Evolution of cosmic string configurations

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(Received 22 July 1993)

We extend and develop our previous work on the evolution of a network of cosmic strings. The new treatment is based on an analysis of the probability distribution of the end-to-end distance, or *extension*, of a randomly chosen segment of left-moving string of given length. The description involves three distinct length scales: ξ, related to the overall string density, $\bar{\xi}$, the persistence length along the string, and ζ, describing the small-scale structure, which is an important feature of the numerical simulations that have been done of this problem. An evolution equation is derived describing how the distribution develops in time due to the combined effects of the universal expansion, of intercommuting and loop formation, and of gravitational radiation. With plausible assumptions about the unknown parameters in the model, we confirm the conclusions of our previous study that if gravitational radiation and small-scale structure effects are neglected the two dominant length scales both scale in proportion to the horizon size. When the extra effects are included, we find that while ξ and $\bar{\xi}$ grow, ζ initially does not. Eventually, however, it does appear to scale, at a much lower level, due to the effects of gravitational back reaction.

PACS number(s): 98.80.Cq

I. INTRODUCTION

Cosmic strings are topological defects that may be formed at a phase transition very early in the history of the Universe [1,2]. Because they are stable, they may survive to a much later epoch and thus provide one of the few direct links between the physics of the very early Universe and recent cosmology. In particular, they may play an important role in generating large-scale structure in the Universe [3–6]. Observational tests of the idea include limits on the gravitational radiation emitted by collapsing loops and oscillating strings [7–9], gravitational lensing of characteristic form [10–12], and predicted anisotropy in the cosmic microwave background [13].

All of these limits depend on our understanding of the process of evolution of a network of cosmic strings. One very important question is whether this evolutionary process leads to a "scaling" regime, in which the characteristic length scales describing the string network increase in proportion to the horizon distance [14–16]. There have been numerical studies of this problem by at least three different groups [17–19]. There is a wide measure of agreement between these groups. All find evidence for scaling of the large-scale structure. There is some disagreement over the predictions concerning the small-scale structure, but all groups agree that there is a substantial amount of structure on scales much less than the scale of the long-string network. It has certainly become apparent that a simple description in terms of a single scale is inadequate.

In earlier work [20,21] we sought to develop an analytic approach to this problem, to complement the numerical studies. In this work, we identified two distinct length scales: ξ, related to the overall density of long strings, and $\bar{\xi}$, the distance over which the strings are correlated in direction. It became apparent, however, that this formalism is inadequate to represent the small-scale structure observed in the simulations. Accordingly, we have extended our treatment by incorporating a third length scale ζ, which describes the structure on the smallest scales.

This treatment, which we present here, is based on a probability analysis of the string configuration, in particular the probability distribution of the end-to-end distance, or *extension*, $\mathbf{r}$, of a randomly chosen segment of string of length l, more precisely, of *left-moving* string. As before, we use null (characteristic) coordinates on the string world sheet. In flat space, the left and right movers are completely decoupled; the universal expansion introduces a weak coupling between them.

We should mention that an alternative approach to obtaining the evolution equations has been proposed by Embacher, in which a path integral formalism is used to obtain the probability distribution for the network of strings in flat space [22]. This promising approach appears to produce similar results to our own in the circumstances where comparisons are possible.

In Sec. II, we review the evolution equations developed earlier and introduce the fundamental probability distribution $p[\mathbf{r}(l)]$ on which our treatment is based. We aim to derive an evolution equation for this quantity, including terms representing the effects of stretching, gravitational radiation, intercommuting of long strings, and formation of loops.

For all except the very smallest scales, it is reasonable to assume that the probability distribution is Gaussian, characterized by the variance $\overline{\mathbf{r}^2} = K(l)$, say. Our evolution equation for the probability p effectively reduces to an equation for this function K. The scales associated with the large- and small-scale structure on the strings, $\bar{\xi}$ and ζ, are defined in terms of K, so their evolution equations follow from that for K.

In Sec. III we analyze the Gaussian ansatz in more detail and derive values of various expectation values that will be required in the subsequent analysis. In particular, we need a number of joint and conditional expectation values.

The basic equations describing the effects of stretching, intercommuting, and loop formation are reviewed in Sec. IV. One of the very important things that emerges from this discussion is that these effects cannot properly be treated in isolation; they interact with each other in very complex ways.

Another important lesson is the crucial significance of the correlations that develop between left and right movers. These are discussed in Sec. V. In our previous work [20,21] we showed that the stretching process generates such a correlation, and derived a relation between the mean value $\alpha = -\overline{\mathbf{p}\cdot\mathbf{q}}$ (where $\mathbf{p}$ and $\mathbf{q}$ are unit vectors along the left- and right-moving strings) and the length scale $\bar{\xi}$. More recently, however, we have realized that this is not the only significant process involved. In particular, loop formation also introduces correlations between the $\mathbf{p}$ and $\mathbf{q}$ vectors, by preferentially eliminating nearly matching pairs. The discussion of this process in Sec. V is based on an analysis of the angular probability distribution of these vectors.

A similar process also introduces angular correlations on somewhat larger scales. This is important in determining the rate of loop formation. In Sec. VI, we begin the construction of the various terms in our evolution equation for the probability distribution $p[\mathbf{r}(l)]$ by treating this process.

Section VII is devoted to the estimation of the parameters appearing in the stretching term. Then in Sec. VIII we deal with the effect of back reaction from gravitational radiation, which operates on a very different scale from the other effects but is ultimately of great importance in the long-term evolution of the system.

Finally, in Sec. IX all these terms are brought together to yield the overall evolution equations for the three length scales. The equations involve four dimensionless functions of the scale ratios as well as several dimensionless constants. Before analyzing the solutions of the evolution equations, we study the behavior of these functions in various regions of parameter space and try to estimate the constants.

We then show that so long as the effects of gravitational radiation are negligible, the system behaves essentially as predicted by our previous study [21]. If we start with all three length scales approximately equal (and somewhat smaller than the horizon), ξ and later $\bar{\xi}$ will start to grow and, under reasonable assumptions, will evolve to a scaling regime in which they are approximately equal and proportional to t. The role of the important parameter q in our previous study is here played by a ratio between two of the dimensionless functions; it is, however, no longer a constant.

There is another important parameter k, also somewhat similar to q (or, more precisely, $q-1$), but defined in terms of the small-scale structure. The value of k is crucial in determining the behavior of the third length scale ζ. If k were large enough, a complete scaling regime could be reached, with ξ, $\bar{\xi}$, and ζ all of comparable magnitude. This is not the type of behavior seen in the simulations. A far more likely scenario is that k is less than the critical value. Then, while ξ and $\bar{\xi} \propto t$, the third length scale ζ does not grow rapidly, and the ratio $\zeta/\bar{\xi}$ becomes very small. Eventually, when it is of order $\Gamma G\mu$, the gravitational radiation effect becomes significant and ζ starts to grow. Here Γ is the numerical factor describing the efficiency of gravitational radiation, estimated to be of order 10^2 [23–28].

Finally, we tackle the important question of whether a new scaling regime is reached, as various authors have already suggested [29–31], due to the effects of gravitational radiation. It turns out that the answer to this question depends crucially on the value of another of the dimensionless constants we have introduced, $\hat{C}$, which relates to the effect of gravitational radiation on small-scale structure. Gravitational back reaction tends to smooth out the small-scale kinks and thus to make ζ grow; $\hat{C}$ defines the rate at which it does so. The essential condition for complete scaling is that $\hat{C}$ exceeds a critical value.

We also consider the question of stability and show that if scaling is achieved it should be stable.

The conclusions are summarized in Sec. X. We discuss in particular a number of remaining open questions and prospects for future study.

II. EVOLUTION EQUATIONS

We shall consider only the case of a *flat* Robertson-Walker universe, with metric

$$ds^2 = dt^2 - R^2(t)\,d\mathbf{x}^2 = R^2(\tau)[d\tau^2 - d\mathbf{x}^2], \qquad (2.1)$$

where R is the Robertson-Walker scale factor, $\tau = x^0$ is the "conformal" time, and $\mathbf{x} = \{x^k\}$ are *comoving* spatial coordinates.

A. The null coordinates

For completeness, we recall here some of the basic formalism described by Kibble and Copeland (KC) [20]. It is convenient to use null (characteristic) coordinates u, v on the world sheet of the string. (In flat space, these are the coordinates $u = t + \sigma$, $v = t - \sigma$, where σ is the length along the string.) We denote partial derivatives with respect to these coordinates by subscripts:

$$x_u = \frac{\partial x}{\partial u}, \qquad x_v = \frac{\partial x}{\partial v}. \qquad (2.2)$$

The null condition is

$$x_u^2 = x_v^2 = 0. \qquad (2.3)$$

In these coordinates the Nambu-Goto action is

$$I = -\mu \int du\, dv\, R^2(\tau) x_u \cdot x_v, \tag{2.4}$$

where $\tau = x^0(u,v)$, and where the dot implies a scalar product in the Minkowski metric.

In terms of the space-time coordinates $(\tau, \mathbf{x})$ the equations of motion of the string may be written as

$$\begin{aligned} x^0_{uv} &= -h_0(x^0_u x^0_v + \mathbf{x}_u \cdot \mathbf{x}_v), \\ \mathbf{x}_{uv} &= -h_0(x^0_u \mathbf{x}_v + \mathbf{x}_u x^0_v), \end{aligned} \tag{2.5}$$

where

$$h_\mu = \frac{\partial_\mu R}{R} = \delta_{\mu 0} \frac{1}{R}\frac{dR}{d\tau} = \delta_{\mu 0} \frac{dR}{dt} = \delta_{\mu 0} H R, \tag{2.6}$$

where H is the Hubble parameter.

It is convenient to define the unit vectors

$$\mathbf{p} = \frac{\mathbf{x}_u}{x^0_u}, \quad \mathbf{q} = \frac{\mathbf{x}_v}{x^0_v}, \tag{2.7}$$

which satisfy the equations of motion

$$\begin{aligned} \mathbf{p}_v &= -h_0 x^0_v (\mathbf{q} - \mathbf{p}\mathbf{p}\cdot\mathbf{q}), \\ \mathbf{q}_u &= -h_0 x^0_u (\mathbf{p} - \mathbf{q}\mathbf{q}\cdot\mathbf{p}). \end{aligned} \tag{2.8}$$

Let us now consider a left-moving segment of string, i.e., a segment bounded by two values of the null coordinate u, say u_1 and u_2.

The total physical extension, or end-to-end distance, of this segment at a given conformal time, τ_0 say, is

$$\begin{aligned} \mathbf{r}_{\rm tot} &= R \int_{u_1}^{u_2} du \left(\frac{\partial \mathbf{x}}{\partial u}\right)_\tau \\ &= R \int_{u_1}^{u_2} du \left[\mathbf{x}_u + \mathbf{x}_v \left(\frac{\partial v}{\partial u}\right)_\tau\right]. \end{aligned} \tag{2.9}$$

In the second term, we can change the variable to v and obtain

$$\mathbf{r}_{\rm tot} = R \int_{u_1}^{u_2} du\, \mathbf{x}_u - R \int_{v_2}^{v_1} dv\, \mathbf{x}_v \equiv \tfrac{1}{2}(\mathbf{r}_{\rm l} - \mathbf{r}_{\rm r}), \tag{2.10}$$

say. (Recall that if $u_1 < u_2$ then $v_1 > v_2$.) The two terms in (2.10) are the left-moving and right-moving extensions, respectively. From now on, we consider only the left-moving term, and write

$$\mathbf{r} \equiv \mathbf{r}_{\rm l}. \tag{2.11}$$

The total physical length of the left-moving segment (measured along the string) may be defined to be

$$l = 2R \int_{u_1}^{u_2} du\, x^0_u(u, v(u, \tau_0)). \tag{2.12}$$

Note that because the time coordinates at the two ends have been chosen equal, we necessarily have, by analogy with (2.10),

$$l = 2R \int_{u_1}^{u_2} du\, x^0_u = 2R \int_{v_2}^{v_1} dv\, x^0_v. \tag{2.13}$$

In this sense, therefore, the lengths of left- and right-moving string are exactly equal.

It is useful to note that that the extension $\mathbf{r}$ can also be written in terms of the unit vector $\mathbf{p}$, defined by (2.7), in the form

$$\mathbf{r} = \int_0^l dy\, \mathbf{p}(y), \tag{2.14}$$

where y is a coordinate along the string defined by

$$dy = 2R x^0_u\, du. \tag{2.15}$$

B. Probability distribution of extension

Consider a large comoving volume V, and let L be the total length of (left-moving) string within V. It is convenient to introduce the characteristic interstring distance ξ defined by

$$\xi^2 = \frac{V}{L}. \tag{2.16}$$

If we introduce a discretization scale δ, then L must be thought of as made up of N small segments, each of length δ, with $N = L/\delta$. (There will be an equal length of right-moving string, but it is convenient to concentrate on one or the other.)

Let us choose a particular length scale l and consider the probability distribution for the end-to-end distance (or *extension*) $\mathbf{r}$: $p[\mathbf{r}(l)]d^3\mathbf{r}$ is the probability that a randomly chosen segment of length l will have extension $\mathbf{r}$ within the small volume $d^3\mathbf{r}$. Note that in contrast with KC [20] and Copeland, Kibble, and Austin (CKA) [21] we are here using real, rather than comoving, lengths, i.e., in terms of our previous notation,

$$\mathbf{r} = R\mathbf{a}, \quad l = Rs. \tag{2.17}$$

The number of possible starting points within V is N, so the expected number of segments with length between l and $l + dl$ and extension $\mathbf{r}$ within $d^3\mathbf{r}$ is

$$\frac{L}{\delta}\frac{dl}{\delta} p[\mathbf{r}(l)] d^3\mathbf{r}. \tag{2.18}$$

Because the distribution is highly nonrandom, there will be many segments with very similar values of $\mathbf{r}$ arising from overlapping selections, especially where at the end points the orientations happen to be similar.

Strictly speaking, some choices of starting point will not yield segments entirely within V; they will extend beyond the boundary, but will be matched by a similar number of segments entering V having originated outside it.

Our object is, first, to derive an equation for the rate of change of $p[\mathbf{r}(l)]$, of the form

$$\frac{\partial p}{\partial t} = \left(\frac{\partial p}{\partial t}\right)_{\rm str} + \left(\frac{\partial p}{\partial t}\right)_{\rm GR} + \left(\frac{\partial p}{\partial t}\right)_{\rm LSI} + \left(\frac{\partial p}{\partial t}\right)_{\rm loops}, \tag{2.19}$$

and then to determine the nature of its solution. The various terms on the right represent respectively the effects of stretching (due to the universal expansion), of gravitational radiation (GR) (back reaction), of long-string intercommuting (LSI), and of loop production.

The separation between the intercommuting and loop-production terms is to some extent arbitrary: there is an upper limit to the size of a "large loop." However, rather than imposing a sharp cutoff, we shall aim to count only those loops that survive the reconnection process. The reconnection probability provides a natural cutoff at a scale determined by the string density. There is no need to consider loops larger than this separately from the long-string network; such loops do not "know" that they are closed. They are rather likely to suffer reconnection. Their formation and reconnection may simply be regarded as instances of long-string intercommuting.

In addition to the equation (2.19) for p, we shall also need an equation for the rate of change of L:

$$\frac{\partial L}{\partial t} = \left(\frac{\partial L}{\partial t}\right)_{\rm str} + \left(\frac{\partial L}{\partial t}\right)_{\rm GR} + \left(\frac{\partial L}{\partial t}\right)_{\rm loops}. \tag{2.20}$$

There is no term representing long-string intercommuting, which has no effect on L.

