$$

$$\boldsymbol{\partial} \cdot \bar{\mathbf{B}} = 0 \ , \tag{3.8}$$

$$\bar{\mathbf{E}}' = \boldsymbol{\partial} \times \bar{\mathbf{B}} - 4\pi \textstyle\sum_s e_s \int f_{s1} \hat{\mathbf{v}} d^3\bar{p} \ , \tag{3.9}$$

$$\bar{\mathbf{B}}' = -\boldsymbol{\partial} \times \bar{\mathbf{E}} \ , \tag{3.10}$$

$$\left(\frac{\partial}{\partial\eta} + \hat{\mathbf{v}} \cdot \boldsymbol{\partial}\right) f_{s1} + e_s(\bar{\mathbf{E}} + \hat{\mathbf{v}} \times \bar{\mathbf{B}}) \cdot \boldsymbol{\partial}_{\bar{\mathbf{p}}} f_{s0} = 0 \ , \tag{3.11}$$

where it has been assumed that charge neutrality holds for the unperturbed system. For an electron-positron plasma with an excess of electrons, this could be achieved by treating nonrelativistic protons as a uniform positive background, as is often done in plasma theory. This prescription relies on the fact that protons are not affected much at the higher frequencies at which the electrons oscillate, due to the large mass ratio.

From this point the analysis follows exactly as for flat space-time, since $\hat{\mathbf{v}}$ is a unit vector, equal to the speed of light in both cases, and all the barred quantities obey the same equations as in flat space-time. The linearized quantities are assumed to have harmonic space and time dependence:

$$f_{s1}, \bar{\mathbf{E}} \ \text{and} \ \bar{\mathbf{B}} \sim e^{i(\bar{\mathbf{k}}\cdot\mathbf{x} - \bar{\omega}\eta)} \ . \tag{3.12}$$

Note that $e^{-i\bar{\omega}\eta}$ is equivalent to the $H_{1/2}(2\omega_i t^{1/2})$ of [2] for the case $R = (t/t_i)^{1/2}$, subject to some multiplicative powers of t which arise from defining the electromagnetic quantities differently. The FIDO observed values of angular frequency and wave number are

$$\omega = R^{-1}\bar{\omega} \ , \tag{3.13}$$

$$\hat{\mathbf{k}} = R^{-1}\bar{\mathbf{k}} \ . \tag{3.14}$$

These are obtained by applying the proper derivatives $R^{-1}\partial/\partial\eta$ and $R^{-1}\boldsymbol{\partial}$ on the logarithm of this space-time dependence.

The kinetic equation (3.11) can then be written

$$f_{s1} = -i\frac{e_s(\bar{\mathbf{E}} + \hat{\mathbf{v}} \times \bar{\mathbf{B}}) \cdot \boldsymbol{\partial}_{\bar{\mathbf{p}}} f_{s0}}{\bar{\omega} - \hat{\mathbf{v}} \cdot \bar{\mathbf{k}}} \ . \tag{3.15}$$

Using (2.49) and (3.10) we obtain

$$\bar{\jmath} = \overleftrightarrow{\bar{\sigma}} \cdot \bar{\mathbf{E}} \ , \tag{3.16}$$

with

$$\begin{aligned} \overleftrightarrow{\bar{\sigma}} &= -i\sum_s e_s^2 \int \frac{d^3\bar{p}}{\bar{\omega} - \hat{\mathbf{v}} \cdot \bar{\mathbf{k}}}\left[\hat{\mathbf{v}} \otimes \boldsymbol{\partial}_{\bar{\mathbf{p}}} f_{s0}\left(1 - \frac{\hat{\mathbf{v}} \cdot \bar{\mathbf{k}}}{\bar{\omega}}\right) \right. \\ &\quad \left. + \hat{\mathbf{v}} \otimes \hat{\mathbf{v}} \frac{\bar{\mathbf{k}} \cdot \boldsymbol{\partial}_{\bar{\mathbf{p}}} f_{s0}}{\bar{\omega}}\right] \\ &= R\overleftrightarrow{\hat{\sigma}} \ , \end{aligned} \tag{3.17}$$

where $\overleftrightarrow{\hat{\sigma}}$ is the FIDO measured conductivity tensor of the plasma. Maxwell's equations (3.9) and (3.10) give

$$\overleftrightarrow{\bar{\epsilon}} \cdot \bar{\mathbf{E}} + \frac{\bar{\mathbf{k}} \times \bar{\mathbf{k}} \times \bar{\mathbf{E}}}{\bar{\omega}^2} = 0 \ , \tag{3.18}$$

where

$$\overleftrightarrow{\bar{\epsilon}} = \overleftrightarrow{\mathbf{1}} + \frac{4\pi i}{\bar{\omega}} \overleftrightarrow{\bar{\sigma}} \ . \tag{3.19}$$

Since the equilibrium distribution f_{s0} is isotropic, the dielectric tensor $\overleftrightarrow{\bar{\epsilon}}$ separates into longitudinal and transverse components,

$$\overleftrightarrow{\bar{\epsilon}} = \bar{\epsilon}_L \frac{\bar{\mathbf{k}} \otimes \bar{\mathbf{k}}}{\bar{k}^2} + \bar{\epsilon}_T \left(\overleftrightarrow{\mathbf{1}} - \frac{\bar{\mathbf{k}} \otimes \bar{\mathbf{k}}}{\bar{k}^2}\right) \ , \tag{3.20}$$

$$\bar{\epsilon}_L = 1 + \sum_s \frac{4\pi e_s^2}{\bar{\omega}\bar{k}^2} \int \frac{d^3\bar{p}}{\bar{\omega} - \hat{\mathbf{v}} \cdot \bar{\mathbf{k}}} (\hat{\mathbf{v}} \cdot \bar{\mathbf{k}})(\boldsymbol{\partial}_{\bar{\mathbf{p}}} f_{s0} \cdot \bar{\mathbf{k}}) \ , \tag{3.21}$$