The last term in (2.20), and the last two in (2.19), are each a combination of a negative term representing the effect of removal or destruction of segments and a positive term representing the corresponding creation of new segments. It is sometimes easier to consider the change in the expected number of segments, (2.18), due to one of these processes. It must be remembered that they affect L as well as p, but we note that from the rate of change of Lp we can easily find those of L and p separately, by using the normalization condition. For example,

$$\left(\frac{\partial L}{\partial t}\right)_{\rm loops} = \int d^3\mathbf{r} \left(\frac{\partial (Lp)}{\partial t}\right)_{\rm loops}, \tag{2.21}$$

and of course

$$\left(\frac{\partial p}{\partial t}\right)_{\rm loops} = \frac{1}{L}\left(\frac{\partial (Lp)}{\partial t}\right)_{\rm loops} - \frac{p}{L}\left(\frac{\partial L}{\partial t}\right)_{\rm loops}. \tag{2.22}$$

The equations we obtain, not surprisingly, turn out to be very complicated. It is unlikely that an exact solution can be found. Our aim, however, is to find approximate solutions valid in special regions of interest, and in particular to establish whether a scaling solution exists. If it does, p should tend asymptotically to a limiting form

$$p[\mathbf{r}(l),t] \sim \frac{1}{t^3} p_{\rm scal}\left[\frac{\mathbf{r}}{t}\left(\frac{l}{t}\right)\right], \qquad t\to\infty. \tag{2.23}$$

C. The Gaussian ansatz

For all except the smallest values of l, it is reasonable to assume that p is a Gaussian:

$$p[\mathbf{r}(l)] = \left(\frac{3}{2\pi K(l)}\right)^{3/2} \exp\left(-\frac{3}{2}\frac{\mathbf{r}^2}{K(l)}\right). \tag{2.24}$$

Here $K(l)$ is the mean square extension:

$$K(l) = \overline{\mathbf{r}^2} = \int d^3\mathbf{r}\, \mathbf{r}^2 p[\mathbf{r}(l)]. \tag{2.25}$$

From Eq. (2.19) for $\partial p/\partial t$ we can derive a corresponding equation for $\partial K/\partial t$. We aim to show that its solution approaches the scaling form

$$K(l,t) \sim t^2 K_{\rm scal}\left(\frac{l}{t}\right), \qquad t\to\infty. \tag{2.26}$$

For very large values of l, $l \gg t$, we expect the string to behave like a Brownian random walk, so that K becomes a linear function of l:

$$K(l,t) \sim 2\bar{\xi}(t) l, \qquad l \gg t, \tag{2.27}$$

with persistence length $\bar{\xi} \propto t$, but for smaller values of l, the variation of K with l will be more rapid, approaching $K \sim l^2$ as $l \to 0$. (Of course, for such very small values, the Gaussian approximation breaks down.)

In the next section, we examine the Gaussian ansatz in more detail and derive some of its consequences. In particular, we evaluate various expectation values that will be needed in the subsequent analysis. Then in the following sections, we discuss each of the terms in (2.19) and (2.20) in turn.

III. THE GAUSSIAN ANSATZ

We shall assume that except for very small values of l the probability distribution of the extension is a Gaussian, (2.24). An important function, introduced by KC [20], is the correlation function

$$f(y) = \overline{\mathbf{p}(0)\cdot\mathbf{p}(y)}, \tag{3.1}$$

where y is the path-length variable introduced in (2.15). This function is clearly related to the variance of $\mathbf{r}$. In fact, we have

$$K(l) \equiv \overline{\mathbf{r}^2} = 2\int_0^l dy (l-y) f(y), \tag{3.2}$$

from which it also follows that

$$K'(l) = 2\int_0^l dy\, f(y) \tag{3.3}$$

and

$$K''(l) = 2f(l), \tag{3.4}$$

where the primes denote derivatives with respect to l.

A. An illustrative model

We shall not make any specific assumption about the form of K as a function of l. However, it is useful to have a specific model in mind, to indicate the kind of behavior we might expect.

The simplest model, described by KC, is based on the hypothesis that there is a single scale in the problem:

$$f(l) = e^{-Dl} \tag{3.5}$$

for some constant D. (Note, however, that because of the change from comoving to real lengths [see Eq. (2.17)], D differs from the corresponding quantity in KC by a factor R.) Then

$$K(l) = \frac{2}{D^2}(e^{-Dl} - 1 + Dl). \tag{3.6}$$

The correlation length $\bar{\xi}$ is defined by

$$\bar{\xi} = \int_0^\infty dy\, f(y), \tag{3.7}$$

so that for large l, K is given by (2.27). In the case of the single-scale model,

$$\bar{\xi} = \frac{1}{D}. \tag{3.8}$$

This model does not describe very well the structure seen in the simulations. A better choice might be a model described by two scales:

$$f(l) = (1-w)e^{-Al} + we^{-Bl}, \tag{3.9}$$

where w is a constant in the range $0 < w < 1$ and we assume $A \gg B$. Here the small and large scales are $1/A$ and $1/B$, respectively. The expression for K is similar to (3.6):

$$K(l) = \frac{2(1-w)}{A^2}(e^{-Al} - 1 + Al) + \frac{2w}{B^2}(e^{-Bl} - 1 + Bl). \tag{3.10}$$

In this case, the correlation length, defined by (3.7), is

$$\bar{\xi} = \frac{1-w}{A} + \frac{w}{B} \approx \frac{w}{B}. \tag{3.11}$$

Note that $\bar{\xi}$ is dominated by the large scale; the existence of the small-scale structure does affect it, via the constant w, but the size of the small scale $1/A$ is more or less immaterial.

It is useful to define another characteristic length scale, ζ, related to the small-scale structure, by

$$\frac{1}{\zeta} = -\left.\frac{\partial f(l)}{\partial l}\right|_{l=0}. \tag{3.12}$$

For the single-scale model, $\zeta = \bar{\xi}$, but in the two-scale case,

$$\frac{1}{\zeta} = (1-w)A + wB, \tag{3.13}$$

so

$$\zeta \approx \frac{1}{(1-w)A}. \tag{3.14}$$

For the two-scale model, we may distinguish three distinct regions, in which the approximate forms of K (including first-order corrections) are

$$\begin{aligned}
&\text{(i)} \quad l \ll \tfrac{1}{A}: && K \approx l^2 - \tfrac{(1-w)Al^3}{3};\\
&\text{(ii)} \quad \tfrac{1}{A} \ll l \ll \tfrac{1}{B}: && K \approx wl^2 - \tfrac{wBl^3}{3} + \tfrac{2(1-w)l}{A};\\
&\text{(iii)} \quad l \gg \tfrac{1}{B}: && K \approx \tfrac{2wl}{B} - \tfrac{2w}{B^2} + \tfrac{2(1-w)l}{A}.
\end{aligned} \tag{3.15}$$

Leading terms for very small and very large l may also be written in the form

$$\begin{aligned}
&\text{(i)} \quad l \ll \tfrac{1}{A}: && K \approx l^2 - \tfrac{l^3}{3\zeta};\\
&\text{(iii)} \quad l \gg \tfrac{1}{B}: && K \approx 2\bar{\xi}l - \tfrac{2\bar{\xi}^2}{w}.
\end{aligned} \tag{3.16}$$

We emphasize again, however, that this model is introduced for purely illustrative purposes; we make no specific assumptions at this stage about the form of K.

B. Higher moments

The requirement that the probability distribution is Gaussian means that all its moments are expressible in terms of the single function K. In particular, the variance of $\mathbf{r}^2$, the function

$$K_{(2)}(l) = \overline{(\mathbf{r}^2)^2} - (\overline{\mathbf{r}^2})^2, \tag{3.17}$$

is given by

$$K_{(2)}(l) = \tfrac{2}{3}K(l)^2. \tag{3.18}$$

The higher cumulants may be found from the cumulant generating function

$$\begin{aligned}
\mathcal{K}(z) \equiv \sum_{n=1}^{\infty} \frac{z^n}{n!} K_{(n)}(l) &= \ln\langle e^{z\mathbf{r}^2}\rangle \\
&= -\tfrac{3}{2}\ln\left[1 - \tfrac{2}{3}zK(l)\right].
\end{aligned} \tag{3.19}$$

C. Joint probabilities

In evaluating the various terms in $\partial p/\partial t$, we shall encounter not only the probability distribution $p[\mathbf{r}(l)]$ but also various more complicated joint probabilities. We need therefore to extend our assumptions to cover these.

Consider for example the configuration of two contiguous segments of lengths l_1 and l_2 illustrated in Fig. 1.

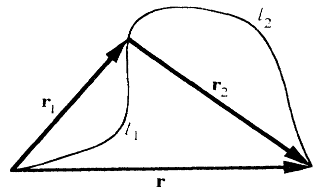

FIG. 1. Two contiguous segments.

We denote the joint probability of extensions $\mathbf{r}_1$ and $\mathbf{r}_2$ within small volumes $d^3\mathbf{r}_1$ and $d^3\mathbf{r}_2$ by

$$p[\mathbf{r}_1(l_1), \mathbf{r}_2(l_2)]\, d^3\mathbf{r}_1\, d^3\mathbf{r}_2. \tag{3.20}$$

For a Brownian process, this probability factorizes into the product $p[\mathbf{r}_1(l_1)] \cdot p[\mathbf{r}_2(l_2)]$, but this is possible only if $K(l)$ is a linear function of l, since it implies that $\overline{\mathbf{r}^2} = \overline{\mathbf{r}_1^2} + \overline{\mathbf{r}_2^2}$. In the general case, where this is not true, $\overline{\mathbf{r}_1 \cdot \mathbf{r}_2}$ is nonzero. In fact,

$$K(l_1, l_2) \equiv \overline{\mathbf{r}_1 \cdot \mathbf{r}_2} = \tfrac{1}{2}[K(l) - K(l_1) - K(l_2)], \tag{3.21}$$

where of course

$$l = l_1 + l_2. \tag{3.22}$$

It is interesting to examine the limiting forms of the expression (3.21) in our two-scale model. The leading terms in the three regions are

$$\begin{aligned} &\text{(i)} \quad l_1, l_2 \ll \tfrac{1}{A}: && K(l_1, l_2) \approx l_1 l_2; \\ &\text{(ii)} \quad \tfrac{1}{A} \ll l_1, l_2 \ll \tfrac{1}{B}: && K(l_1, l_2) \approx w l_1 l_2; \\ &\text{(iii)} \quad l_1, l_2 \gg \tfrac{1}{B}: && K(l_1, l_2) \approx \tfrac{w}{B^2}. \end{aligned} \tag{3.23}$$

The constant value as l_1 and l_2 approach infinity is noteworthy.

It is useful to note that a similar formula to (3.21) holds for the mean value of the scalar product of overlapping segments; for example, in the configuration of Fig. 1,

$$\overline{\mathbf{r}_1 \cdot \mathbf{r}} = \tfrac{1}{2}[K(l) + K(l_1) - K(l_2)]. \tag{3.24}$$

This expression of course is nonzero even for linear K.

In line with the Caussian ansatz for $p[\mathbf{r}(l)]$, we shall assume that the joint probability (3.20) is also Gaussian, except of course for very small values of l_1 or l_2. Its form is then completely determined by the covariance matrix

$$\mathbf{K} \equiv \begin{bmatrix} \overline{\mathbf{r}_1^2} & \overline{\mathbf{r}_1 \cdot \mathbf{r}_2} \\ \overline{\mathbf{r}_1 \cdot \mathbf{r}_2} & \overline{\mathbf{r}_2^2} \end{bmatrix} = \begin{bmatrix} K(l_1) & K(l_1, l_2) \\ K(l_1, l_2) & K(l_2) \end{bmatrix}. \tag{3.25}$$

The distribution may be written

$$p[\mathbf{r}_1, \mathbf{r}_2] = \left(\frac{3}{2\pi}\right)^3 \frac{1}{(\det \mathbf{K})^{3/2}} \exp[-\tfrac{3}{2}\tilde{\mathbf{R}} \cdot \mathbf{K}^{-1}\mathbf{R}], \tag{3.26}$$

where

$$\mathbf{R} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}. \tag{3.27}$$

The determinant in (3.26) may be written, using (3.21), as a symmetric function of the three variables

$$K = K(l), \quad K_1 = K(l_1), \quad K_2 = K(l_2), \tag{3.28}$$

namely

$$\det \mathbf{K} = \tfrac{1}{4}[2KK_1 + 2KK_2 + 2K_1K_2 - K^2 - K_1^2 - K_2^2]. \tag{3.29}$$

We note that the determinant is automatically positive provided that K increases as a function of l faster than linearly but less than quadratically, so that

$$K_1 + K_2 < K < (\sqrt{K_1} + \sqrt{K_2})^2. \tag{3.30}$$

This is assured if $f(l)$ is a positive, monotonically decreasing function.

It is useful to note that the joint probability (3.20) could equally well be written in terms of the variables $\mathbf{r}_1$ and $\mathbf{r}$, say, rather than $\mathbf{r}_1$ and $\mathbf{r}_2$, provided of course that $\mathbf{K}$ were replaced by the appropriate covariance matrix:

$$\mathbf{K} \to \begin{bmatrix} \overline{\mathbf{r}_1^2} & \overline{\mathbf{r}_1 \cdot \mathbf{r}} \\ \overline{\mathbf{r}_1 \cdot \mathbf{r}} & \overline{\mathbf{r}^2} \end{bmatrix}. \tag{3.31}$$

D. Conditional expectation values

In later sections, we shall need various conditional expectation values, for example the expectation value of the extension $\mathbf{r}_1$ of the l_1 segment in Fig. 1 for a given value of the overall extension $\mathbf{r}$ of the composite segment of length l. Let us denote this conditional averaging, over the ensemble of segments with given values of $\mathbf{r}$ and l, by angle brackets.

From the joint probability (3.26), it is straightforward to evaluate this conditional expectation value:

$$\langle \mathbf{r}_1 \rangle = \int d^3\mathbf{r}_1\, \mathbf{r}_1 p[\mathbf{r}_1 | \mathbf{r}], \qquad p[\mathbf{r}_1 | \mathbf{r}] = \frac{p[\mathbf{r}_1, \mathbf{r}]}{p[\mathbf{r}]}. \tag{3.32}$$

We find

$$\langle \mathbf{r}_1 \rangle = \frac{\overline{\mathbf{r}_1 \cdot \mathbf{r}}}{\overline{\mathbf{r}^2}}\mathbf{r} \tag{3.33}$$

$$= \frac{K(l) + K(l_1) - K(l_2)}{2K(l)}\mathbf{r}. \tag{3.34}$$

This is not at all surprising: a selected portion of a segment of extension $\mathbf{r}$ is obviously more likely to have its own extension $\mathbf{r}_1$ in the same direction as $\mathbf{r}$.

For large values of l and l_1, for which K is linear, this yields, as one might expect

$$\langle \mathbf{r}_1 \rangle \approx \frac{l_1}{l}\mathbf{r} \qquad (\text{large } l_1, l). \tag{3.35}$$

From (3.33) it follows of course that

$$\langle \mathbf{r}\cdot\mathbf{r}_1 \rangle = \frac{\overline{\mathbf{r}\cdot\mathbf{r}_1}}{\overline{\mathbf{r}^2}}\mathbf{r}^2. \tag{3.36}$$

Less obvious, but also useful, is the conditional expectation value of $\mathbf{r}_1^2$. A straightforward calculation yields

$$\langle \mathbf{r}_1^2 \rangle = \frac{\overline{\mathbf{r}^2}\,\overline{\mathbf{r}_1^2} - (\overline{\mathbf{r}\cdot\mathbf{r}_1})^2}{\overline{\mathbf{r}^2}} + \frac{(\overline{\mathbf{r}\cdot\mathbf{r}_1})^2}{(\overline{\mathbf{r}^2})^2}\mathbf{r}^2. \tag{3.37}$$

From (3.34) and (3.37) we obtain the conditional variance of $\mathbf{r}_1$:

$$\langle \mathbf{r}_1^2 \rangle - (\langle \mathbf{r}_1 \rangle)^2 = \frac{\overline{\mathbf{r}^2}\,\overline{\mathbf{r}_1^2} - (\overline{\mathbf{r}\cdot\mathbf{r}_1})^2}{\overline{\mathbf{r}^2}}. \tag{3.38}$$

It is remarkable that this conditional variance is in fact independent of the actual value of $\mathbf{r}$. Note however that for *any* given value of $\mathbf{r}$, it is less than the *unconditional* variance $\overline{\mathbf{r}_1^2}$.

A useful corollary of (3.34) yields the conditional expectation value of the unit vector $\mathbf{p}$. We can write

$$\mathbf{p}(l_1) = \frac{\partial \mathbf{r}_1}{\partial l_1}, \tag{3.39}$$

whence

$$\overline{\mathbf{r}\cdot\mathbf{p}(l_1)} = \tfrac{1}{2}[K'(l_1) + K'(l-l_1)], \tag{3.40}$$

and

$$\langle \mathbf{p}(l_1) \rangle = \frac{\overline{\mathbf{r}\cdot\mathbf{p}(l_1)}}{\overline{\mathbf{r}^2}}\mathbf{r} = \frac{K'(l_1) + K'(l-l_1)}{2K(l)}\mathbf{r}. \tag{3.41}$$

For large values of l and l_1, this reduces to

$$\langle \mathbf{p}(l_1) \rangle \approx \frac{\mathbf{r}}{l} \quad (\text{large } l_1, l), \tag{3.42}$$

again as one might expect.

It is useful to note that these results, in particular (3.41), continue to hold for *negative* values of l_1, provided that for negative l, $K(l)$ is interpreted as meaning $K(|l|)$. In other words, because the correlation of direction extends over a finite distance, a small segment close to but outside our chosen segment will still be correlated with it, though clearly less strongly than if it were inside. Of course, as $l_1 \to -\infty$, $K'(l_1) \to -2\xi$, while $K'(l-l_1) \to 2\xi$, so eventually $\langle \mathbf{p}(l_1) \rangle$ does approach zero.

E. Triple joint probabilities

We shall also need to consider more complex joint probabilities, such as the triple joint probability of the configuration shown in Fig. 2.

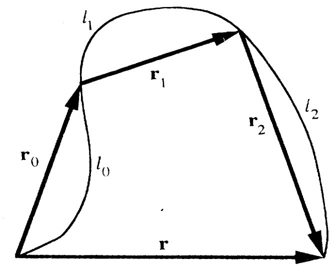

FIG. 2. Three contiguous segments.

This may be written as

$$p[\mathbf{r}_0,\mathbf{r}_1,\mathbf{r}] = \left(\frac{3}{2\pi}\right)^{9/2} \frac{1}{(\det \mathbf{K})^{3/2}} \exp[-\tfrac{3}{2}\tilde{\mathbf{R}}\cdot\mathbf{K}^{-1}\mathbf{R}], \tag{3.43}$$

where now

$$\mathbf{R} = \begin{bmatrix} \mathbf{r}_0 \\ \mathbf{r}_1 \\ \mathbf{r} \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} \overline{\mathbf{r}_0^2} & \overline{\mathbf{r}_0\cdot\mathbf{r}_1} & \overline{\mathbf{r}_0\cdot\mathbf{r}} \\ \overline{\mathbf{r}_0\cdot\mathbf{r}_1} & \overline{\mathbf{r}_1^2} & \overline{\mathbf{r}_1\cdot\mathbf{r}} \\ \overline{\mathbf{r}_0\cdot\mathbf{r}} & \overline{\mathbf{r}_1\cdot\mathbf{r}} & \overline{\mathbf{r}^2} \end{bmatrix}. \tag{3.44}$$

Integrating over all values of $\mathbf{r}_0$ yields a joint probability distribution for $\mathbf{r}_1$ and $\mathbf{r}$ identical in form to the previous one, save for the fact that the expression for $\overline{\mathbf{r}_1\cdot\mathbf{r}}$ is different: namely,

$$\overline{\mathbf{r}_1\cdot\mathbf{r}} = \tfrac{1}{2}[K(l_0+l_1) + K(l-l_0) - K(l_0) - K(l-l_0-l_1)]. \tag{3.45}$$

With this change, the previous equations remain valid.

Another useful result can be obtained from (3.43). We can find an expression for the conditional average of a scalar product of unit vectors, $\langle \mathbf{p}_0\cdot\mathbf{p}_2 \rangle$, where $\mathbf{p}_0 = \mathbf{p}(l_0)$ and $\mathbf{p}_2 = \mathbf{p}(l_0+l_1)$, by using analogues of (3.39): namely,

$$\mathbf{p}(l_0) = \frac{\partial \mathbf{r}_0}{\partial l_0}, \quad \mathbf{p}(l_2) = \frac{\partial \mathbf{r}_2}{\partial l_2}. \tag{3.46}$$

We find

$$\begin{aligned} \langle \mathbf{r}_0(l_0)\cdot\mathbf{r}_2(l_2) \rangle &= \overline{\mathbf{r}_0\cdot\mathbf{r}_2} + \frac{\overline{\mathbf{r}_0\cdot\mathbf{r}}\,\overline{\mathbf{r}_2\cdot\mathbf{r}}}{(\overline{\mathbf{r}^2})^2}(\mathbf{r}^2 - \overline{\mathbf{r}^2}) \\ &= \tfrac{1}{2}[K(l) + K(l-l_0-l_2) - K(l-l_0) - K(l-l_2)] \\ &\quad + \tfrac{1}{4}[K(l) + K(l_0) - K(l-l_0)][K(l) + K(l_2) - K(l-l_2)]\frac{\mathbf{r}^2 - K(l)}{K(l)^2}. \end{aligned} \tag{3.47}$$

Hence, differentiating with respect to both l_0 and l_2, we get

$$\langle \mathbf{p}_0 \cdot \mathbf{p}_2 \rangle = \tfrac{1}{2} K''(l - l_0 - l_2) + \tfrac{1}{4}[K'(l_0) + K'(l - l_0)][K'(l_2) + K'(l - l_2)] \frac{\mathbf{r}^2 - K(l)}{K(l)^2}. \tag{3.48}$$

Comparing with (3.41), we see that the conditional covariance function of $\mathbf{p}_0$ and $\mathbf{p}_2$ is again independent of $\mathbf{r}$ (but smaller than the unconditional value):

$$\langle \mathbf{p}(l_0) \cdot \mathbf{p}(l_2) \rangle - \langle \mathbf{p}(l_0) \rangle \cdot \langle \mathbf{p}(l_2) \rangle = \tfrac{1}{2} K''(l_1) - \frac{1}{4K(l)}[K'(l_0) + K'(l - l_0)][K'(l_2) + K'(l - l_2)]. \tag{3.49}$$

Note that as before these equations continue to hold even when either l_0 or l_2 is negative.

F. Small-l behavior

The Gaussian approximation obviously breaks down for very small values of l. To obtain information about the time evolution of the smallest-scale structures, we need to have approximate formulas that can be used in that region too.

For very small l, the expectation value of $\mathbf{r}^2$ takes the form given in (i) of (3.16):

$$K \approx l^2 - \frac{l^3}{3\zeta}. \tag{3.50}$$

We shall also need to consider the probability distribution for a small segment of length l_1 within a larger segment of length l and extension $\mathbf{r}$. From (3.41), it follows that the *direction* of $\mathbf{r}_1$ is correlated with that of $\mathbf{r}$. However, the length is not; the expression (3.37) does *not* hold in the limit of small l_1. In fact, in that limit the length is essentially fixed; the probability distribution for $\mathbf{r}_1$ is concentrated in a thin shell near $|\mathbf{r}_1| = l_1$. Therefore,

$$\langle \mathbf{r}_1^2 \rangle \approx \overline{\mathbf{r}_1^2} \approx l_1^2, \qquad l_1 \to 0. \tag{3.51}$$

Another important difference from the Gaussian case concerns the variance of $\mathbf{r}^2$, which is no longer given by (3.18). In fact, for small l, the leading term in $K_{(2)}$ is clearly of order l^5. To obtain a more specific result, consider for example a model in which small-angle kinks are randomly distributed on the string, with D kinks per unit length, and suppose the kink angles are distributed according to a (two-dimensional) distribution with (small) variance $\overline{\theta^2}$. Then for values of l such that $Dl \ll 1$, we find

$$K(l) = l^2 - \tfrac{1}{3} D l^3 (1 - \overline{\cos\theta}) \approx l^2 - \tfrac{1}{6} D l^3 \overline{\theta^2}. \tag{3.52}$$

The characteristic scale ζ of the small-scale structure, defined by (3.12), is

$$\zeta \approx \frac{2}{\overline{\theta^2} D}. \tag{3.53}$$

Similarly, we obtain

$$K_{(2)}(l) \approx \tfrac{1}{30} D l^5 \overline{\theta^4}, \qquad l \to 0. \tag{3.54}$$

The ratio $K_{(2)}/K^2$, which is $\frac{2}{3}$ in the Gaussian case, becomes

$$\frac{K_{(2)}(l)}{K(l)^2} \approx \frac{\overline{\theta^4}}{30\overline{\theta^2}^2} \frac{l}{\zeta} \to 0 \quad \text{as } l \to 0. \tag{3.55}$$

In particular, if the distribution of kink angles is Gaussian, then

$$\frac{K_{(2)}(l)}{K(l)^2} \approx \frac{\overline{\theta^2}}{15} \frac{l}{\zeta}. \tag{3.56}$$

IV. DERIVATION OF BASIC RATE EQUATIONS

The evolution of the system of strings is a complicated process. The mechanisms represented by the various terms in (2.19) do not act independently. At least to a first approximation, gravitational radiation is separable from the others, because the dominant scale involved is much smaller. We shall therefore postpone its discussion. As we shall see, however, the remaining three act in a complex synergy.

We begin this section by deriving the basic equations for these processes, reviewing for completeness the discussion of KC and CKA [20,21].

A. Rates of change of length and extension

From the equations of motion (2.5), we can derive expressions for the expected rates of change of the length l and extension $\mathbf{r}$ of a chosen segment, expressing $\dot{l}$ and $\dot{\mathbf{r}}$ as functions of l and $\mathbf{r}$. Here, unlike KC, we use dots to denote derivatives with respect to real time; in particular, $\dot{l}$ stands for the average value of dl/dt over the ensemble of segments of given length and extension.

Similarly, we can derive an expression for $(\dot{L})_{\rm str}$, the contribution of stretching to the rate of increase in the overall length of string within a given large comoving volume.

It is very important to note that there are consistency requirements. As before, let us denote by an overbar the average value of any function of $\mathbf{r}$ over the probability distribution $p[\mathbf{r}(l)]$ for given length l: e.g.,

$$\bar{\dot{l}} = \int d^3\mathbf{r}\, \dot{l} p[\mathbf{r}(l)]. \tag{4.1}$$

The lengths l of different segments will stretch by different amounts, depending on the value of $\mathbf{r}$ and other factors. However, the entire string can be chopped up conceptually into segments of any prescribed length l, and its overall growth is obviously independent of l. Thus for every value of l we must have

$$\frac{\bar{\dot{l}}}{l} = \frac{(\dot{L})_{\rm str}}{L}. \tag{4.2}$$

Deriving the expression for $(\partial p/\partial t)_{\rm str}$ from $\dot{l}$ and $\dot{\mathbf{r}}$ is actually quite subtle, because of the dependence of $\dot{l}$ on $\mathbf{r}$; a sample of segments initially of equal lengths does not remain so.

Consider a small time interval dt during which the expected changes of l and $\mathbf{r}$ are

$$l \to l' = l + \dot{l}dt, \qquad \mathbf{r} \to \mathbf{r}' = \mathbf{r} + \dot{\mathbf{r}}dt. \tag{4.3}$$

The total length will also change of course, according to

$$L \to L' = L + (\dot{L})_{\rm str}dt. \tag{4.4}$$

Now suppose that within V we select segments at random by choosing independently a random starting point and a random length l, chosen from a uniform distribution from 0 up to a large upper limit, say L. The changes of length and extension of the chosen segments over a short time interval dt will vary randomly, with expectation values given by (4.3). However, because of the consistency requirement (4.2), the final distribution will still be *uniformly distributed* in l' (at least so long as $l \ll L$, so that the upper cutoff of lengths is irrelevant). Hence we have the important equality

$$p[\mathbf{r}'(l'), t']\frac{dl'}{L'}d^3\mathbf{r}' = p[\mathbf{r}(l), t]\frac{dl}{L}d^3\mathbf{r}. \tag{4.5}$$

Now

$$dl' = \left(1 + dt\frac{\partial \dot{l}}{\partial l}\right) dl \tag{4.6}$$

and

$$d^3\mathbf{r}' = \left(1 + dt\frac{\partial}{\partial \mathbf{r}}\cdot\dot{\mathbf{r}}\right) d^3\mathbf{r}. \tag{4.7}$$

Hence it follows that

$$\left(\frac{\partial p}{\partial t}\right)_{\rm str} = -\frac{\partial}{\partial l}(\dot{l}p) - \frac{\partial}{\partial \mathbf{r}}\cdot(\dot{\mathbf{r}}p) + \frac{(\dot{L})_{\rm str}}{L}p. \tag{4.8}$$

This is consistent with the normalization condition because, from (4.2),

$$\frac{\partial}{\partial l}\bar{\dot{l}} = \frac{(\dot{L})_{\rm str}}{L}. \tag{4.9}$$

As before, let us consider a left-moving segment of string, defined by the inequalities $u_1 < u < u_2$. The length of our chosen segment at time t is

$$l(t) = 2R\int_{u_1}^{u_2} du\, x_u^0(u, v(u,t)), \tag{4.10}$$

while its extension is

$$\mathbf{r}(t) = 2R\int_{u_1}^{u_2} du\, \mathbf{x}_u(u, v(u,t)). \tag{4.11}$$

To find the rates of change, we use

$$\left(\frac{\partial v}{\partial t}\right)_u = \frac{1}{R}\left(\frac{\partial v}{\partial \tau}\right)_u = \frac{1}{R}\left(\frac{\partial \tau}{\partial v}\right)_u^{-1} = \frac{1}{Rx_v^0}. \tag{4.12}$$

Hence from the equations of motion (2.5),

$$\begin{aligned}\frac{dl}{dt} &= \frac{\dot{R}}{R}l + 2\int_{u_1}^{u_2} du\, x_{uv}^0\frac{1}{x_v^0}\\ &= \frac{\dot{R}}{R}l - 2\dot{R}\int_{u_1}^{u_2} du\, x_u^0(1 + \mathbf{p}\cdot\mathbf{q})\\ &= -2\dot{R}\int_{u_1}^{u_2} du\, x_u^0\mathbf{p}\cdot\mathbf{q}.\end{aligned} \tag{4.13}$$

As before, we denote averaging over the ensemble of segments with given values of l and $\mathbf{r}$ by angular brackets. We then have

$$\dot{l} = -2\dot{R}\int_{u_1}^{u_2} du\, \langle x_u^0\mathbf{p}\cdot\mathbf{q}\rangle \equiv \alpha(\mathbf{r}, l)\frac{\dot{R}}{R}l, \tag{4.14}$$

say. Note that in order to satisfy the consistency requirement (4.2), the average value of α over the $\mathbf{r}$ distribution must be a constant $\bar{\alpha}$, independent of l. We must have

$$(\dot{L})_{\rm str} = \bar{\alpha}\frac{\dot{R}}{R}L. \tag{4.15}$$

Thus $\bar{\alpha}$ may be identified with the constant α of KC [20] and CKA [21]. More generally, we shall find later that α has the form

$$\alpha(\mathbf{r}, l) = \bar{\alpha} + \hat{\alpha}(l)\frac{\mathbf{r}^2 - K(l)}{K(l)}. \tag{4.16}$$

Similarly, we obtain

$$\frac{d\mathbf{r}}{dt} = \frac{\dot{R}}{R}\mathbf{r} - 2\dot{R}\int_{u_1}^{u_2} du\, x_u^0(\mathbf{p} + \mathbf{q}) \tag{4.17}$$

and hence

$$\dot{\mathbf{r}} = -2\dot{R}\int_{u_1}^{u_2} du\, \langle x_u^0\mathbf{q}\rangle \equiv \beta(\mathbf{r}, l)\frac{\dot{R}}{R}\mathbf{r}. \tag{4.18}$$

The parameters α and β are related but distinct. In particular, there is no special condition on the mean value of β. It will emerge later that β may be taken to be a function of l alone, independent of $\mathbf{r}$.

It is convenient to rewrite (4.14) and (4.18) in terms of the path-length variable y introduced in (2.15). We then have

$$\alpha l = -\int_0^l dy\, \langle \mathbf{p}\cdot\mathbf{q}\rangle \tag{4.19}$$

and

$$\beta\mathbf{r} = -\int_0^l dy\, \langle \mathbf{q}\rangle. \tag{4.20}$$

We shall return to the computation of these averages in the next section.

Substituting $\dot{l}$ and $\dot{\mathbf{r}}$ into (4.8), and using (4.16), we thus find

$$\left(\frac{\partial p}{\partial t}\right)_{\rm str} = -H\bar{\alpha}l\frac{\partial p}{\partial l} - H\frac{\partial}{\partial l}\left(l\hat{\alpha}(l)\frac{\mathbf{r}^2 - K}{K}p\right) -H\beta(l)\frac{\partial}{\partial \mathbf{r}}\cdot(\mathbf{r}p), \qquad (4.21)$$

where as before H is the Hubble parameter, $H = \dot{R}/R$.

Taking a moment of (4.21), we find for the rate of change of K:

$$\left(\frac{\partial K}{\partial t}\right)_{\rm str} = -H\frac{\partial}{\partial l}\left(\bar{\alpha}lK + \hat{\alpha}l\frac{K_{(2)}}{K}\right) + 2H\beta K + H\bar{\alpha}K = H\left[2\beta K - \bar{\alpha}l\frac{\partial K}{\partial l} - \frac{\partial}{\partial l}\left(\hat{\alpha}l\frac{K_{(2)}}{K}\right)\right]. \qquad (4.22)$$

B. Intercommuting probability

Next we review briefly the derivation of the intercommuting probability given by KC [20]. This also provides an opportunity to refine the argument.