$$\begin{aligned} \bar{\epsilon}_T = 1 + \sum_s \frac{2\pi e_s^2}{\bar{\omega}\bar{k}^2} \int \frac{d^3\bar{p}}{\bar{\omega} - \hat{\mathbf{v}} \cdot \bar{\mathbf{k}}} &[(\hat{\mathbf{v}} \cdot \boldsymbol{\partial}_{\bar{\mathbf{p}}} f_{s0})\bar{k}^2 \\ &-(\hat{\mathbf{v}} \cdot \bar{\mathbf{k}})(\boldsymbol{\partial}_{\bar{\mathbf{p}}} f_{s0} \cdot \bar{\mathbf{k}})] \ , \end{aligned} \tag{3.22}$$

which yield the dispersion relations for longitudinal and transverse modes of oscillation:

$$\bar{\epsilon}_L = 0 \ , \tag{3.23}$$

$$\bar{\epsilon}_T = \frac{\bar{\mathbf{k}}^2}{\bar{\omega}^2} \ . \tag{3.24}$$

The above integrals are evaluated using the Landau prescription of adding a small positive imaginary part to $\bar{\omega}$ to make the boundary conditions causal [14,15], and then using the Plemelj formula to obtain the real and imaginary parts:

$$\bar{\epsilon}_L = 1 - \sum_s \frac{8\pi^2 e_s^2}{\bar{k}^2}\left[\frac{\bar{\omega}}{\bar{k}}\ln\left|\frac{\bar{k}+\bar{\omega}}{\bar{k}-\bar{\omega}}\right| - 2 - i\pi\frac{\bar{\omega}}{\bar{k}}\theta(\bar{k}-\bar{\omega})\right] I_s \ , \tag{3.25}$$

$$\bar{\epsilon}_T = 1 - \sum_s \frac{4\pi^2 e_s^2}{\bar{k}^2}\left[\frac{\bar{k}^2-\bar{\omega}^2}{\bar{k}\bar{\omega}}\ln\left|\frac{\bar{k}+\bar{\omega}}{\bar{k}-\bar{\omega}}\right| + 2 - i\pi\left(\frac{\bar{k}^2-\bar{\omega}^2}{\bar{k}\bar{\omega}}\right)\theta(\bar{k}-\bar{\omega})\right] I_s, \tag{3.26}$$

$$I_s = -\int_0^\infty \frac{\partial f_{s0}}{\partial \bar{p}}\bar{p}^2 d\bar{p} \ . \tag{3.27}$$

Note that the form of the dielectric tensor to this approximation is totally fixed by $\sum_s e_s^2 I_s$. This is due to the fact that all particles are traveling at the same speed (c). The θ function indicates that no Landau damping occurs when the phase velocity of the wave is greater than light, which is because there are no particles with such velocities. Thus the long wavelength limit is completely free of this source of damping, although a correct quantum treatment [16] indicates damping above the pair-production threshold ($\omega > \sqrt{4m^2 + \hat{k}^2}$). As seen from (3.40) this only occurs at temperatures above 10 MeV. The mode equations (3.23) and (3.24) give, in the limit of small $\bar{k}$,

$$\text{longitudinal } \bar{\omega}^2 = \bar{\omega}_p^2 + \tfrac{3}{5}\bar{k}^2 + \cdots \ , \tag{3.28}$$

$$\text{transverse } \bar{\omega}^2 = \bar{\omega}_p^2 + \tfrac{6}{5}\bar{k}^2 + \cdots \ , \tag{3.29}$$

where

$$\bar{\omega}_p^2 = \frac{16\pi^2}{3}\sum_s e_s^2 I_s \ . \tag{3.30}$$

Now it remains to evaluate $\sum_s e_s^2 I_s$. Integrating by parts and substituting the fermion distribution (3.1) gives

$$\sum_s e_s^2 I_s = e^2\int_0^\infty \frac{2(2S+1)}{(2\pi)^3}\left[\frac{1}{e^{\bar{\beta}(\bar{p}-\bar{\mu})}+1} + \frac{1}{e^{\bar{\beta}(\bar{p}+\bar{\mu})}+1}\right]\bar{p}d\bar{p} \tag{3.31}$$

$$= \frac{4e^2}{\bar{\beta}^2(2\pi)^3}\left[f_2(z) + f_2(z^{-1})\right] \ , \tag{3.32}$$

$$z = e^{\bar{\beta}\bar{\mu}} \ , \tag{3.33}$$

where

$$f_\alpha(z) = \frac{1}{\Gamma(\alpha)}\int_0^\infty \frac{x^{\alpha-1}}{z^{-1}e^x+1}dx \ , \tag{3.34}$$

$$= \sum_{j=1}^\infty \frac{(-1)^{j+1}z^j}{j^\alpha} \text{ if } 0 \le z < 1 \ , \tag{3.35}$$

and the electron spin $S = 1/2$ has been substituted.

The number equation (3.5) gives

$$\Delta\bar{n} = \frac{16\pi}{\bar{\beta}^3(2\pi)^3}\left[f_3(z) - f_3(z^{-1})\right] \ . \tag{3.36}$$

For the most likely situation where the excess of particles over antiparticles is small compared to the total density, $z \approx 1$. An expansion of these functions about $z = 1$ has been done by Hore and Frankel [17]. The result is

$$f_\alpha(e^{\bar{\beta}\bar{\mu}}) = \sum_{j=0}^\infty \frac{(\bar{\beta}\bar{\mu})^j}{j!}\tau(\alpha - j) \quad |\bar{\beta}\bar{\mu}| < \pi \ , \tag{3.37}$$

$$\tau(z) = (1 - 2^{1-z})\zeta(z) \ , \tag{3.38}$$

where $\zeta(z)$ is the Riemann zeta function. Using the above expansion, z can be eliminated from (3.32) and (3.36) to obtain

$$\sum_s e_s^2 I_s = \frac{2\pi^2 e^2}{3\bar{\beta}^2(2\pi)^3}\left(1 + \frac{27}{\pi^2}(\bar{\beta}^3\Delta\bar{n})^2 + \cdots\right) \ , \tag{3.39}$$

and hence

$$\bar{\omega}_p^2 = \frac{4\pi e^2}{9\bar{\beta}^2}\left(1 + \frac{27}{\pi^2}(\bar{\beta}^3\Delta\bar{n})^2 + \cdots\right) \ . \tag{3.40}$$

For a hypothetical plasma of spin-0 bosons which has an imbalance of particles and antiparticles the statistical mechanics is complicated by the presence of a Bose condensate, which is particularly important in the ultrarelativistic limit [18–20]. We leave the treatment of this plasma to a future paper, noting that an analysis similar to the above yields the same plasma frequency as for fermions in the balanced ($z = 1$) case.