Consider a large volume V, which, according to (2.16), contains a length $L = V/\xi^2$ of long string. In other words, each volume ξ^3 contains on average a length ξ of string. The model introduced in KC was to regard this string as formed of N independently moving straight segments, each of length ξ, where

$$N = \frac{L}{\xi} = \frac{V}{\xi^3}. \qquad (4.23)$$

Choose any pair of these segments. For simplicity, assume that at the relevant time, the spatial coordinate σ along the string is chosen to coincide with the variable y which measures length along the string. The two segments will intersect at some time during a short interval δt if there is a solution to

$$\mathbf{x}_{01} + y_1\mathbf{x}'_1 + t\dot{\mathbf{x}}_1 = \mathbf{x}_{02} + y_2\mathbf{x}'_2 + t\dot{\mathbf{x}}_2, \qquad (4.24)$$

with

$$0 < y_1 < \xi, \qquad 0 < y_2 < \xi, \qquad 0 < t < \delta t. \qquad (4.25)$$

Equivalently, if the starting point of one of the segments, $\mathbf{x}_{01}$, is fixed, intercommuting will occur if the other starting point, $\mathbf{x}_{02}$, lies within a small volume

$$\delta V = \xi^2\delta t|\mathbf{x}'_1 \wedge \mathbf{x}'_2\cdot(\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2)|. \qquad (4.26)$$

[The additional factor of $\frac{1}{4}$ appearing in the paper by KC, Eq. (4.26), was an error.] The probability of intercommuting between this pair of segments is $\delta V/V$. To obtain the probability that a chosen segment undergoes intercommuting, we multiply by the number of segments, N, given by (4.23). Thus the probability that a string segment of length l will undergo intercommuting during a time interval dt is

$$\chi\frac{l\,dt}{\xi^2}, \qquad (4.27)$$

where χ is the average value of the scalar triple product $|\mathbf{x}'_1 \wedge \mathbf{x}'_2\cdot(\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2)|$.

The rms values of $|\mathbf{x}'|$ and $|\dot{\mathbf{x}}|$ are $\sqrt{(1+\alpha)/2}$ and $\sqrt{(1-\alpha)/2}$, respectively. One way of estimating χ is to use these as typical values. Then averaging over all angles, maintaining the orthogonality of $\mathbf{x}'$ and $\dot{\mathbf{x}}$, introduces a factor of $2/\pi$. Thus typically

$$\chi = \frac{1+\alpha}{\pi}\sqrt{\frac{1-\alpha}{2}} \approx 0.24. \qquad (4.28)$$

Although χ therefore has a weak α dependence, this is probably not sufficiently important to make it necessary to use an $\mathbf{r}$-dependent value. We shall see that this is in fact probably an overestimate of χ.

Another way of estimating χ is to rewrite the scalar triple product in terms of $\mathbf{p}$ and $\mathbf{q}$ vectors, as

$$\tfrac{1}{4}|\mathbf{p}_1 \wedge \mathbf{p}_2\cdot(\mathbf{q}_1 - \mathbf{q}_2) + (\mathbf{p}_1 - \mathbf{p}_2)\cdot\mathbf{q}_1 \wedge \mathbf{q}_2|. \qquad (4.29)$$

Averaging over all directions of the $\mathbf{p}$ and $\mathbf{q}$ vectors, assuming that they are independently and isotropically distributed, yields

$$\chi = \frac{2\pi}{35} \approx 0.18. \qquad (4.30)$$

[It is easy to see why this result should be smaller than our first estimate (4.28). Using typical average values of the magnitudes of $\mathbf{x}'$ and $\dot{\mathbf{x}}$ ignores the anticorrelation between them; including it would tend to reduce the estimate.] We can improve the estimate (4.30) by allowing for the correlation between the $\mathbf{p}$ and $\mathbf{q}$ vectors, corresponding to the nonzero value of α. This gives a corrected value

$$\chi = \frac{2\pi}{35} + \frac{4\pi\alpha}{105} \approx 0.20. \qquad (4.31)$$

It might be argued that the model used here, assuming straight segments of length ξ, is inaccurate. The length scale on which strings are roughly straight is not ξ but $\bar{\xi}$. A better approximation might be to assume that the string is composed of straight segments of length $\bar{\xi}$. However, this actually makes no difference to the final answer. The small volume δV is then proportional to $\bar{\xi}^2\delta t$, while N becomes $N = V/\xi^2\bar{\xi}$, so the probability of intercommuting is still proportional to the length of the segment, in this case $\bar{\xi}$.

The effects of small-scale structure on the strings are, however, more problematic. When we view the string configuration on a sufficiently small scale, the kinkiness is irrelevant. Consider two kinky segments of string, each of length $\bar{\xi}$. For simplicity, suppose that one of them is instantaneously at rest, while the other is moving. During a short time interval δt, the moving segment will trace out a thin ribbon of width $|\dot{\mathbf{x}}|\delta t$. Intercommuting will occur if the other segment intersects the ribbon. So long as δt is small compared to the small-scale structure, the probability of this is clearly proportional to the lengths of the segments and is in no way reduced by the fact that they are kinky.

However, although this calculation should give a correct estimate of the total number of intercommuting events, it tells us nothing about their distribution. If the strings were really composed of independently moving straight sections, the intercommuting events would be uncorrelated. But this is not true for kinky strings. When two kinky strings approach each other, it is clear that there are likely to be several intercommuting events within a small region. It could be argued that only one of these should really count as a long-string intercommuting event; the others are either loop formation or reconnection events. Indeed, bursts of loop formation in the vicinity of a long-string intercommuting have been observed in the simulations.

There has been one estimate of the number density of long-string intercommuting events, by Shellard and Allen [19], which gives a much lower figure than ours, equivalent to $\chi \approx 0.03$. This arises because the large-scale coherent velocities of long segments of string tend to be quite low. With our definition, however, that is clearly an underestimate of χ. We have chosen to define as a loop those sections of string that become permanently detached from the network. We should therefore count as intercommuting events those at which transient loops are formed and reattached.

We conclude that the correct value of χ with our definition is probably somewhat below 0.2 but substantially greater than 0.03. It would be safe to say that $\chi \approx 0.1$ within a factor of 2.

C. Effect of intercommuting

We now turn to the effect of long-string intercommuting on the probability distribution $p[\mathbf{r}(l)]$.

Clearly there is a negative term in $[\partial(Lp)/\partial t]_{\rm LSI}$ due to the destruction of segments by intercommuting, equal to

$$-\chi\frac{l}{\xi^2}Lp[\mathbf{r}(l)]. \qquad (4.32)$$

In this case, the corresponding positive term is rather more complicated. For each segment destroyed, a new segment of equal length l is created. However, its extension $\mathbf{r}$ is the sum of the extensions $\mathbf{r}_1$ and $\mathbf{r}_2$ of the two, generally speaking uncorrelated, segments thus brought together.

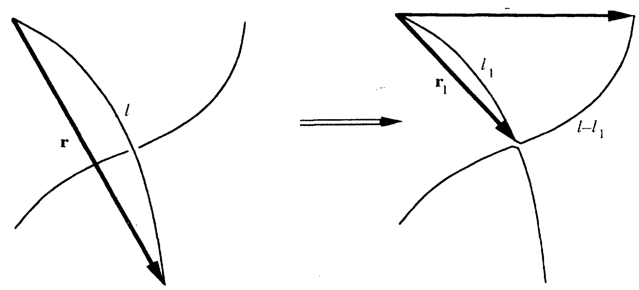

FIG. 3. Effect of intercommuting.

The number of segments created is exactly equal to the number destroyed. The probability that one of these, chosen at random, is composed of lengths l_1 and $l - l_1$, within an interval dl_1, is dl_1/l. Then the probability that the corresponding extensions are $\mathbf{r}_1$ and $\mathbf{r}_2$, within intervals $d^3\mathbf{r}_1$ and $d^3\mathbf{r}_2$ (see Fig. 3), is

$$p[\mathbf{r}_1(l_1)]d^3\mathbf{r}_1\, p[\mathbf{r}_2(l-l_1)]d^3\mathbf{r}_2. \qquad (4.33)$$

To find the probability that the total extension is $\mathbf{r}$, we have to set $\mathbf{r}_2 = \mathbf{r} - \mathbf{r}_1$ and integrate over $\mathbf{r}_1$. Thus the positive term in $[\partial(Lp)/\partial t]_{\rm LSI}$ is

$$+\frac{\chi l L}{\xi^2}\int_0^l \frac{dl_1}{l}\int d^3\mathbf{r}_1\, p[\mathbf{r}_1(l_1)]\, p[(\mathbf{r}-\mathbf{r}_1)(l-l_1)]. \qquad (4.34)$$

In contrast to the case of loop formation, here the assumption that the two probabilities are independent should be a good one, because the two segments involved generally come from regions that are far apart along the string.

In this case, we evidently have

$$\left(\frac{\partial L}{\partial t}\right)_{\rm LSI} = 0. \qquad (4.35)$$

Hence, putting (4.32) and (4.34) together, we obtain

$$\left(\frac{\partial p}{\partial t}\right)_{\rm LSI} = \frac{\chi}{\xi^2}\left\{\int_0^l dl_1 \int d^3\mathbf{r}_1\, p[\mathbf{r}_1(l_1)]\, p[(\mathbf{r}-\mathbf{r}_1)(l-l_1)] - lp[\mathbf{r}(l)]\right\}. \qquad (4.36)$$

It is again straightforward to find an expression for the rate of change of K. When we multiply by $\mathbf{r}^2$ and integrate, in the first term it is best to go back to using $\mathbf{r}_1$ and $\mathbf{r}_2$ as integration variables. Since there is assumed to be no correlation the mean value of $\mathbf{r}_1\cdot\mathbf{r}_2$ vanishes. Thus we find

$$\left(\frac{\partial K}{\partial t}\right)_{\rm LSI} = \frac{\chi}{\xi^2}\left\{\int_0^l dl_1[K(l_1) + K(l-l_1)] - lK(l)\right\}. \qquad (4.37)$$

Note that in general we expect

$$K(l) > K(l_1) + K(l - l_1), \tag{4.38}$$

so the effect of long-string intercommuting is to *reduce* the value of K (unless K is a linear function of l, in which case it is unchanged).

By symmetry, we can simplify the expression (4.37) slightly:

$$\left(\frac{\partial K}{\partial t}\right)_{\text{LSI}} = \frac{\chi}{\xi^2}\left\{2\int_0^l dl_1 K(l_1) - lK(l)\right\}. \tag{4.39}$$

From this, we can find the rates of change of the various length scales. Intercommuting has no effect on L, or, therefore, on ξ.

The length scale $\bar{\xi}$ is defined by (2.27). To find its rate of change we differentiate (4.39) and then allow l to approach infinity:

$$2\frac{d\bar{\xi}}{dt} = \lim_{l\to\infty}\frac{\partial K'(l)}{\partial t}. \tag{4.40}$$

This yields

$$\frac{\dot{\bar{\xi}}_{\text{LSI}}}{\bar{\xi}} = -\frac{\chi}{w}\frac{\bar{\xi}}{\xi^2}. \tag{4.41}$$

Similarly, from (3.50) we find

$$\frac{2\dot{\zeta}}{\zeta^2} = \lim_{l\to 0}\frac{\partial K'''(l)}{\partial t}. \tag{4.42}$$

Of course, for this to be consistent, we also have to verify that the rates of change of K, K', and K'' all vanish in the limit. This is easy to do. Thus we find

$$\frac{\dot{\zeta}_{\text{LSI}}}{\zeta} = -\frac{\chi\zeta}{\xi^2}. \tag{4.43}$$

D. Effect of loop formation

Next we review and revise the derivation of the probability of loop formation. This is perhaps the most problematic aspect of our analysis; we shall approach it in several stages.

Consider a segment of string of length l and extension $\mathbf{r}$. We want to evaluate the probability that this segment will form a loop during a short time interval. To be specific, let $\Theta(\mathbf{r}, l)\, dl\, dy\, dt$ be the probability that a loop will form in the time interval dt, of length between l and $l + dl$, and with starting point within the small interval dy. This is the function we aim to estimate.

From Θ, we can compute rates of change due to loop formation of all the quantities we need. In particular, the probability that any particular point on the string will be incorporated into a loop within the time interval dt is clearly λdt, where

$$\lambda = \int_0^\infty dl\, l\Lambda(l), \tag{4.44}$$

and

$$\Lambda(l) = \int d^3\mathbf{r}\, p[\mathbf{r}(l)]\Theta(\mathbf{r}, l). \tag{4.45}$$

The parameter λ determines the overall rate of loss of length to loop formation. Consider a very long section of string, of total length L. Then the expected rate of change of L is

$$\left(\frac{\partial L}{\partial t}\right)_{\text{loops}} = -\lambda L. \tag{4.46}$$

The dependence of Θ on $\mathbf{r}$ is unknown and may well be complicated. However, consistent with our Gaussian approximation, it seems reasonable to assume that, except for very small values of l, this is also Gaussian, parametrized by a variance function $Q(l)$. Specifically, we assume that

$$\Theta(\mathbf{r}, l) \approx \Lambda(l)\left(\frac{K(l) + Q(l)}{Q(l)}\right)^{3/2} e^{-3\mathbf{r}^2/2Q(l)}. \tag{4.47}$$

We now turn to the expression for the effect of loop formation on the evolution of $p[\mathbf{r}(l)]$. Let us consider the expression for $(\partial(Lp)/\partial t)_{\text{loops}}$. It consists of a negative term representing the number of segments lost to loop formation and a corresponding positive one representing the number of new segments created by the process.

Consider a segment of length l and extension $\mathbf{r}$. It will disappear if a loop is formed within it, overlapping either end, or enclosing it entirely. Let us ask for the probability that this happens within a short time interval dt due to the formation of a loop of length l_1, within a range dl_1.

We denote the starting point of the loop segment relative to that of our chosen segment of length l by y_0 (see Fig. 4). Allowing for all possible overlaps, the range of possible starting points is

$$-l_1 < y_0 < l. \tag{4.48}$$

[The total number of starting points, with discretization scale δ, is $(l + l_1)/\delta$.] The probability that a segment of length l_1 starting from one of these points has extension $\mathbf{r}_1$ within a volume δ^3 is $p[\mathbf{r}_1(l_1)|\mathbf{r}(l), y_0]\delta^3$, where the symbol $p[\ldots|\ldots]$ denotes a probability conditional on both the value of $\mathbf{r}(l)$ and on the position of the starting point y_0. Rather than using this conditional probability, it is more convenient to work with the corresponding joint probability, using the identity

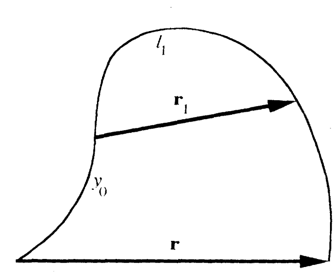

FIG. 4. Position of excised loop segment.

$$p[\mathbf{r}_1(l_1),\mathbf{r}(l)|y_0] = p[\mathbf{r}_1(l_1)|\mathbf{r}(l);y_0]\,p[\mathbf{r}(l)]. \qquad (4.49)$$

However, the joint probability is still conditional on the position y_0 of the starting point.

The probability that within a time interval dt a loop is formed of length between l_1 and l_1+dl_1, starting between y_0 and y_0+dy_0 is, according to the discussion of Sec. V, $\Theta(\mathbf{r}_1,l_1)\,dl_1\,dy_0\,dt$.

Thus we find for the negative term the expression

$$-L\int_0^\infty dl_1 \int d^3\mathbf{r}_1\,\Theta(\mathbf{r}_1,l_1)\int_{-l_1}^{l} dy_0\,p[\mathbf{r}_1(l_1),\mathbf{r}(l)|y_0]. \qquad (4.50)$$

In evaluating the corresponding positive term, we have to consider excision of a loop entirely within a segment of length $l+l_1$. Thus it is

$$L\int_0^\infty dl_1 \int d^3\mathbf{r}_1\,\Theta(\mathbf{r}_1,l_1)\int_0^l dy_0\,p[\mathbf{r}_1(l_1),(\mathbf{r}+\mathbf{r}_1)(l+l_1)|y_0]. \qquad (4.51)$$

Putting (4.50) and (4.51) together, we have

$$\left(\frac{\partial(Lp)}{\partial t}\right)_{\rm loops} = L\int_0^\infty dl_1 \int d^3\mathbf{r}_1\,\Theta(\mathbf{r}_1,l_1)\left\{\int_0^l dy_0\,p[\mathbf{r}_1(l_1),(\mathbf{r}+\mathbf{r}_1)(l+l_1)|y_0] - \int_{-l_1}^{l} dy_0\,p[\mathbf{r}_1(l_1),\mathbf{r}(l)|y_0]\right\}. \qquad (4.52)$$

It is straightforward to perform the integration over $\mathbf{r}$ to find $(\partial L/\partial t)_{\rm loops}$, because clearly

$$\int d^3\mathbf{r}\,p[\mathbf{r}_1(l_1),\mathbf{r}(l)|y_0] = p[\mathbf{r}_1(l_1)], \qquad (4.53)$$

independent of y_0. This of course reproduces (4.46) with λ given by (4.44).