Now we are in a position to make a comparison between the classical and quantum descriptions of the plasma. The relevant flat space-time work can be found in Tsytovich [16] for spin-$\frac{1}{2}$ fermions and Kowalenko, Frankel and Hines [21] for spin-0 bosons. For both fermions and bosons, the longitudinal and transverse response functions are equal in the $\mathbf{k} \to 0$ limit. For fermions,

$$\epsilon_L(0,\omega) = \epsilon_T(0,\omega) = 1 - \frac{4\pi e^2}{\omega^2}\int (f_{+0} + f_{-0})\frac{4E_p(1 - \frac{p^2}{3E_p^2})}{4E_p^2 - \omega^2}d^3p \ , \tag{3.41}$$

and, for bosons,

$$\epsilon_L(0,\omega) = \epsilon_T(0,\omega) = 1 - \frac{4\pi e^2}{\omega^2}\int (f_{+0} + f_{-0}) \times \frac{\omega^2 - 4E_p^2(1 - \frac{p^2}{3E_p^2})}{E_p(\omega^2 - 4E_p^2)}d^3p \ , \tag{3.42}$$

where

$$E_p = \sqrt{p^2 + m^2} \ . \tag{3.43}$$

If it is assumed that $\omega \ll 4E_p^2$, which is justified, to first order in the fine structure constant, by the result (3.40), both of the above expressions simplify to

$$\epsilon = 1 - \frac{4\pi e^2}{\omega^2}\int (f_{+0} + f_{-0})\frac{1 - \frac{p^2}{3E_p^2}}{E_p} d^3p \ . \quad (3.44)$$

Finally we take the ultrarelativistic limit $p \gg m$ and obtain (3.30).

B. Hydrodynamic treatment

We conclude this section by giving the corresponding hydrodynamic treatment of the ultrarelativistic plasma. This has a twofold purpose, reproducing the results of Holcomb and Tajima [2] using our slightly different formalism, and providing a point of comparison with the next section which uses the hydrodynamic equations exclusively. As suggested by Ref. [2] and the foregoing calculations, the form of the equations follows that of flat space-time when the conformalized variables ($\bar{\mathbf{E}}$, $\bar{\mathbf{B}}$, and so on) are used. However as in flat space-time, the dispersion relations are incorrect.

In the hydrodynamic treatment the plasma is represented as a perfect fluid — no viscosity or heat conduction — governed by Eqs. (2.36) and (2.37), where ϵ, $\hat{\mathbf{S}}$, and $\hat{\overleftrightarrow{\mathbf{W}}}$ are given by

$$\epsilon = \Gamma^2(\rho + p\hat{v}^2) \ , \quad (3.45)$$

$$\hat{\mathbf{S}} = \Gamma^2(\rho + p)\hat{\mathbf{v}} \ , \quad (3.46)$$

$$\hat{\overleftrightarrow{\mathbf{W}}} = \Gamma^2(\rho + p)\hat{\mathbf{v}} \otimes \hat{\mathbf{v}} + p\,\overleftrightarrow{\mathbf{1}} \ , \quad (3.47)$$

and now $\hat{\mathbf{v}}$ and the boost factor Γ correspond to the bulk motion of the plasma, ρ is the total (rest plus internal) energy density in the rest frame and p is the pressure in the rest frame. These are given in terms of the temperature T by

$$\rho = \frac{2S+1}{2\pi^2}\Gamma(4)\left\{\begin{array}{c}\tau(4)\\ \zeta(4)\end{array}\right\}T^4 \ , \quad (3.48)$$

$$p = \frac{1}{3}\rho \ , \quad (3.49)$$

where particles and antiparticles are treated separately, and the τ (ζ) applies to fermions (bosons). We have assumed equal numbers of particles and antiparticles for simplicity. $\tau(z)$ is defined by (3.38). In contrast with the nonrelativistic case, these are not proportional to the particle number density, which is given by

$$n = \frac{2S+1}{2\pi^2}\Gamma(3)\left\{\begin{array}{c}\tau(3)\\ \zeta(3)\end{array}\right\}T^3 \ . \quad (3.50)$$

It is convenient to define

$$\bar{T} = RT \ , \quad (3.51)$$

$$\bar{n} = R^3 n \ . \quad (3.52)$$

$\bar{T}$, n, and $\hat{\mathbf{v}}$ are linearized for each species; thus,

$$\bar{T} = \bar{T}_0 + \bar{T}_1 \ , \quad (3.53)$$

$$\bar{n} = \bar{n}_0 + \bar{n}_1 \ , \quad (3.54)$$

$$|\hat{\mathbf{v}}| \ll 1 \ . \quad (3.55)$$

The equation of particle conservation is

$$\bar{n}_1' + \bar{n}_0 \boldsymbol{\partial}\cdot\hat{\mathbf{v}} = 0 \ . \quad (3.56)$$

The energy equation (2.36) together with this gives

$$\frac{\bar{T}_1'}{\bar{T}_0} = \frac{1}{3}\frac{\bar{n}_1'}{\bar{n}_0} \ , \quad (3.57)$$

which is just the adiabatic equation. The momentum equation (2.37) gives

$$\hat{\mathbf{v}}' = \frac{\tau(3)}{4\tau(4)\bar{T}_0}e\bar{\mathbf{E}} - \frac{\boldsymbol{\partial}\bar{T}_1}{\bar{T}_0} \ , \quad (3.58)$$

for fermions.

These equations are written down for each species, and are connected by the Maxwell equations with

$$\bar{\rho}_e = \sum_s e_s \bar{n}_{s1} \ , \quad (3.59)$$

$$\bar{\jmath} = \sum_s e_s \bar{n}_{s0}\hat{\mathbf{v}} \ . \quad (3.60)$$

As before we assume harmonic space and time dependence $e^{i(\bar{\mathbf{k}}\cdot\mathbf{x} - \bar{\omega}\eta)}$ for the linearized quantities, including $\bar{\mathbf{E}}$ and $\bar{\mathbf{B}}$. For longitudinal oscillations ($\hat{\mathbf{v}}$, $\bar{\mathbf{k}}$, and $\bar{\mathbf{E}}$ all parallel), only Poisson's equation is needed from the Maxwell set, and we obtain, for fermions,

$$\bar{\omega}^2 = \bar{\omega}_p^2 + \tfrac{1}{3}\bar{k}^2 \ , \quad (3.61)$$

$$\bar{\omega}_p^2 = \frac{4\pi e^2}{\bar{\beta}^2}\frac{2S+1}{2\pi^2}\frac{\tau(3)^2}{\tau(4)} \ , \quad (3.62)$$

which is clearly not the same as the leading term of (3.40). The boson result is the same as the above with τ replaced by ζ. For transverse modes ($\bar{\mathbf{k}}$ perpendicular to $\hat{\mathbf{v}}$ and $\bar{\mathbf{E}}$, which are parallel), we obtain

$$\bar{\omega}^2 = \bar{\omega}_p^2 + \bar{k}^2, \quad (3.63)$$

with $\bar{\omega}_p^2$ as above. These results are of the same form as the flat space-time ones, as observed in [2], with t replaced by η and the variables replaced by their "conformalized" forms.

To summarize the results of this section, in the ultrarelativistic limit, the equations scale so that the solutions are obtainable in the same way as in flat space-time, and the plasma modes redshift in the same way as a free photon. The quantum and classical treatments yield the same plasma frequency to first order in the fine structure constant, but the classical treatment does not allow for pair production damping which occurs at high temperatures. The fluid treatment gives incorrect dispersion relations for ultrarelativistic plasmas even in flat space-time, however the general features (time dependence, structure of the dispersion relations) are the same as for the kinetic theory treatment.

IV. NONRELATIVISTIC LIMIT

A. Formulation

Now we turn to the limit of nonrelativistic particles. There are two scenarios we consider here, "prerecombination," where the temperature and the expansion are both dominated by the photons,

$$T \sim R^{-1} \quad R = \left(\frac{t}{t_i}\right)^{\frac{1}{2}} = \frac{\eta}{2t_i} \ , \tag{4.1}$$

and "postrecombination," where both are dominated by the plasma, which has now mostly decoupled from thermal photons,

$$T \sim R^{-2} \quad R = \left(\frac{t}{t_i}\right)^{\frac{2}{3}} = \frac{\eta^2}{9t_i^2} \ . \tag{4.2}$$

In the above equations t_i is an arbitrary fixed time. We will assume for simplicity that it is of the same order as t, so that R is of order unity, the comoving coordinates x, y, and z correspond to approximately physical (proper) distances, and $\bar{\omega}$ and $\bar{\mathbf{k}}$ are of the same order as the physically measured ω and $\hat{\mathbf{k}}$. This makes the simplifying assumptions we will make, for example (4.23), more intuitively transparent.

There are a number of interesting effects in the nonrelativistic limit, related to the different rates of cooling of decoupled photons and matter (above). Contrary to Holcomb [3] the modes of an unmagnetized plasma do not simply redshift as free photons. This is evident from the different time dependences of the plasma frequency and free photon:

$$\text{plasma } \omega_p = \sqrt{\frac{4\pi n e^2}{m}} \sim R^{-\frac{3}{2}} \ , \tag{4.3}$$

$$\text{photon } \omega \sim R^{-1} \ . \tag{4.4}$$

In the nonrelativistic limit the expression for $\hat{\mathbf{v}}$ (2.44) becomes

$$\hat{\mathbf{v}} = \frac{\bar{\mathbf{P}}}{mR} \ . \tag{4.5}$$

This leads to factors of R in the kinetic equation (2.45) which are clearly not present in the flat space-time case, and are not removable by any coordinate or variable transformation known to the authors. Rather than attempting to solve the coupled Maxwell-Vlasov set of integro-differential equations numerically, we use a simpler, fluid approach as did Holcomb [3], which leads to coupled linear ordinary differential equations (ODE's) that are solvable analytically.

As in the last part of the previous section, the plasma is represented as a perfect fluid, governed by Eqs. (2.36), (2.37), (3.45)–(3.47). The expressions for ρ and p in an ideal nonrelativistic gas are

$$\rho = n(m + \tfrac{3}{2}T) \ , \tag{4.6}$$

$$p = nT \ , \tag{4.7}$$

$$m \gg T \ , \tag{4.8}$$

where T is the temperature and n is the FIDO observed number density, and obeys the continuity equation (3.56). A couple of important points should be made regarding the hydrodynamic approximation in the nonrelativistic limit, which follow on from the flat space-time treatment.

Firstly, although the fluid approach in flat space-time yields the correct functional form of the longitudinal modes of oscillation (Chap. 4 of [22])

$$\omega^2 = \omega_p^2 + O\left(\frac{k^2 T}{m}\right) \ , \tag{4.9}$$

the coefficient of k^2T/m term is found to be incorrect (5/3 instead of 3). This is due to the fluid approximation, and will follow on to our curved space-time calculation. A kinetic treatment would yield the correct coefficient, but as we have stated above, this does not appear feasible in curved space-time. Our results (below) indicate that the expansion of the Universe has no effect on this coefficient in the fluid approximation, so it appears likely that a kinetic treatment would give a coefficient of 3, unaffected by the expansion of the Universe.