Combining (4.52) and (4.46) we obtain

$$\left(\frac{\partial p}{\partial t}\right)_{\rm loops} = \int_0^\infty dl_1 \int d^3\mathbf{r}_1\,\Theta(\mathbf{r}_1,l_1)\left\{\int_0^l dy_0\,p[\mathbf{r}_1(l_1),(\mathbf{r}+\mathbf{r}_1)(l+l_1)|y_0] - \int_{-l_1}^{l} dy_0\,p[\mathbf{r}_1(l_1),\mathbf{r}(l)|y_0] + l_1 p[\mathbf{r}_1(l_1)]p[\mathbf{r}(l)]\right\}. \qquad (4.54)$$

E. Probability of loop formation

Now let us turn to the calculation of $\Theta(\mathbf{r},l)$, or equivalently $\Lambda(l)$ and $Q(l)$. Our segment of length l will form a loop during a short time interval dt if the corresponding total extension $\mathbf{r}_{\rm tot}$, given by (2.10), vanishes at some instant during that interval. In other words, the corresponding left- and right-moving segments, each of length l, must have the same extension, $\mathbf{r}$.

To simplify the counting, let us imagine that space-time is partitioned into cells each of volume δ^4, and moreover that the string is partitioned into small segments of length δ.

The probability that a left-moving segment of length l has an extension $\mathbf{r}$ within a δ^3 volume labeled j is

$$p_j = p[\mathbf{r}(l)]\delta^3. \qquad (4.55)$$

The probability that a loop is formed is essentially the probability that the corresponding right-moving segment also has the same extension. Assuming for the moment that the probabilities are uncorrelated (which as we shall see is by no means true), this probability is also p_j. We have to multiply it by factors relating to dl, dt, etc. The number of length steps within a given range dl is dl/δ. The number of time steps within dt is $2dt/\delta$. (The factor of 2 arises, as explained by KC [20], because the segments are moving with the speed of light in opposite directions, so that each encounters a new segment after a time $\delta/2$.) The number of starting points on dy is dy/δ.

However, we should not merely multiply these factors. Particularly for small loops, the angles between the various vectors are small.

A typical configuration is shown in Fig. 5. The question is, by how much can we vary the length l, the starting point y and the time t without moving the difference in extensions of the two segments out of the δ^3 volume. This is essentially the same calculation that we did ear-

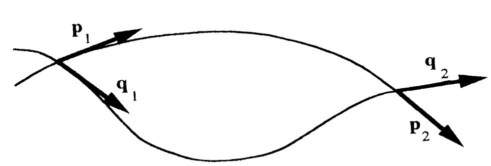

FIG. 5. Excision of a small loop.

lier in estimating the parameter χ that determines the rate of long-string intercommuting. The volume swept out by the total extension $\mathbf{r}_{\rm tot}$ when the parameters vary over ranges dl, dy, and dt is

$$|\mathbf{x}_1' \wedge \mathbf{x}_2' \cdot (\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2)|\, dl\, dy\, dt. \quad (4.56)$$

Thus we should include in our expression for the loop-formation probability a factor

$$\Delta(\mathbf{r}, l) = \tfrac{1}{4}\langle |\mathbf{p}_1 \wedge \mathbf{p}_2 \cdot (\mathbf{q}_1 - \mathbf{q}_2) + (\mathbf{p}_1 - \mathbf{p}_2) \cdot \mathbf{q}_1 \wedge \mathbf{q}_2| \rangle. \quad (4.57)$$

The number of cells within the volume $d^3\mathbf{r}$ is $d^3\mathbf{r}/\delta^3$. Hence our required probability is

$$p[\mathbf{r}(l)]\Theta(\mathbf{r}, l)\, d^3\mathbf{r}\, dl\, dy\, dt = p_j^2 \frac{d^3\mathbf{r}}{\delta^3} \frac{dl}{\delta} \frac{2dt}{\delta} \frac{dy}{\delta} \Delta(\mathbf{r}, l), \quad (4.58)$$

i.e.,

$$\Theta(\mathbf{r}, l) = 2p[\mathbf{r}(l)]\Delta(\mathbf{r}, l). \quad (4.59)$$

Note that the factors of δ cancel, as they must.

Inclusion of the factor Δ eliminates the problem of multiple counting of loops noted by KC [20]. This was avoided by CKA [21] by imposing a small-scale cutoff and treating the contribution of small loops separately. However, that procedure introduced additional problems associated with the choice of the cutoff. It is better to treat loop production within a unified framework.

So long as the loops are reasonably large, it is reasonable to assume that the unit vectors involved are independently randomly distributed on the unit sphere. Then we can replace Δ by its average, $\overline{\Delta} = \chi \approx 0.1$.

For very small loops we expect Δ to be small. Consider for example the model in which the strings are composed of straight segments joining randomly distributed kinks. For very short loops one would expect to find only a single kink on each of the left- and right-moving segments forming the loop. However, it is easy to see that in this case the scalar triple product (4.56) vanishes identically, because the three vectors are coplanar. This is because a triangular loop is necessarily planar. The point is that the loop-formation condition in this situation is satisfied only on a set of configurations of measure zero.

This is of course an idealized model, but even in a more realistic model we should expect that the Δ factor would be very small for loops with only a single pair of kinks.

It follows from this argument that Δ should vanish rapidly as $l \to 0$. In the simple model, the leading contribution would come from loops with at least three kinks all together. If the kinks are randomly distributed, with a separation of order ζ, then the number of kinks on a segment of length l is Poisson distributed, with mean $\approx l/\zeta$, so for three or more kinks we expect a factor of at least $(l/\zeta)^3$.

Also for small loops, the $\mathbf{r}$ dependence is likely to be important. For values of $|\mathbf{r}|$ close to l, the segments forming the loop must be nearly straight with a high degree of correlation between the $\mathbf{p}$ vectors at the two ends, and also between the $\mathbf{p}$ and $\mathbf{q}$ vectors. In that case the scalar triple product will be small. On the other hand smaller values of $\mathbf{r}$ imply large-angle kinks and correspondingly less correlation between the unit vectors. So we expect Δ to decrease with increasing $|\mathbf{r}|$.

Except for very small loops, where Δ is in any case small, $\mathbf{r}^2$ is generally much less than l^2. Thus it would seem reasonable to assume, consistent with our earlier assumptions, that it has the form of a Gaussian,

$$\Delta(\mathbf{r}, l) \approx \Delta_0 e^{-a\mathbf{r}^2/l^2}, \quad (4.60)$$

where a is a constant of order unity. The overall factor Δ_0 approaches χ at large l and vanishes at least like $(l/\zeta)^3$ as $l \to 0$. The Gaussian factor yields a contribution to $1/Q(l)$ of magnitude $2a/3l^2$.

The formula (4.59) is still not accurate, for several reasons, but particularly because it neglects all correlations between the left- and right-moving strings. The rate of change of L due to loop formation is to be computed using (4.46) and (4.44). As it stands, this expression would give problems at both ends of the range of integration over l. At the upper end, we have to consider the probability of reconnection; we aim to include only loops that do not reconnect. At the lower end, the integral would diverge, because of the neglect of a very significant angular correlation effect.

We deal first with reconnection. For a loop of size l, the probability of reconnection within a short time interval dt is $\chi l dt/\xi^2$. The probability that the loop will survive reconnection to a much later time is therefore

$$\exp\left(-\chi \int_t^\infty dt' \frac{l}{\xi(t')^2}\right). \quad (4.61)$$

If we assume that over the relevant period ξ at least approximately scales, i.e., $\xi(t) \propto t$, then the integral yields

$$\exp\left(-\frac{\chi l t}{\xi(t)^2}\right). \quad (4.62)$$

(It is not necessary here to allow for the variation of l with time, due to gravitational radiation, which occurs over a very much longer time scale.) To allow for the probability of reconnection, we should replace (4.59) by

$$\Theta(\mathbf{r}, l) = 2p[\mathbf{r}(l)]e^{-\chi l t/\xi^2}\Delta(\mathbf{r}, l). \quad (4.63)$$

The exponential provides an effective upper cutoff in (4.44) at a scale of order ξ^2/t. Above that scale, loops are rather likely to reconnect; below it, they mostly survive.

It should be noted that the assumption of approximate scaling is a very weak one. Even if ξ does not exactly scale, the dominant contribution to the integral (4.61) will come from close to the lower limit so (4.62) will change at most by a factor of order unity.

Even with the modifications described, the expression (4.63) is still not entirely correct, because it neglects any possible correlation between the left- and right-moving segments. This turns out to be the most important effect of all. These correlations form the subject of the next section.

V. CORRELATIONS BETWEEN LEFT AND RIGHT MOVERS

Taking account of the correlations between left- and right-moving segments, we should really write

$$\Theta(\mathbf{r},l)p[\mathbf{r}(l)] = 2p[\mathbf{r}(l);\mathbf{r}(l)]e^{-\chi l t/\xi^2}\Delta(\mathbf{r},l), \qquad (5.1)$$

where $p[\mathbf{r}(l);\mathbf{r}'(l)]\,d^3\mathbf{r}\,d^3\mathbf{r}'$ is the joint probability that the corresponding left- and right-moving segments have extensions $\mathbf{r}$ and $\mathbf{r}'$ respectively.

One approach might be to extend the Gaussian ansatz, representing the joint probability distribution by a six-dimensional Gaussian with an appropriate covariance function $\overline{\mathbf{r}\cdot\mathbf{r}'}$. However, this is not a good representation of $p[\mathbf{r}(l);\mathbf{r}'(l)]$. The covariance is actually quite small (and negative), suggesting that the correlation effect yields merely a small reduction in the loop-formation probability. But this is quite false: the joint probability distribution is in fact very sharply reduced in the forward ($\mathbf{r}=\mathbf{r}'$) direction.

We begin by considering the correlation of the individual $\mathbf{p}$ and $\mathbf{q}$ vectors. There are two quite separate processes that generate such correlations. In CKA [21] we considered only the effect of stretching, but in fact loop production also plays an important role, and indeed intercommuting cannot be ignored.

Consider a particular $\mathbf{p}$ segment, $\mathbf{p}(u_0)$, and the approaching $\mathbf{q}$ segments, in particular a segment $\mathbf{q}(v_0)$ which encounters it at time t_0. (See Fig. 6.) We wish to estimate how the angular probability distribution of $\mathbf{q}$ relative to the direction of $\mathbf{p}$ changes as the vectors approach one another. It is again convenient to use the path-length variable y defined in (2.15). Let us define $\Phi(y,z)$, so that the probability that $\mathbf{p}(y_0)\cdot\mathbf{q}(y_1) = z$, within the range dz, is

$$\Phi(y,z)\frac{dz}{2}, \qquad (5.2)$$

where $y = y_0 - y_1$. With this normalization, the initial condition for Φ at large $y_0 - y_1$ is

$$\Phi(\infty,z) = 1, \qquad (5.3)$$

representing a completely random distribution when the segments are far apart.

Let us now seek to write down an evolution equation for Φ, of the form

$$\frac{\partial\Phi}{\partial y} = \left(\frac{\partial\Phi}{\partial y}\right)_{\rm str} + \left(\frac{\partial\Phi}{\partial y}\right)_{\rm LSI} + \left(\frac{\partial\Phi}{\partial y}\right)_{\rm loops}. \qquad (5.4)$$

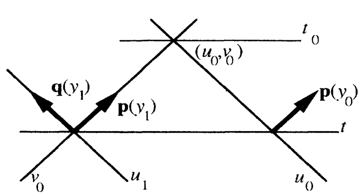

FIG. 6. Coordinates on the string world sheet.

We are assuming here that the process of establishing an angular correlation as the vectors $\mathbf{p}$ and $\mathbf{q}$ approach takes a relatively short time, compared to the time scales for evolution, so that Φ may be regarded as a function only of z and of the path length y between them, not explicitly of the time. This is reasonable because the angular correlation sets in only when the vectors are already quite close. For similar reasons, we are justified in neglecting the effect of gravitational radiation which is small on the scales of interest here.

We also introduce the corresponding angular probability distribution for the $\mathbf{p}$ vector that encounters $\mathbf{q}$ at y_1: the probability that $\mathbf{p}(y_0)\cdot\mathbf{p}(y_1) = z$ within the range dz is

$$\Psi(y,z)\,\frac{dz}{2}. \qquad (5.5)$$

Let us recall that

$$\bar{z}_\Psi \equiv \int_{-1}^{1}\frac{dz}{2}\,z\Psi(y,z) = f(y), \qquad (5.6)$$

where f is the function defined in (3.1). (When it is necessary to distinguish, we denote the mean value of z with respect to the distribution Φ by $\bar{z}_\Phi$ and that with respect to Ψ by $\bar{z}_\Psi$.)

A. The exponential ansatz

To reduce the problem to manageable proportions, we make a simplifying assumption concerning the angular distribution function Ψ, analogous to the Gaussian ansatz, namely that (except when y is very small) it takes the form of an exponential:

$$\Psi(y,z) = \frac{k}{\sinh k}e^{kz}, \qquad (5.7)$$

where k is a function of y. The relation between k and f is

$$\bar{z}_\Psi = f(y) = \coth k - \frac{1}{k}. \qquad (5.8)$$

(As we shall see the ansatz breaks down for very small values of y.)

It will be useful to examine the limiting cases of large and small values of y. First, when y is large, $f \ll 1$ and consequently also $k \ll 1$. In that case we have

$$f \approx \frac{k}{3} - \frac{k^3}{45} \qquad (y \text{ large}). \qquad (5.9)$$

In most cases, therefore, a linear approximation $k \approx 3f$ will be adequate. Neglecting k^2, we may write

$$\Psi \approx 1 + kz \approx 1 + 3fz \qquad (y \text{ large}). \qquad (5.10)$$

In the opposite limit, where y is small (but not small enough to render the ansatz invalid), we have $1 - f \ll 1$ and hence $k \gg 1$. In this case, the leading approximation is

$$k \approx \frac{1}{1-f} \approx \frac{\zeta}{y} \qquad (y \text{ small}). \tag{5.11}$$

Here we may write

$$\Psi \approx 2ke^{-k(1-z)} \qquad (y \text{ small}). \tag{5.12}$$

The distribution becomes concentrated near $z = 1$ within a range of order $1/k$.

It is interesting to note that in the intermediate region, both approximations are in fact reasonably good. For example for $f = \frac{1}{2}$, the leading large-y and small-y approximations give respectively $k = \frac{3}{2}$ and $k = 2$; the correct answer is $k = 1.8$.

When the exponential ansatz is valid, all the moments of the distribution are of course determined by the expectation value. In particular,

$$\begin{aligned}\left(\overline{z^2}\right)_\Psi &= 1 - \frac{2\coth k}{k} + \frac{2}{k^2}\\ &\approx \frac{1}{3} + \frac{2k^2}{45} \approx \frac{1}{3} + \frac{2\bar{z}_\Psi^2}{5} \qquad (y \text{ large}).\end{aligned} \tag{5.13}$$

Indeed, so long as $k \ll 1$ it is a good approximation to set $\left(\overline{z^2}\right)_\Psi \approx \frac{1}{3}$.

The exponential ansatz can also be applied to the $\mathbf{q}$ distribution Φ, in the modified form

$$\Phi(y,z) = \frac{b}{\sinh b} e^{-bz}. \tag{5.14}$$

(Since the directions of $\mathbf{p}$ and $\mathbf{q}$ vectors are *anti*correlated, with this definition b is positive.) However, as we shall see, the ansatz ceases to be a good approximation for very small values of y. When the approximation is valid, the evolution equation for Φ effectively reduces to an equation for b or equivalently $\bar{z}_\Phi$.

B. Equation for Φ

Now let us consider the ensemble of approaching $\mathbf{q}$ vectors, in particular those that at a given time t fall within a small interval δy_1 at y_1.

In the time interval dt, these vectors either move closer to y_0 or else are eliminated by being incorporated into a loop. The expected distance by which they move closer is

$$dy = -(2 + \lambda y)dt, \tag{5.15}$$

with $y = y_0 - y_1$. Here the 2 is the normal velocity of approach and the extra term $\lambda y dt$ represents the expected loss of length between y_1 and y_0 due to loop formation. Integrating this relation, we find

$$y = \frac{2}{\lambda}\left(e^{\lambda(t_0 - t)} - 1\right), \tag{5.16}$$

where t_0 is the time at which the segments coincide. Note that the distance y becomes exponentially large for time differences larger than $1/\lambda$. Loop formation effectively shuffles the string segments on this time scale, providing a very effective long-distance cutoff.