The second point is that, as in the previous section, the perfect fluid (3.45)–(3.47), continuity (3.56), and the energy equation (2.36) imply the adiabatic equation

$$R^2 T = \kappa (R^3 n)^{\frac{2}{3}} \ , \tag{4.10}$$

which is easier to use. Note that if the photons are in thermal equilibrium with the plasma and have a greater heat capacity (as expected for the prerecombination era), then the temperature is determined by the photons (4.1) and the plasma is not strictly adiabatic, being continually heated by the thermal photons. For small oscillations at time scales much shorter than these processes, however, adiabaticity is a good approximation.

Equations (3.45)–(3.47) are linearized with respect to $\hat{\mathbf{v}}$ and substituted into (2.37), noting that $\mathbf{E}$ and $\mathbf{B}$ are also considered to be small quantities. As mentioned previously for the relativistic case, one possible extension of this work is to include an external magnetic field which is not assumed to be small. n and T are also linearized:

$$R^3 n = \bar{n} = \bar{n}_0 + \bar{n}_1 \ , \tag{4.11}$$

$$T = T_0 + T_1 \ , \tag{4.12}$$

so that

$$\frac{\bar{n}_1}{\bar{n}_0} = \frac{3}{2}\frac{T_1}{T_0} \ . \tag{4.13}$$

The force and continuity equations become

$$\bar{\mathbf{v}}' + i\bar{\mathbf{k}}R\frac{\bar{n}_1}{\bar{n}_0}\frac{5}{3}\frac{T_0}{m} = \frac{e}{m}\mathbf{E} \ , \tag{4.14}$$

$$i\bar{\mathbf{k}} \cdot \bar{\mathbf{v}} R^{-1} = -\frac{\bar{n}_1'}{\bar{n}_0} \ , \tag{4.15}$$

$$\text{with } \bar{\mathbf{v}} = R\hat{\mathbf{v}} \ , \tag{4.16}$$

after eliminating T_1 using the adiabatic equation (4.10) and assuming an $e^{i\bar{\mathbf{k}}\cdot\mathbf{x}}$ dependence in the linearized variables. These equations must then be solved in conjunction with the Maxwell equations. If more than one component of the plasma is being considered, then the above equations are written for each species. We consider only electrons; the generalization is quite straightforward.

B. Longitudinal oscillations

It is convenient to treat the longitudinal and transverse oscillations separately. For longitudinal oscillations ($\bar{\mathbf{v}}, \bar{\mathbf{k}}$, and $\bar{\mathbf{E}}$ all parallel), only Poisson's equation is needed from the Maxwell set:

$$i\bar{\mathbf{k}} \cdot \bar{\mathbf{E}} = 4\pi e \bar{n}_1 \quad . \tag{4.17}$$

This is substituted into (4.15), and the resulting expression for $\bar{\mathbf{v}}$ substituted in (4.14) to obtain

$$\bar{E}'' + \frac{R'}{R}\bar{E}' + \left(\frac{5}{3}\frac{T_0}{m}\bar{k}^2 + \frac{\bar{\omega}_p^2}{R}\right)\bar{E} = 0 \ , \tag{4.18}$$

$$\bar{\omega}_p^2 = \frac{4\pi\bar{n}e^2}{m} \quad . \tag{4.19}$$

The above equation may be solved in terms of Bessel functions for either of the cases (4.1) and (4.2), but the form of the solution is quite different:

$$\text{prerecombination } \bar{E} = Z_0\left[\sqrt{8t_i\left(\frac{5}{3}\frac{\bar{T}_0}{m}\bar{k}^2 + \bar{\omega}_p^2\right)}\eta^{\frac{1}{2}}\right] \ , \tag{4.20}$$

postrecombination

$$\bar{E} = \eta^{-1/2} Z_{\sqrt{\frac{1}{4}-9\bar{\omega}_p^2 t_i^2}}\left(\sqrt{135\frac{\bar{T}_0}{m}\bar{k}^2 t_i^4 \eta^{-1}}\right) \ , \tag{4.21}$$

where Z is any Bessel function and $\bar{T}_0$ is the time-independent value of T_0, that is,

$$\bar{T}_0 = T_0 R^{1 \text{ or } 2} \quad . \tag{4.22}$$

Any two linearly independent Bessel functions of the appropriate order may be used to solve (4.18). We choose Hankel functions $H^{(1)}$ and $H^{(2)}$, which most resemble positive and negative frequency exponentials. Noting that the plasma frequency is much higher than the reciprocal of the age of the Universe, that is,

$$\bar{\omega}_p \eta \gg 1, \tag{4.23}$$

$$\bar{\omega}_p t_i \gg 1, \tag{4.24}$$

the above solutions may be replaced by their asymptotic forms. The required expansions of Bessel functions for large argument and/or order are found by Watson [23], pp. 198 and 262–268, respectively.

For all ν, and $z \gg 1$ we have

$$H_\nu^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}}e^{i(z-\nu\pi/2-\pi/4)}\sum_{m=0}^{\infty}\frac{\Gamma(\nu+m+1/2)}{(-2iz)^m m!\Gamma(\nu-m+1/2)} \ , \tag{4.25}$$

$$H_\nu^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}}e^{-i(z-\nu\pi/2-\pi/4)}\sum_{m=0}^{\infty}\frac{\Gamma(\nu+m+1/2)}{(2iz)^m m!\Gamma(\nu-m+1/2)} \ , \tag{4.26}$$

where the "$\sim$" denotes that the series is asymptotic, and so must be truncated, with an error of the same order as the first omitted term.