During the time interval dt, the small interval δy_1 on the string effectively becomes a little shorter, because some members of the ensemble are eliminated by incorporation into loops. In fact, $d\delta y_1/dt = -\lambda \delta y_1$, so δy_1 is replaced by $\delta y_1 - \lambda dt\, \delta y_1$. The angular distribution of the remaining vectors changes from $\Phi(y,z)$ to $\Phi(y+dy,z)$, with dy given by (5.15). To find the difference between these, we need to know the angular distribution, $X(y,z)$ say, of the vectors that have been eliminated. Here $X(y_0 - y_1, z)\frac{dz}{2}$ is the probability that a vector $\mathbf{q}$ which is incorporated into a loop within a short time interval has $\mathbf{p}(y_0)\cdot\mathbf{q}(y_1) = z$ within the range dz. If we can estimate X, then we can obtain a differential equation for Φ. In fact (ignoring stretching and intercommuting contributions),

$$\begin{aligned}\delta y_1\, \Phi(y,z) = (\delta y_1 - \lambda dt\, \delta y_1)\, \Phi(y+dy,z)&\\ +\lambda dt\, \delta y_1\, X(y,z),&\end{aligned} \tag{5.17}$$

or equivalently

$$(2+\lambda y)\left(\frac{\partial \Phi}{\partial y}\right)_{\text{loops}} = -\lambda\Phi + \lambda X. \tag{5.18}$$

Note that this is consistent with maintenance of the normalization condition $\int \frac{dz}{2}\Phi = 1$ provided that X is also normalized.

The key observation is that since loops are formed by matching segments of left- and right-moving string, the angular distributions of excised $\mathbf{p}$ and $\mathbf{q}$ segments should be identical. Clearly, the probability of obtaining a matching pair must depend on the probability distributions of both $\mathbf{p}$ and $\mathbf{q}$. It seems reasonable to assume that X is proportional to the *product* $\Psi\Phi$. Normalization then requires

$$X(y,z) = N(y)\Psi(y,z)\Phi(y,z), \tag{5.19}$$

where

$$N^{-1}(y) = \int_{-1}^{1} \frac{dz}{2}\Psi(y,z)\Phi(y,z). \tag{5.20}$$

Thus we find

$$(2+\lambda y)\left(\frac{\partial \Phi}{\partial y}\right)_{\text{loops}} = -\lambda\Phi + \lambda N\Psi\Phi. \tag{5.21}$$

So far we have ignored both intercommuting and stretching, both of which will also have an effect on the angular distribution Φ. Consider first the effect of intercommuting. The probability that an intercommuting event occurs between y_1 and y_0 within the time interval dt is $\chi y dt/\xi^2$. If it does occur the relevant members of the ensemble of $\mathbf{q}$ vectors are deleted and replaced by new vectors drawn from an essentially independent random distribution. In other words, $\Phi(y,z)$ on the left-hand side of (5.17) is replaced by

$$\Phi(y,z)\left(1-\frac{\chi y dt}{\xi^2}\right)+\frac{\chi y dt}{\xi^2}. \qquad (5.22)$$

Thus the effect of intercommuting is described by

$$(2+\lambda y)\left(\frac{\partial\Phi}{\partial y}\right)_{\rm LSI}=-\frac{\chi y}{\xi^2}(1-\Phi). \qquad (5.23)$$

Finally, let us consider the effect of stretching. The equation of motion (2.8) for $\mathbf{q}$ may be written

$$\frac{\partial\mathbf{q}}{\partial t}=-H(\mathbf{p}_1-\mathbf{q}\mathbf{q}\cdot\mathbf{p}_1), \qquad (5.24)$$

where $\mathbf{p}_1$ denotes the vector $\mathbf{p}(y_1)=\mathbf{p}(u_1,v_0)$ (see Fig. 6). Strictly speaking, the Hubble parameter H is a variable quantity here, but, since the correlations extend over distances which are small compared to the horizon distance, it should be a good approximation to treat it as a constant.

We now have to average this result over the angular distribution of $\mathbf{p}(y_1)$. In principle, this vector is correlated with both $\mathbf{p}(y_0)$ and $\mathbf{q}(y_1)$. However, any contribution proportional to $\mathbf{q}(y_1)$ cancels out in (5.24). Thus it is reasonable to consider only its correlation with $\mathbf{p}(y_0)$, and to replace $\mathbf{p}(y_1)$ by $\bar{z}_\Psi\mathbf{p}(y_0)$.

There is an exactly similar expression for the rate of change of $\mathbf{p}(y_0)$, which yields an identical contribution. Taking account of both we thus find

$$(2+\lambda y)\left(\frac{\partial\Phi}{\partial y}\right)_{\rm str}=\frac{\partial}{\partial z}\left(\frac{\partial z}{\partial t}\Phi\right)$$
$$=-2H\bar{z}_\Psi\frac{\partial}{\partial z}\left[(1-z^2)\Phi\right]. \qquad (5.25)$$

Bringing all three contributions together, we may write the equation for Φ as

$$(2+\lambda y)\frac{\partial\Phi}{\partial y}=-2H\bar{z}_\Psi\frac{\partial}{\partial z}\left[(1-z^2)\Phi\right]-\frac{\chi y}{\xi^2}(1-\Phi)$$
$$-\lambda\Phi+\lambda N\Psi\Phi. \qquad (5.26)$$

When the exponential ansatz is valid, all we need is the equation for the rate of change of $\bar{z}_\Phi$: namely,

$$(2+\lambda y)\frac{\partial\bar{z}_\Phi}{\partial y}=2H\bar{z}_\Psi\left[1-\left(\overline{z^2}\right)_\Phi\right]+\frac{\chi y}{\xi^2}\bar{z}_\Phi-\lambda\bar{z}_\Phi+\lambda\bar{z}_X, \qquad (5.27)$$

where $\bar{z}_X$ is the mean with respect to the distribution $X=N\Psi\Phi$.

C. Equation for Ψ

Before trying to solve the equation for Φ, some parenthetical remarks about the possibility of deriving a similar equation for Ψ may be in order.

The effects of stretching and intercommuting on Ψ are very similar, and we have assumed that the distribution X of excised segments is the same for both. So we should be able to write down a very similar equation for the evolution of Ψ. However, there is a very important difference, concerning the rate at which one $\mathbf{p}$ segment approaches another. The most obvious difference is that the distance $y=y_0-y_1$ between the two $\mathbf{p}$ vectors decreases *only* because of loop formation. In other words the term 2 in the factor on the left of (5.18) is absent in the corresponding equation for Ψ.

But there is a further point: it is no longer reasonable to assume that dy/dt is independent of z. In fact, if it were, the string would never develop the long-range directional correlation that it does. The formation of loops (except for the very smallest) depends strongly on the large-scale configuration of the strings. If the left- and right-moving sections are relatively straight, the large-loop formation probability is low, because the values of $\mathbf{r}$ for given l are large. On the other hand, if the strings are curled up tightly, typical values of $\mathbf{r}$ are small and large-loop formation becomes highly probable. In the former case, positive values of $\mathbf{p}(y_0)\cdot\mathbf{p}(y_1)$ are clearly favored. Conversely, if $\mathbf{p}(y_0)\cdot\mathbf{p}(y_1)$ is positive, the expected value of $-dy/dt$ will be smaller than if it is negative. Roughly speaking, we may expect

$$\frac{dy}{dt}\approx-[\lambda y-(z-\bar{z}_\Psi)v(y)], \qquad (5.28)$$

where $v(y)$ is a function that could in principle be determined from the later discussion of the detailed effects of loop formation. There is no doubt a similar effect even for the cross correlation between $\mathbf{p}$ and $\mathbf{q}$, but in that case it seems likely to be negligibly small.

It follows that for Ψ, in place of (5.26), we would have

$$\frac{\partial}{\partial y}\Big(\left[\lambda y-(z-\bar{z}_\Psi)v(y)\right]\Psi\Big)$$
$$=-2H\bar{z}_\Phi\frac{\partial}{\partial z}\left[(1-z^2)\Psi\right]-\frac{\chi y}{\xi^2}(1-\Psi)+\lambda N\Phi\Psi. \qquad (5.29)$$

Similarly, the analogue of (5.27) is

$$\frac{\partial}{\partial y}\Big(\lambda y\bar{z}_\Psi-\left[\left(\overline{z^2}\right)_\Psi-\bar{z}_\Psi^2\right]v(y)\Big)$$
$$=2H\bar{z}_\Phi\left[1-\left(\overline{z^2}\right)_\Psi\right]+\frac{\chi y}{\xi^2}\bar{z}_\Psi+\lambda\bar{z}_X. \qquad (5.30)$$

D. Solution of equation for Φ

Having set up the equation (5.26) for Φ we now set about solving it. For the moment at least, we shall treat Ψ as given, via the exponential ansatz, in terms of the correlation function $\bar{z}_\Psi=f(y)$.

Consider first the region of large y where the linear approximation (5.10) for Ψ and Φ should be valid. In this case, we need consider only the evolution equation for $\bar{z}_\Phi$. So long as $\bar{z}_\Phi$ remains small, we can also use the approximation $\overline{(z^2)}_\Phi \approx \frac{1}{3}$. Note also that under these conditions, the distribution X is also linear, with

$$\bar{z}_X = \bar{z}_\Psi + \bar{z}_\Phi. \tag{5.31}$$

Thus we find

$$(2+\lambda y)\frac{\partial \bar{z}_\Phi}{\partial y} = \frac{\chi y}{\xi^2}\bar{z}_\Phi + \left(\lambda + \frac{4}{3}H\right)\bar{z}_\Psi. \tag{5.32}$$

It is straightforward to integrate this equation using an integrating factor, to obtain

$$\bar{z}_\Phi(y) = -\frac{\lambda+\frac{4}{3}H}{2+\lambda y}\int_y^\infty dy' \left(\frac{2+\lambda y'}{2+\lambda y}\right)^{n-1} \times e^{-\chi(y'-y)/\lambda\xi^2}\bar{z}_\Psi(y'), \tag{5.33}$$

where

$$n = \frac{2\chi}{\lambda^2\xi^2}. \tag{5.34}$$

As we noted earlier, the exponential ansatz breaks down near $y=0$, but if we ignore that for the moment, we can estimate the value of $\bar{\alpha} = -\bar{z}_\Phi(0)$, as

$$\bar{\alpha} \approx \frac{\lambda+\frac{4}{3}H}{2}\int_0^\infty dy\left(1+\frac{\lambda y}{2}\right)^{n-1} e^{-\chi y/\lambda\xi^2}\bar{z}_\Psi(y). \tag{5.35}$$

For large values of y, $\bar{z}_\Psi$ is expected to fall off exponentially. The model described in Sec. II suggests that $\bar{z}_\Psi \approx we^{-By}$, with $B = w/\bar{\xi}$. In that case, $\bar{z}_\Phi$ is expressible in terms of the incomplete Γ function:

$$\bar{z}_\Phi(y) = -\frac{w(\lambda+\frac{4}{3}H)}{\lambda}x^{-n}e^{x-By}\Gamma(n,x) \tag{5.36}$$

where

$$x = \left(\frac{B}{\lambda}+\frac{n}{2}\right)(2+\lambda y). \tag{5.37}$$

Expanding for large x, the leading term is

$$\bar{z}_\Phi(y) \approx -\frac{(\lambda+\frac{4}{3}H)we^{-By}}{(B+\frac{1}{2}n\lambda)(2+\lambda y)}. \tag{5.38}$$

As a consistency check, we may substitute this solution into the equation (5.30) for the large-y Ψ distribution and verify that it can be satisfied with a reasonable form of the unknown velocity-distortion function $v(y)$. In fact, we find, for the leading approximation,

$$v(y) = 3\left(w\lambda + \frac{\chi\bar{\xi}}{\xi^2}\right)ye^{-By}, \tag{5.39}$$

which seems entirely reasonable.

For $y \gg 1/\lambda$, $-\bar{z}_\Phi$ is small compared to $\bar{z}_\Psi$. But as y falls it grows rapidly. For moderate values of y the two are of comparable magnitude, assuming that $1/\lambda$ is of the same order as ξ and $\bar{\xi}$.

We know from the simulations that even at $y=0$, $\bar{z}_\Phi$ does not become large [9,19]:

$$\bar{z}_\Phi(0) = -\bar{\alpha} \approx -0.14 \tag{5.40}$$

(in the radiation-dominated era). Hence it is reasonable to assume for all values of y that $\bar{z}_\Phi \ll 1$. This does not necessarily mean that the exponential ansatz is valid (a point we shall return to shortly), but so long as it is we can still use (5.27), with $\overline{(z^2)}_\Phi \approx \frac{1}{3}$. However, as $y \to 0$, we can no longer use (5.31); instead, we have both $\bar{z}_\Psi \to 1$ and $\bar{z}_X \to 1$. The net effect is that in the small-y region, the exponent $n-1$ in (5.33) becomes n. But since λy and $\lambda y' \ll 2$ in that region, the effect is minimal. The expressions obtained above should still give a good approximation to $\bar{z}_\Phi$.

E. Behavior near $y = 0$

To get at least a rough estimate of the value of $\bar{z}_\Phi(0) = -\bar{\alpha}$, let us first assume that $\lambda\xi \gg 1$ and $\lambda\bar{\xi} \gg 1$. Then in (5.36) both x and n are small. The leading term in the expansion of $\Gamma(n,x)$ for small x and n yields

$$\bar{\alpha} \approx w\left(1+\frac{4H}{3\lambda}\right)\ln(\lambda\bar{\xi}) \qquad (\lambda\bar{\xi} \gg 1). \tag{5.41}$$

This is, however, much too large, clearly inconsistent with our assumption that $\bar{z}_\Phi$ always remains small.

It is perhaps more plausible to assume that $\lambda\xi \ll 1$ and $\lambda\bar{\xi} \ll 1$. Then *both* x and n are large, and roughly equal, and the asymptotic form of the incomplete gamma function gives

$$\bar{\alpha} \approx \frac{w}{2}\sqrt{\frac{\pi}{\chi}}\left(1+\frac{2H}{3\lambda}\right)\lambda\xi \qquad (\lambda\xi \ll 1). \tag{5.42}$$

Although $\bar{z}_\Phi$ remains small, the exponential ansatz is not in fact a good approximation near $y=0$, because the distribution Ψ becomes so sharply peaked near $z=1$. The distribution functions in the last two terms of (5.26) are of course both normalized, but the second one is negligibly small over most of the angular range, becoming very large near $z=1$. Thus, while a linear approximation to Φ remains good for most values of z, near $z=1$ it becomes very poor. As $y \to 0$, Φ acquires a deep hole in the forward direction. For this reason, our estimate of $\bar{z}_\Phi(0)$ requires some correction.

To estimate the likely size of the effect, let us assume that the exponential ansatz is at least qualitatively reasonable down to values of y of order ζ. So when we come to consider the equation (5.26) in the region of small y we can use the exponential (or indeed linear) form as our initial condition at $y \sim \zeta$.

When $y \ll \zeta$ we can use the approximation (5.12) for Ψ, with $\bar{z}_\Psi \approx 1-(y/\zeta)$. Then our equation (5.26) for Φ becomes

$$(2+\lambda y)\frac{\partial\Phi}{\partial y} = -2H\left(1-\frac{y}{\zeta}\right)\frac{\partial}{\partial z}\left[(1-z^2)\Phi\right] - \frac{\chi y}{\xi^2}(1-\Phi) - \lambda\Phi + \lambda N\frac{2\zeta}{y}e^{-(1-z)\zeta/y}\Phi. \tag{5.43}$$

It is convenient to change from y to $k \approx \zeta/y$ as the independent variable. The equation involves three small parameters, $\lambda\zeta$, $H\zeta$, and $\chi\zeta^2/\xi^2$. Since the last of these is very small indeed, the intercommuting term will give a very small contribution in this region. Neglecting this term, and also neglecting λy in comparison with 2, we find

$$2k^2\frac{\partial\Phi}{\partial k} = 2H\zeta\left(1-\frac{1}{k}\right)\frac{\partial}{\partial z}\left[(1-z^2)\Phi\right] + \lambda\zeta(1-2Nke^{-k(1-z)})\Phi. \tag{5.44}$$

We now seek to solve this equation, starting with some initial value k_0 of k, at which we assume that Φ has the exponential form (5.14), with a value b_0 of b. Since we expect both $H\zeta$ and $\lambda\zeta$ to be small compared to unity, we may adopt a perturbation approach, taking as our zero-order solution, $\Phi_0(k,z) = \Phi(k_0,z)$, independent of k. We regard the $H\zeta$ and $\lambda\zeta$ terms in (5.44) as separate small perturbations, so we may treat their contributions one at a time.

Let us consider first the $H\zeta$ term. Substituting the zeroth-order solution into this term we can obtain a first-order solution, $\Phi_0 + \Phi_1$, where

$$\Phi_1(k,z) = H\zeta\left[\left(\frac{1}{k_0}-\frac{1}{k}\right) - \frac{1}{2}\left(\frac{1}{k_0^2}-\frac{1}{k^2}\right)\right] \times[2z + b_0(1-z^2)]\Phi(k_0,z). \tag{5.45}$$

We are particularly interested in the average value $\bar{z}_\Phi(y=0) = -\bar{\alpha}$, obtained by multiplying by z and integrating. This gives a contribution equal to

$$\bar{\alpha}_1 \approx -\frac{2}{3}H\zeta\left(\frac{1}{k_0} - \frac{1}{2k_0^2}\right). \tag{5.46}$$

This is of course a small term. Its dependence on k_0 must of course be canceled by terms in the contribution of Φ_0. In fact, $\bar{\alpha}_1$ can be made to vanish by appropriate choice of k_0 (e.g., $k_0 = \frac{1}{2}$). It will be convenient to make such a choice.