If ν/i is real and much greater than 1, a careful reading of [23] gives

$$H_\nu^{(1)}(\nu \operatorname{sech}\gamma) \sim \sqrt{\frac{2i}{\pi\nu\tanh\gamma}}e^{\nu(\tanh\gamma-\gamma)-i\pi/4}\sum_{m=0}^{\infty}\frac{A_m\Gamma(m+1/2)}{(\nu\tanh\gamma/2)^m\Gamma(1/2)} \ , \tag{4.27}$$

$$H_\nu^{(2)}(\nu \operatorname{sech}\gamma) \sim \sqrt{\frac{2i}{\pi\nu\tanh\gamma}}e^{-\nu(\tanh\gamma-\gamma)+i\pi/4}\sum_{m=0}^{\infty}\frac{A_m\Gamma(m+1/2)}{(-\nu\tanh\gamma/2)^m\Gamma(1/2)} \ . \tag{4.28}$$

Here, A_m is an even polynomial in $\coth\gamma$. The first few are

$$A_0 = 1 \ , \tag{4.29}$$

$$A_1 = \tfrac{1}{8} - \tfrac{5}{24}\coth^2\gamma \ , \tag{4.30}$$

$$A_2 = \tfrac{3}{128} - \tfrac{77}{576}\coth^2\gamma + \tfrac{385}{3456}\coth^4\gamma \ . \tag{4.31}$$

Note that since $\operatorname{sech}\gamma$ is pure imaginary, $\tanh\gamma$ is real and greater than 1.

Omitting the irrelevant constant factors we find that the electric field is given by

$$\text{prerecombination } \bar{E} \sim \eta^{-1/4}e^{\pm i\sqrt{8t_i\eta[\bar{\omega}_p^2+(5\bar{T}_0\bar{k}^2)/(3m)]}} \ , \tag{4.32}$$

$$\text{postrecombination } \bar{E} \sim \eta^{-1/2}\left(9\bar{\omega}_p^2 t_i^2 + 135\frac{\bar{T}_0}{m}\bar{k}^2 t_i^4\eta^{-2}\right)^{-1/4}$$
$$\times \exp \pm i\left\{\sqrt{9\bar{\omega}_p^2 t_i^2 + 135\frac{\bar{T}_0}{m}\bar{k}^2 t_i^4 \eta^{-2}} - 3\bar{\omega}_p t_i \operatorname{arcsinh}\left[\frac{3\bar{\omega}_p t_i \eta}{\sqrt{135\bar{T}_0\bar{k}^2 t_i^4/m}}\right]\right\} \ . \tag{4.33}$$

These expressions may be related back to the physically measured electric field $\hat{E}$ and time t by (2.38), (4.1), and (4.2). Because we are using a fluid treatment, the above equations for the longitudinal modes are only valid for

$$\bar{\omega}_p^2 \gg \frac{\bar{T}_0}{m}\bar{k}^2 \ , \tag{4.34}$$

$$\bar{\omega}_p^2 \gg \frac{\bar{T}_0}{m}\bar{k}^2\frac{t_i^2}{\eta^2} \ . \tag{4.35}$$

From the above equations we can see that the amplitude of longitudinal oscillations in both radiation- and matter-dominated universes decays slightly faster than a free photon, for which $\bar{E}$ is constant.

The locally measured frequency ω is obtained by differentiating the argument of the exponential with respect to t. The result is

$$\text{prerecombination } \omega = \sqrt{\bar{\omega}_p^2 + \frac{5}{3}\frac{\bar{T}_0}{m}\bar{k}^2}\left(\frac{t}{t_i}\right)^{-3/4}$$

$$= \sqrt{\omega_p^2 + \frac{5}{3}\frac{T_0}{m}\hat{k}^2} \ , \tag{4.36}$$

$$\text{postrecombination } \omega = \sqrt{\bar{\omega}_p^2 + \frac{5\bar{T}_0\bar{k}^2}{3m}\left(\frac{t}{t_i}\right)^{-2/3}}\left(\frac{t}{t_i}\right)^{-1}$$

$$= \sqrt{\omega_p^2 + \frac{5}{3}\frac{T_0}{m}\hat{k}^2} \ . \tag{4.37}$$

In both cases the result is simply the flat space-time expression, which varies with time as the plasma becomes less dense. Note that although it is possible to guess this expression from the outset, based on arguments from general relativity, it is not possible to determine the decay of the amplitude, Eqs. (4.32) and (4.33) without going through the full calculation involving the asymptotics of the Bessel functions.

C. Transverse oscillations

Now we turn to the transverse oscillations ($\bar{\mathbf{k}}$ perpendicular to $\bar{\mathbf{v}}$ and $\bar{\mathbf{E}}$, which are parallel). The equation of continuity (4.15) immediately implies that

$$\bar{n}_1' = 0 \ , \tag{4.38}$$

and hence

$$\bar{n}_1 = 0 \ , \tag{4.39}$$

since any constant in $\bar{n}_1$ would appear in $\bar{n}_0$ by definition. Equation (4.14), and two Maxwell equations become

$$\bar{v}' = \frac{e}{m}\bar{E} \ , \tag{4.40}$$

$$\bar{E}' = -i\bar{k}\bar{B} - 4\pi\bar{n}_0 e\frac{\bar{v}}{R} \ , \tag{4.41}$$

$$\bar{B}' = -i\bar{k}\bar{E} \ , \tag{4.42}$$

noting the relative orientation of $\bar{\mathbf{E}}$ and $\bar{\mathbf{B}}$. Equations (4.40) and (4.42) show that

$$\bar{B}' = -\frac{i\bar{k}m}{e}\bar{v}' \ , \tag{4.43}$$

and hence

$$\bar{B} = -\frac{i\bar{k}m}{e}\bar{v} \ , \tag{4.44}$$

where the constant of integration has been dropped for a similar reason to the $\bar{n}_1$ case above — we are not considering a plasma with an overall magnetic field $\bar{\mathbf{B}}_0$ or a drift velocity $\bar{\mathbf{v}}_0$. $\bar{E}$ is eliminated using (4.40) to obtain

$$\bar{v}'' + \left(\bar{k}^2 + \frac{\bar{\omega}_p^2}{R}\right)\bar{v} = 0 \ , \tag{4.45}$$

with $\bar{\omega}_p^2$ defined as in (4.19). The expressions for the electric and magnetic fields can be obtained from the solutions in terms of $\bar{v}$ by differentiation, using the above equations. Note that if the expansion of the Universe is ignored ($R = 1$), the equation for $\bar{v}$ reduces to the cold plasma result,

$$\omega^2 = \omega_p^2 + k^2 \ , \tag{4.46}$$

which is accurate up to temperature corrections.