Now let us turn to the effect of the $\lambda\zeta$ perturbation, ignoring the $H\zeta$ term. The zero-order value of N is given by

$$N_0^{-1} = \int_{-1}^{1}\frac{dz}{2}2ke^{-k(1-z)}\Phi(k_0,z) \approx \Phi(k_0,1) = \frac{b_0e^{-b_0}}{\sinh b_0}. \tag{5.47}$$

Now we can substitute this into the right-hand side of (5.44) and integrate, obtaining in first order,

$$\begin{aligned}\Phi(k,z) &= \Phi(k_0,z)\exp\left(\frac{\lambda\zeta}{2k_0} - \frac{\lambda\zeta}{2k} - \lambda\zeta N_0\int_{k_0}^{k}\frac{dk'}{k'}e^{-k'(1-z)}\right) \\ &= \Phi(k_0,z)\exp\left(\frac{\lambda\zeta}{2k_0} - \frac{\lambda\zeta}{2k} + \lambda\zeta N_0\{\mathrm{Ei}[-k_0(1-z)] - \mathrm{Ei}[-k(1-z)]\}\right),\end{aligned} \tag{5.48}$$

where Ei is the exponential integral.

It is interesting to examine the special case $z = 1$. At that point, we have

$$\Phi(k,1) \approx N_0^{-1}\exp\left(\frac{\lambda\zeta}{2k_0} - \frac{\lambda\zeta}{2k}\right)\left(\frac{k}{k_0}\right)^{-\lambda\zeta N_0}. \tag{5.49}$$

Note that $\Phi(k,1)$ approaches zero as $k\to\infty$ (or $y\to 0$), but very slowly.

The most interesting limit of course is $k\to\infty$. Strictly speaking, in this limit the first-order approximation in $\lambda\zeta$ breaks down, but in fact it does so only at such large values of k that $\partial\Phi/\partial k$ is already negligibly small. There seems no need to go beyond first order. For $z \neq 1$, we have, in that limit,

$$\Phi(k=\infty,z) = \Phi(k_0,z)\exp\left(\frac{\lambda\zeta}{2k_0} + \lambda\zeta N_0\mathrm{Ei}[-k_0(1-z)]\right). \tag{5.50}$$

The behavior of this function near $z = 1$ is given by

$$\Phi(k=\infty,z) \approx N_0^{-1}e^{-b_0(1-z)}\exp\left(\frac{\lambda\zeta}{2k_0}\right)[k_0(1-z)]^{\lambda\zeta N_0} \qquad (1-z \ll 1). \tag{5.51}$$

This clearly exhibits the expected sharp dip near $z = 1$: the last factor vanishes at that point, but is close to unity over most of its range.

Finally let us examine $-\bar{\alpha}$, obtained by multiplying (5.50) by z and integrating. Since $\Phi(y=0,z)$ is a product of two factors, each of which separately gives a small value of $\bar{z}$, the two effects are approximately additive, and we may write

$$\bar{\alpha} \approx \bar{\alpha}_{\rm lin} + \bar{\alpha}_{\rm nl}, \tag{5.52}$$

where the subscripts stand for "linear" and "nonlinear." Here the linear contribution $\bar{\alpha}_{\rm lin}$ is the value given by (5.41), (5.42), or something in between, while the nonlinear term $\bar{\alpha}_{\rm nl}$ comes from the second factor in (5.50). To first order in $\lambda\zeta$, it is

$$\bar{\alpha}_{\rm nl} = -(\bar{z}_\Phi)_{\rm nl} \approx \frac{\lambda\zeta}{2}. \tag{5.53}$$

VI. ANGULAR CORRELATION AND LOOP FORMATION

We are now in a position to return to the calculation of the probability of loop production, $\Theta(\mathbf{r}, l)$. The provisional formula (4.63) is wrong because it assumes that the probability distributions of the left- and right-moving segments are independent. It should be replaced by (5.1).

To complete our program of expressing all the terms in the equation for $\partial p[\mathbf{r}(l)]/\partial t$ in terms of p itself, we have to reexpress the joint probability here in terms of individual probabilities.

A. Small loops

Let us first consider the case of very small loops. We shall deal separately with larger loops in the next subsection.

In the case of small loops, the essential effect is due to the angular correlation between $\mathbf{p}$ and $\mathbf{q}$ vectors. To be completely correct, we should consider the joint probability distribution of all the $\mathbf{p}$ and $\mathbf{q}$ vectors forming this section of string. This would obviously be a very complicated object; we are not in a position to deal with it. However, in the case of small loops, the internal correlation between $\mathbf{p}$ vectors at different points is very strong (as is that between $\mathbf{q}$ vectors), so it seems reasonable, in order to represent the effect of the $\mathbf{p}$-$\mathbf{q}$ correlation, to choose a single representative vector from each class. We choose the pair for which the effect is strongest, namely the vectors at the midpoints of the segments, namely $\mathbf{p}(l/2)$ and $\mathbf{q}(l/2)$. (We have considered as an alternative averaging over the chosen position; this makes little difference.)

Thus we take

$$p[\mathbf{r}(l);\mathbf{r}(l)] \approx \int \frac{d^2\mathbf{p}}{4\pi}\frac{d^2\mathbf{q}}{4\pi}\,\Phi(0,\mathbf{p}\cdot\mathbf{q}) \times p[\mathbf{r}(l)|\mathbf{p}(l/2)]\,p[\mathbf{r}(l)|\mathbf{q}(l/2)], \tag{6.1}$$

where $p[\mathbf{r}(l)|\mathbf{p}(l/2)]$ is the probability distribution of extension conditional on the direction of the vector $\mathbf{p}$ at the midpoint, $d^2\mathbf{p}$ denotes an integration over the unit sphere and of course Φ is the angular distribution function for $z = \mathbf{p}\cdot\mathbf{q}$ evaluated at the point where the two vectors meet, namely $y = 0$. It is reasonable to assume that, apart from their mutual correlation, the $\mathbf{p}$ and $\mathbf{q}$ vectors are uniformly distributed on the sphere.

The Gaussian approximation should be valid, both for the conditional probabilities and for the loop-production function $\Theta(\mathbf{r}, l)$, since there is an effective cutoff in the region of very small loops due to the behavior of the angular distribution function and the volume factor. Then $p[\mathbf{r}(l)|\mathbf{p}(l/2)]$ is the Gaussian distribution with appropriate values of $\bar{\mathbf{r}}$ and $\overline{\mathbf{r}^2}$. By (3.40),

$$\bar{\mathbf{r}} = S\mathbf{p}, \tag{6.2}$$

where

$$S = \overline{\mathbf{r}\cdot\mathbf{p}} = K'(l/2). \tag{6.3}$$

The value of $\overline{\mathbf{r}^2}$ is unaffected by $\mathbf{p}$: $\overline{\mathbf{r}^2} = K(l)$. This means of course that the variance of $\mathbf{r}$ is reduced:

$$\hat{K}(l) \equiv \overline{\mathbf{r}^2} - \bar{\mathbf{r}}^2 = K - S^2. \tag{6.4}$$

It is worth remarking that for moderately small values of l the Gaussian ansatz is a much better approximation for the conditional probability $p[\mathbf{r}(l)|\mathbf{p}(l/2)]$ than it is for the unconditional probability $p[\mathbf{r}(l)]$. For small l, the probability distribution is of course concentrated near the sphere $\mathbf{r}^2 = l^2$ and is nothing like a Gaussian centered at $\mathbf{r} = \mathbf{0}$. However, the Gaussian approximation to $p[\mathbf{r}(l)|\mathbf{p}(l/2)]$ is centered near that sphere and has much smaller variance, since, for small l,

$$S \approx l - \frac{l^2}{4\zeta}, \qquad \hat{K} \approx \frac{l^3}{6\zeta} \qquad (\text{small } l). \tag{6.5}$$

We can now compute Θ, or the variance function Q in the Gaussian approximation.

Substituting (6.1) into (5.1), we obtain

$$\Theta(\mathbf{r},l)p[\mathbf{r}(l)] \approx \Lambda(l)\left(\frac{3}{2\pi Q(l)}\right)^{3/2} e^{-3\mathbf{r}^2/2Q(l)}\left(\frac{3}{2\pi K(l)}\right)^{3/2} e^{-3\mathbf{r}^2/2K(l)}$$
$$\approx 2e^{-\chi lt/\xi^2}\Delta(\mathbf{r},l)\int\frac{d^2\mathbf{p}}{4\pi}\frac{d^2\mathbf{q}}{4\pi}\,\Phi(0,\mathbf{p}\cdot\mathbf{q})\,p[\mathbf{r}(l)|\mathbf{p}(l/2)]\,p[\mathbf{r}(l)|\mathbf{q}(l/2)]. \tag{6.6}$$

It is easy to carry out the integrations over $\mathbf{p}$ and $\mathbf{q}$, leaving only a single integration which may be written

$$\Theta(\mathbf{r},l)p[\mathbf{r}(l)] \approx 2e^{-\chi lt/\xi^2}\Delta(\mathbf{r},l)\left(\frac{3}{2\pi\hat{K}}\right)^3 e^{-3(\mathbf{r}^2+S^2)/\hat{K}}\,\frac{\hat{K}}{3Sr}\int_0^1 du\,\Phi(0,2u^2-1)\sinh\left(\frac{6Sr}{\hat{K}}u\right). \tag{6.7}$$

Without having a specific form for Φ, it is not possible to proceed further. However, we can get a good idea of the likely effect by assuming that Φ has a sharp step-function cutoff, i.e., $\Phi = 0$ for $z > z_0 \equiv 1 - 2\bar{\alpha}_{\rm nl}$ and $\Phi =$ const for $z < z_0$. We then find

$$\Theta(\mathbf{r}, l)p[\mathbf{r}(l)] \approx 2e^{-\chi lt/\xi^2}\Delta(\mathbf{r}, l)\left(\frac{3}{2\pi\hat{K}}\right)^3 e^{-3(\mathbf{r}^2+S^2)/\hat{K}}\frac{\sinh^2 X}{X^2}, \tag{6.8}$$

with

$$X = \sqrt{1-\alpha_{\rm nl}}\frac{3Sr}{\hat{K}}. \tag{6.9}$$

Using (4.60), we can now perform the integration over $\mathbf{r}$ in (4.45) to find Λ. For small l, $S^2/\hat{K} \approx 6\zeta/l$, so there is a strong exponential cutoff for $l < \zeta$. Thus we can legitimately set $\Delta(\mathbf{r}, l) \approx \Delta_0$. Then we obtain

$$\Lambda \approx \frac{\Delta_0}{4\pi(1-\bar{\alpha}_{\rm nl})}\left(\frac{3}{\pi\hat{K}}\right)^{1/2}\frac{1}{S^2}\left(e^{-3\bar{\alpha}_{\rm nl}S^2/\hat{K}} - e^{-3S^2/\hat{K}}\right). \tag{6.10}$$

B. The intermediate-scale region

The formula (6.1) incorporates our model of the **p**-**q** correlation which, as we have seen, provides a very effective cutoff for values of l of order ζ or less. However, it would still predict an impossibly large value of λ. If we substitute from (6.10) into (4.44), this arises from the contribution of the intermediate region, where

$$\zeta \ll l \ll \bar{\xi}. \tag{6.11}$$

In this region, $\hat{K} \sim w(1-w)l^2$, while $S^2/\hat{K} \sim w/(1-w)$ is of order unity, so the integrand in (4.44) behaves like $1/l^2$. Thus we find $\lambda \sim 1/\zeta$, up to a constant of order one. This would mean an extremely rapid decrease of L, on a time scale of order ζ, which is clearly not consistent with the results of the simulations.

The explanation for this discrepancy again lies in the angular correlation effect, but of longer segments of string, not merely individual **p** and **q** vectors.

Think of a section of string of length l and extension $\mathbf{r}$, and suppose that $\zeta \ll l \ll \bar{\xi}$. In other words, the section contains many kinks, but viewed on a large scale it is likely to be fairly straight; i.e., $|\mathbf{r}|$ is a sizable fraction of l. Now consider the collection of **p** vectors on this section. Their ensemble average is of course $\langle \mathbf{p} \rangle = \mathbf{r}/l$. So the distribution of **p** vectors will be strongly skewed, concentrated into a cone around the direction of $\mathbf{r}$.

The **q** vectors on the corresponding right-moving section will also be concentrated in a cone, around the direction of $\mathbf{r}_{\rm r}$. In most cases, the two cones will not overlap much, and rather few loops will form. On the other hand, where the cones do overlap, many loops will form. Such a section will disappear rapidly in a burst of loop formation. After a short time, regions where the distributions overlap will be rare (cf. [32]).

Of course, regions of overlap are continually being reformed, as sections of string meet new partners. In particular, the intercommuting process brings together segments of string that had been far apart and are therefore more or less uncorrelated. In some fraction of cases, depending on the size of the cones, these segments will have overlapping distributions, so a fresh burst of loop formation will be triggered. This burst phenomenon has indeed been observed in the simulations.

It is important to note that the angular concentration effect will be even more pronounced for somewhat larger loops. One may think of the string section as composed of short straight segments of length $\sim \zeta$, with randomly varying orientation within the cone around $\mathbf{r}$. Thus their transverse extensions are essentially a two-dimensional random walk. If we select a length $n\zeta$, its overall extension will be close to $n\zeta\mathbf{r}/l$, with a transverse spread proportional to $\sqrt{n}\zeta$. Hence the angular distribution of such sections will be concentrated in a cone whose angle is reduced by a factor $1/\sqrt{n}$ compared to the cone of **p** vectors.

The question we want to answer is: how does the loop formation probability depend on the three scale lengths $\zeta, \bar{\xi}, \xi$?

In approaching this question, let us begin by considering an initial state in which all three scales are comparable in magnitude, as would be expected shortly after the string-forming phase transition: $\zeta \sim \bar{\xi} \sim \xi < t$. In this case, the upper cutoff of the loop-formation integral, at $\xi^2/\chi t$, may well be smaller than the lower cutoff, ζ. What this means is simply that almost all loops formed reconnect to the network. There is then no reason why either ζ or $\bar{\xi}$ should grow rapidly. If we consider only the stretching terms, we expect (in the radiation era)

$$\frac{\dot{\xi}}{\xi} = \frac{3-\alpha}{2}H = \frac{3-\alpha}{4t}. \tag{6.12}$$

Here α may be dependent on the ratios of length scales, but its value is presumably small, say around 0.2. This implies that $\xi \propto t^{0.7}$. Thus we reach a regime where $\zeta \sim \bar{\xi} \ll \xi < t$. The upper cutoff does grow, if only rather slowly: $\xi^2/\chi t \propto t^{0.4}$.

The next stage is easy to describe, at least qualitatively. Large numbers of small loops start to form, without reconnection, increasing the rate of growth of ξ. But also $\bar{\xi}$ starts to grow. This is because of the selective nature of loop formation. Some sections of string will be relatively straight. On those, the distributions of **p** and **q** vectors will be confined to cones. In a few such cases, the cones may overlap, leading to rapid disappearance; in most, they will not. By contrast, on the more wiggly sections of string, $\mathbf{p}$ or $\mathbf{q}$ vectors will readily find partners. Those sections will disappear too, though not perhaps quite as rapidly as where there are overlapping cones. The net result will be to eliminate selectively the more wiggly sections of string, leading to growth of $\bar{\xi}$.

One consequence of this that will be important later is that the function $\Theta(\mathbf{r}, l)$ representing the probability of loop formation will be more concentrated towards $\mathbf{r} = \mathbf{0}$ than would otherwise have been expected. In other words, the variance function Q defined in (4.47) will be smaller than the original form (4.59) would suggest.

Although $\bar{\xi}$ starts to grow, there is still no reason for ζ to do so. Every intercommuting event and every loop formation introduces new kinks, keeping down the average interkink distance. (As we shall see later, there is good reason to believe that ζ does eventually start to grow, but only when gravitational radiation becomes important.)

Thus, within a few expansion times, we may expect to reach a regime where $\zeta \ll \bar{\xi} \ll \xi < t$. How, in such a regime, does the loop formation rate depend on the various length scales? In other words, what correction factor do we need to apply to the rate (5.1) calculated neglecting the angular correlation effects?