The solutions to the above for the (4.1) and (4.2) cases may be found to be

prerecombination $\bar{v}$

$$= (-2i\bar{k}\eta)e^{i\bar{k}\eta}M\left(1 - i\frac{t_i\bar{\omega}_p^2}{\bar{k}}, 2, -2i\bar{k}\eta\right) \ , \tag{4.47}$$

or

$$(-2i\bar{k}\eta)e^{i\bar{k}\eta}U\left(1 - i\frac{t_i\bar{\omega}_p^2}{\bar{k}}, 2, -2i\bar{k}\eta\right) \ , \tag{4.48}$$

$$\text{postrecombination } \bar{v} = \sqrt{\eta}Z_{\sqrt{\frac{1}{4}-9\bar{\omega}_p^2 t_i^2}}(\bar{k}\eta) \ , \tag{4.49}$$

where Z is any Bessel function, M is a confluent hypergeometric function, and U is a logarithmic solution to the confluent hypergeometric differential equation. M and U are linearly independent solutions. For more information about these functions, the reader is referred to [24] and pp. 504–535 of [25].

In the long wavelength limit we require expansions of $M(a,b,z)$ and $U(a,b,z)$ for large (complex) a. The appropriate analysis of these functions and their differential equations is contained in pp. 76–81 of [24]. The result is

$$M\left(\frac{u^2}{4} + \frac{b}{2}, b, z^2\right) \sim \Gamma(b)u^{1-b}z^{1-b}2^{b-1}e^{z^2/2}\left(I_{b-1}(uz)\sum_{s=0}^{\infty}\frac{A_s(z)}{u^{2s}} + \frac{1}{u}I_b(uz)\sum_{s=0}^{\infty}\frac{B_s(z)}{u^{2s}}\right) \ , \tag{4.50}$$

$$U\left(\frac{u^2}{4}+\frac{b}{2},b,z^2\right)\sim\frac{u^{b-1}z^{1-b}2^{2-b}e^{z^2/2}}{\Gamma(u^2/4+b/2)}\left(K_{b-1}(uz)\sum_{s=0}^{\infty}\frac{A_s(z)}{u^{2s}}-\frac{1}{u}K_b(uz)\sum_{s=0}^{\infty}\frac{B_s(z)}{u^{2s}}\right)\quad, \tag{4.51}$$

where I and K are modified Bessel functions, and the coefficients A_s and B_s are obtained using recursion relations; the first few are

$$A_0(z)=1\quad, \tag{4.52}$$

$$B_0(z)=\frac{z^3}{6}\quad, \tag{4.53}$$

$$A_1(z)=(b-2)\frac{z^2}{6}+\frac{z^6}{72}\quad, \tag{4.54}$$

$$B_1(z)=-b(b-2)\frac{z}{3}-\frac{z^5}{15}+\frac{z^9}{1296}\quad. \tag{4.55}$$

Substituting the variables pertaining to our problem, we find that the M solution is written in terms of $I_1\left[i4\bar{\omega}_p t_i\left(\frac{t}{t_i}\right)^{1/4}\right]$ and $I_2\left[i4\bar{\omega}_p t_i\left(\frac{t}{t_i}\right)^{1/4}\right]$ multiplied by an expansion which is useful (judging by its most divergent terms) when

$$\bar{k}\eta\ll\left[2\bar{\omega}_p t_i\left(\frac{t}{t_i}\right)^{1/4}\right]^{1/2}\quad. \tag{4.56}$$

The U solution is of the same form with I replaced by K. These expressions are written in terms of $H^{(1)}$ and $H^{(2)}$, linear combinations are taken so that one solution is purely $H^{(1)}$, and the other purely $H^{(2)}$, and the large argument expansions of these functions (4.25) and (4.26) are used to find the decay of the amplitude and frequency, as before. Care must be taken to include the second term in the Hankel function expansions, even though only terms of leading order in $(\bar{\omega}_p\sqrt{\eta t_i})^{-1}$ are retained in the final expression. The result is

$$\bar{v}_{\rm rms}\sim\eta^{1/4}\left[1+O\left(\frac{\bar{k}^2}{\bar{\omega}_p^4t_i^2}(\bar{k}\eta)^{0\text{ or }2}\right)\right]\quad, \tag{4.57}$$

$$\begin{aligned}\omega&=\bar{\omega}_p\left(\frac{t}{t_i}\right)^{-3/4}\left[1+\frac{1}{2}\left(\frac{\bar{k}}{\bar{\omega}_p}\right)^2\left(\frac{t}{t_i}\right)^{1/2}\right.\\&\qquad\left.-\frac{1}{8}\left(\frac{\bar{k}}{\bar{\omega}_p}\right)^4\left(\frac{t}{t_i}\right)+\cdots\right]\\&=\omega_p\left[1+\frac{\hat{k}^2}{2\omega_p^2}-\frac{\hat{k}^4}{8\omega_p^4}+\cdots\right]\quad.\end{aligned} \tag{4.58}$$

This result requires some explanation. There are two dimensionless parameters in this problem, for example, $\bar{k}\eta$ and $\bar{\omega}_p\sqrt{\eta t_i}$. The value of t/t_i is irrelevant, since t_i is arbitrary, as discussed previously. The original series in [24] contains terms of differing order in these parameters at every order in the expansion. All these terms are small if the condition (4.56) is met. However, when the logarithm is taken to separate out the value of ω, cancellations occur, leading to the expansion given above, in the single parameter $(\bar{k}^2/\bar{\omega}_p^2)(t/t_i)^{1/2}$, together with very small corrections of order $(\bar{\omega}_p\sqrt{\eta t_i})^{-1}$, which are not shown above. The expression $(\bar{k}^2/\bar{\omega}_p^2)(t/t_i)^{1/2}$ is a physically reasonable parameter, with a well-understood nonrelativistic limit, unlike the parameter appearing in (4.56). Similarly, the expansion for the magnitude of $\bar{v}$ appears to cancel completely, leaving only the first term. The order of the next possible term is given, depending on the value of $\bar{k}\eta$, which gives, up to a constant factor, the number of wavelengths in the visible Universe. Thus the 0 is applicable if $\bar{k}\eta\ll 1$, and the 2 is applicable if $\bar{k}\eta\gg 1$. The expression for ω above reduces to the small k expansion of the flat space-time result (4.46).

The following expansions of M and U, required for the short wavelength (photonlike) limit, are found on p. 508 of [25].

$$\begin{aligned}M(a,b,z)&\sim\frac{\Gamma(b)}{\Gamma(b-a)}e^{\pm i\pi a}z^{-a}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(1+a-b+n)}{\Gamma(a)\Gamma(1+a-b)\Gamma(n+1)}(-z)^{-n}\\&\quad+\frac{\Gamma(b)}{\Gamma(a)}e^z z^{a-b}\sum_{n=0}^{\infty}\frac{\Gamma(b-a+n)\Gamma(1-a+n)}{\Gamma(b-a)\Gamma(1-a)\Gamma(n+1)}(z)^{-n}\quad,\end{aligned} \tag{4.59}$$

$$U(a,b,z)\sim z^{-a}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(1+a-b+n)}{\Gamma(a)\Gamma(1+a-b)\Gamma(n+1)}(-z)^{-n}\quad. \tag{4.60}$$

Here, the plus sign applies to $-\pi/2<\arg z<3\pi/2$ and the minus sign applies to $-3\pi/2<\arg z\le-\pi/2$. These asymptotic series are useful (judging by the most divergent terms) under the condition

$$\bar{k}\eta\gg\left[2\bar{\omega}_p t_i\left(\frac{t}{t_i}\right)^{1/4}\right]^{4/3}\quad. \tag{4.61}$$

Taking the above expansions to the $n=1$ term, and using an analysis similar to that used in the long wavelength case, we obtain, for the amplitude and frequency,

$$\bar{v}_{\rm rms} \sim 1 - \frac{\bar{\omega}_p^2}{4\bar{k}^2}\left(\frac{t}{t_i}\right)^{-1/2} , \tag{4.62}$$

$$\omega = \bar{k}\left(\frac{t}{t_i}\right)^{-1/2}\left[1 + \frac{\bar{\omega}_p^2}{2\bar{k}^2}\left(\frac{t}{t_i}\right)^{-1/2} - \frac{\bar{\omega}_p^4}{8\bar{k}^4}\left(\frac{t}{t_i}\right)^{-1} + \cdots\right]$$
$$= \hat{k}\left[1 + \frac{\omega_p^2}{2\hat{k}^2} - \frac{\omega_p^4}{8\hat{k}^4} + \cdots\right] . \tag{4.63}$$

Again there is a single expansion parameter, and the result reduces to the flat space-time result (4.46), at least, to this order in the expansion. In this case there are also small corrections to the amplitude, showing that the photon's amplitude does not decay quite as rapidly as a free photon, but that higher energy photons behave more like free photons, as physically one would expect.

For the postrecombination case the expansions (4.27)–(4.31) may be used, resulting in

$$\text{postrecombination } \bar{v} \sim \eta^{1/2}\left(9\bar{\omega}_p^2 t_i^2 + \bar{k}^2\eta^2\right)^{-1/4} \exp \pm i\left\{\sqrt{9\bar{\omega}_p^2 t_i^2 + \bar{k}^2\eta^2} - 3\bar{\omega}_p t_i \text{arcsinh}\left[\frac{3\bar{\omega}_p t_i}{\bar{k}\eta}\right]\right\} . \tag{4.64}$$

This is fairly similar to the longitudinal result (4.33). The FIDO observed angular frequency ω, is found to be

$$\omega = \sqrt{\bar{\omega}_p^2 + \bar{k}^2\left(\frac{t}{t_i}\right)^{2/3}}\left(\frac{t}{t_i}\right)^{-1} = \sqrt{\omega_p^2 + \hat{k}^2} . \tag{4.65}$$

Again we see that the locally measured frequency is given by the flat space-time result, which varies with time as the Universe expands.

V. CONCLUSION

We have reformulated the general relativistic Vlasov-Maxwell kinetic equations and also the equations for a perfect fluid with electromagnetic interactions in the spatially flat Robertson-Walker metric, in a form suitable for analytic solution.

At ultrarelativistic temperatures ($T \gg m$) the main result of [2] is recovered, that is, that the plasma modes redshift in exactly the same manner as a free photon. We have shown that the same result holds when a more detailed kinetic treatment is used, and made comparisons between the use of kinetic and fluid equations, and between classical and quantum descriptions of the plasma.

At nonrelativistic temperatures ($T \ll m$) we do not recover the zero magnetic field results of [3], in which it is claimed that the plasma modes redshift like a free photon in this regime, also. Rather, the rates of redshifting of the plasma frequency (4.3) and free photon (4.4) compete, leading to complicated solutions (4.20), (4.21), (4.47)–(4.49), which we have derived for both radiation- and matter-dominated expansion rates. In each of these cases we have shown that the locally measured frequency reduces to the flat space-time result. This follows from general physical arguments, however, in contrast to the results of [2] and [3], a full analysis of the plasma modes, including the effect of the expansion on the amplitude of oscillation, requires the complete solutions, which involve Bessel functions and confluent hypergeometric functions not expressible in terms of elementary functions.

The most straightforward extension of this work is to include an external magnetic field, thus generalizing the five modes of oscillation found in nonrelativistic magnetized plasmas [22] to the general relativistic arena, and providing insight into the dynamics of cosmological magnetic fields.

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