Consider a particular loop size l within the range $\zeta \ll l \ll \xi^2/\chi t$. We also assume that $l \ll \bar{\xi}$. Typically then a segment of length l will be part of a longer section that is roughly straight on a scale of order $\bar{\xi}$, with many kinks separated by distances of order ζ. The segments of length l will be concentrated around this overall direction. Unless this is a region where there is a burst of loop formation, the corresponding right-moving segments will be concentrated in a similar, but essentially nonoverlapping, cone—simply because regions where cones overlap quickly disappear.

In these circumstances, loop formation will occur only when segments meet new partners with a different overall direction. This happens for two reasons—intercommuting and the steady progression of left and right movers.

Consider first intercommuting. The probability that a segment of length l experiences intercommuting within a time interval δt is $\chi l \delta t/\xi^2$, or, to put it another way, the average time between such events is $\xi^2/\chi l$.

The steady progression will bring a given segment alongside ones with a different overall orientation when the left and right movers have moved relatively by a distance of order $\bar{\xi}$, i.e., after a time $\bar{\xi}/2$. In the regime we are presently considering, this is a much shorter time, so the steady progression is the important effect. (The intercommuting time would be shorter only for segments well above the upper cutoff length.) It is therefore reasonable to ignore the intercommuting effect.

It is important to realize that this does not mean intercommuting plays no role; quite the contrary. What we are saying is that the *direct* effect of intercommuting *within* the chosen segment is unimportant. The *indirect* effect of nearby intercommuting is one of the things that keeps $\bar{\xi}$ down, and is in fact clearly crucial.

If the orientations of the segments of length l were random, a given segment would meet another of essentially different orientation every time it moved on by a distance l, i.e., after a time $l/2$. As it is, because of the long-range correlation, it will meet a segment with an essentially different orientation only after a time $\bar{\xi}/2$. On this ground, the rate of loop formation should be suppressed by a factor of approximately $l/\bar{\xi}$ (for $l < \bar{\xi}$; there is no suppression for $l > \bar{\xi}$).

This is not the end of the story, however. We have to consider what happens when a new segment is encountered. The new right-moving section of string may have any orientation relative to the left-moving section. We may assume that the relative orientation is random. Thus if α is the angle between the two, then $\cos\alpha$ should be uniformly distributed between -1 and 1.

As we saw, the left-moving segments of length l lie within a cone whose semivertical angle β is proportional to $1/\sqrt{n}$, where n is, roughly speaking, the number of sizable kinks on the segment, namely $n \approx l/\zeta$. The solid angle within the cone is of order $\pi\beta^2 \sim \pi\zeta/l$.

Now clearly, if α is significantly larger than β, then our chosen segment is very unlikely to meet a matching partner. On the other hand, in the relatively rare cases in which $\alpha < \beta$, loop formation is extremely probable. In other words, we may expect *another* suppression factor, roughly equal to the solid angle of the cone divided by 4π, namely $\zeta/4l$.

Putting these two suppression factors together, we obtain an additional factor in $\Lambda(l)$ of about $\zeta/4\bar{\xi}$. We now insert the expression for $\Lambda(l)$, given by (6.10) with this factor, into the formula (4.44) for the overall loop-formation rate λ. Here we have to integrate from the lower limit ζ to the upper limit $\xi^2/(\chi t)$. Since $\Lambda(l) \propto 1/l^3$, the integral behaves like $\int dl/l^2$, and it is the lower limit that dominates—as must be true if most loops produced are indeed small. Hence the integral is of order $1/\zeta$. Taking account of the extra suppression factor, we see from (6.10) and (4.44) that

$$\lambda = \frac{c}{\bar{\xi}}, \tag{6.13}$$

where c is a dimensionless parameter rather less than unity. This parameter was treated as a constant by KC [20] and by CKA [21] was expressed as a sum of separate small- and large-loop terms. Here, however, we regard it as a function of the ratios of length scales and the horizon. It is expected to be less than unity, perhaps of order 0.1.

Note that the index n defined in (5.34) is

$$n = \frac{2\chi}{c^2}\frac{\bar{\xi}^2}{\xi^2}. \tag{6.14}$$

So n is expected to be larger than unity except in the rather unlikely event that $\bar{\xi}$ is very much larger than ξ.

C. Rates of change of ξ and $\bar{\xi}$

We now take up the calculation of the effect of loop formation on the rates of change of the length scales in the problem.

First, we consider ξ. It is of course defined in terms of L by (2.16), from which it follows that

$$\frac{\dot{\xi}}{\xi} = \frac{3H}{2} - \frac{\dot{L}}{2L}. \tag{6.15}$$

Thus the rate of change of ξ due to loop formation is

$$\frac{\dot{\xi}_{\text{loops}}}{\xi} = \frac{\lambda}{2} = \frac{c}{2\bar{\xi}}, \tag{6.16}$$

where the dimensionless function c is given by

$$c\left(\frac{\zeta}{\xi}, \frac{\bar{\xi}}{\xi}, \frac{t}{\xi}\right) = \lambda\bar{\xi} = \bar{\xi}\int_0^\infty dl\, l\Lambda(l). \tag{6.17}$$

For the other two scales, we have to examine the rate of change of the variance function K. The Gaussian approximation should be valid, since there is an effective cutoff in the region of very small loops due to the behavior of the angular distribution function and the volume factor.

It is then straightforward to derive an expression for the rate of change of K by performing the integrations over $\mathbf{r}_1$ and $\mathbf{r}$ in (4.54) explicitly.

In the Gaussian approximation, the joint probabilities are given by expressions of the form (3.26) with $\mathbf{K}$ replaced by the covariance matrix (3.31) and $\overline{\mathbf{r}\cdot\mathbf{r}_1}$ given by (3.45).

For the first term of (4.54), the appropriate covariance matrix is

$$\mathbf{K} = \begin{bmatrix} K_1 & K_a \\ K_a & K_b \end{bmatrix}, \tag{6.18}$$

where

$$\begin{aligned} K_1 &= K(l_1), \\ K_a &= \tfrac{1}{2}[K(l_1+y_0) + K(l+l_1-y_0) \\ &\quad - K(y_0) - K(l-y_0)], \\ K_b &= K(l+l_1), \end{aligned} \tag{6.19}$$

while for the second it is

$$\mathbf{K} = \begin{bmatrix} K_1 & K_c \\ K_c & K \end{bmatrix}, \tag{6.20}$$

where

$$\begin{aligned} K_c &= \tfrac{1}{2}[K(l_1+y_0) + K(l-y_0) \\ &\quad - K(y_0) - K(l-l_1-y_0)], \\ K &= K(l). \end{aligned} \tag{6.21}$$

The exponent in the first term of the integrand in (4.54) contains the inverse of (6.18) plus a single-entry contribution from the extra factor $\Theta(\mathbf{r}_1, l_1)$. In other words, the inverse of $\mathbf{K}$ is replaced by the inverse of a new matrix $\mathbf{L}$, given by

$$\mathbf{L}^{-1} = \mathbf{K}^{-1} + \frac{1}{Q_1}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{6.22}$$

with $Q_1 = Q(l_1)$. Thus

$$\mathbf{K}\,\mathbf{L}^{-1} = \frac{1}{Q_1}\begin{bmatrix} Q_1 + K_1 & 0 \\ K_a & Q_1 \end{bmatrix}, \tag{6.23}$$

whence we find

$$\mathbf{L} = \frac{1}{Q_1+K_1}\begin{bmatrix} Q_1K_1 & Q_1K_a \\ Q_1K_a & K_b(Q_1+K_1) - K_a^2 \end{bmatrix}. \tag{6.24}$$

It is now straightforward to take moments of (4.54) to find $(\partial K/\partial t)_{\text{loops}}$. When we perform the $\mathbf{r}$ and $\mathbf{r}_1$ integrations, we obtain a determinantal factor, which essentially cancels the normalization constants, and a factor corresponding to the expectation value of $\mathbf{r}^2$ in the appropriate distribution.

Let us consider the expectation value factor. In performing the $\mathbf{r}$ and $\mathbf{r}_1$ integrations in the first term, it is simplest to change variables from $\mathbf{r}$ to $\mathbf{r}' = \mathbf{r} + \mathbf{r}_1$, so that $\mathbf{r}^2$ becomes $(\mathbf{r}' - \mathbf{r}_1)^2$. Hence the appropriate expectation value comprises a sum of elements of the matrix $\mathbf{L}$. In the second term, we have only a single element contributing.

In this way we obtain

$$\begin{aligned} \left(\frac{\partial K}{\partial t}\right)_{\text{loops}} = \int_0^\infty dl_1\,\Lambda(l_1)\Bigg[& l\left(K_b - K + \frac{Q_1K_1}{Q_1+K_1}\right) \\ & -\frac{2Q_1}{Q_1+K_1}\int_0^l dy_0\, K_a - \frac{1}{Q_1+K_1}\int_0^l dy_0\, K_a^2 + \frac{1}{Q_1+K_1}\int_{-l_1}^l dy_0\, K_c^2\Bigg]. \end{aligned} \tag{6.25}$$

From (4.40) and (6.25), we now obtain

$$\frac{\dot{\bar{\xi}}_{\text{loops}}}{\bar{\xi}} = \frac{1}{2\bar{\xi}} I\left(\frac{\zeta}{\xi}, \frac{\bar{\xi}}{\xi}, \frac{t}{\xi}\right), \tag{6.26}$$

where I is another dimensionless function of the scale ratios defined by

$$I = \int_0^\infty dl_1\,\Lambda(l_1)\left(2\bar{\xi}l_1 + \frac{Q_1(K_1 - 4\bar{\xi}l_1)}{Q_1+K_1}\right). \tag{6.27}$$

D. Rate of change of ζ

Estimating the rate of change of ζ is less straightforward. The region of the integrand where l_1 is very small is strongly suppressed by the $\Lambda(l_1)$ factor, so we need not be particularly concerned with the behavior of the remaining factors in this region. However, the same is not true of the region where the length l of our original segment becomes small. In the small-l case there is a special feature that requires separate attention.

We noted in Sec. III F that for small values of l_1 the conditional expectation value of $\mathbf{r}_1^2$ is no longer given by the expression (3.37) but rather by (3.51); it reduces to the *un*conditional expectation. The same argument applies to the conditional expectation value of $\mathbf{r}^2$ for small l. It follows that in the contribution to (6.25) arising from the second (negative) term of (4.52), the integrand should in that limit be simply $K \approx l^2$, rather than $K - K_c^2/(Q_1 + K_1)$.

The same argument applies to the first (positive) term. It is true that neither l_1 nor $l + l_1$ goes to zero in the small-l limit, so the mean values of $\mathbf{r}_1^2$, $\mathbf{r}'^2$ or $\mathbf{r}_1 \cdot \mathbf{r}'$ individually should be well approximated by the Gaussian form. However, we are interested in the expectation value of $\mathbf{r}^2$, which is a small difference of these large quantities. Clearly, it too must approach l^2 in the limit $l \to 0$.

If we simply set the mean value of $\mathbf{r}^2$ everywhere equal to $K(l)$, the entire contribution would cancel out. However, this may not be quite correct.

Suppose a loop is formed and that we choose a very small segment of length l on the loop. What is the variance of its extension? Clearly the leading term for small l is l^2, but we could easily have

$$\overline{\mathbf{r}_{\text{loop}}^2} = l^2 - \frac{(1+k)l^3}{3\zeta}, \qquad (6.28)$$

with a value of $k \neq 0$. Indeed, it seems likely that loops are generally kinkier on small scales than long strings, in which case we would expect $k > 0$.

In this case, we easily find

$$\frac{\dot{\zeta}_{\text{loops}}}{\zeta} = k\lambda. \qquad (6.29)$$

VII. STRETCHING

To complete our central task of deriving the various terms in the evolution equations for $p[\mathbf{r}(l)]$ and $K(l)$, we have to estimate the various parameters and unknown functions that appear in them. For the stretching terms, we need to examine the parameters α and β, which are defined by (4.19) and (4.20).

To match the notation of the preceding section, we denote the integration variable in these expressions by y_0. Note that the expectation values appearing here are those conditional on the extension $\mathbf{r}$ of our chosen segment of length l.

A. Evaluation of β

First consider β, given by (4.20). To estimate the expectation value $\langle \mathbf{q}(y_0) \rangle$ conditional on the values of $\mathbf{r}$, we again use the equation of motion for $\mathbf{q}$, (2.8). We could now derive an equation, similar to that for Φ in the preceding section, but now for the complete angular distribution, $\Phi[\mathbf{q}(y_1)]$ say, of $\mathbf{q}(y_1)$.

However, we can use a simplifying ansatz. The effect of $\mathbf{r}$ is important mainly in the region of relatively large values of $y_0 - y_1$, in which it is reasonable to use a generalization of the exponential ansatz (5.14): namely,

$$\Phi[\mathbf{q}(y_1)] = \frac{|\mathbf{b}|}{\sinh|\mathbf{b}|} e^{\mathbf{b}\cdot\mathbf{q}}, \qquad (7.1)$$

where, as in (5.10),

$$\mathbf{b} \approx 3\langle \mathbf{q}(y_1) \rangle. \qquad (7.2)$$

Thus all we need is an equation for the expectation value $\langle \mathbf{q}(y_1) \rangle$.

The required equation is a simple generalization of (5.27): namely,

$$\begin{aligned}(2+\lambda y)\frac{\partial\langle \mathbf{q}(y_1)\rangle}{\partial y} &= H\langle \mathbf{p}(y_1) - \mathbf{q}\,\mathbf{q}\cdot\mathbf{p}(y_1)\rangle \\ &\quad + \frac{\chi y}{\xi^2}\langle \mathbf{q}(y_1)\rangle - \lambda\langle \mathbf{q}(y_1)\rangle \\ &\quad + \lambda\langle \mathbf{q}(y_1)\rangle_X, \end{aligned} \qquad (7.3)$$

with $y = y_0 - y_1$ as before. Note that because $\mathbf{p}(y_0)$ does not appear here, there is only a single stretching term on the right-hand side; the factor of 2 that appeared in (5.27) is absent.

Consider first the region of large y, where the expectation values are small. Within this region we may write, as in (5.31),

$$\langle \mathbf{q}(y_1)\rangle_X = \langle \mathbf{p}(y_1)\rangle + \langle \mathbf{q}(y_1)\rangle \qquad (7.4)$$

and replace $\langle \mathbf{p}(y_1) - \mathbf{q}\,\mathbf{q}\cdot\mathbf{p}(y_1)\rangle$ by $\frac{2}{3}\langle \mathbf{p}(y_1)\rangle$. Then, as in (5.33), we find

$$\begin{aligned}\langle \mathbf{q}(y_0)\rangle \approx -\frac{\lambda + \frac{2}{3}H}{2}\int_0^\infty dy\left(1+\frac{\lambda y}{2}\right)^{n-1} \\ \times e^{-\chi y/\lambda\xi^2}\langle \mathbf{p}(y_1)\rangle,\end{aligned} \qquad (7.5)$$

where n is again given by (5.34).

The conditional expectation value $\langle \mathbf{p}(y_1)\rangle$ which appears here was evaluated in the Gaussian approximation in Sec. III. It is given by (3.41).

As before, there will also be a nonlinear contribution to $\langle \mathbf{q}(y_0)\rangle$ arising from the sharp hole in the angular distribution for z close to 1. So long as the contributions to this quantity are both reasonably small (which appears to be the case), it should again be well represented by a sum

$$\langle \mathbf{q}(y_0)\rangle \approx \langle \mathbf{q}(y_0)\rangle_{\text{lin}} + \langle \mathbf{q}(y_0)\rangle_{\text{nl}}. \qquad (7.6)$$

Moreover, the nonlinear contribution to $\mathbf{q}(y_0)\cdot\mathbf{p}(y_0)$ is not much affected by the value of $\mathbf{r}$, so it is reasonable to set

$$\langle \mathbf{q}(y_0)\rangle_{\text{nl}} \approx -\bar{\alpha}_{\text{nl}}\langle \mathbf{p}(y_0)\rangle. \qquad (7.7)$$

Substituting from (7.6), (7.5), and (7.7), we thus obtain

$$\langle \mathbf{q}(y_0)\rangle = -\bar{\alpha}_{\rm nl}\langle \mathbf{p}(y_0)\rangle - \frac{\lambda+\frac{2}{3}H}{2}\int_0^\infty dy\left(1+\frac{\lambda y}{2}\right)^{n-1} \times e^{-\chi y/\lambda\xi^2}\langle \mathbf{p}(y_0-y)\rangle. \tag{7.8}$$

Now, by (3.41),

$$\langle \mathbf{p}(y_1)\rangle = \frac{K'(y_1)+K'(l-y_1)}{2K(l)}\mathbf{r}. \tag{7.9}$$

Thus, (4.20) yields

$$\beta(l) = \bar{\alpha}_{\rm nl} + \frac{\lambda+\frac{2}{3}H}{4K(l)}\int_0^l dy_0\int_0^\infty dy\,[K'(y_0-y)+K'(l-y_0+y)]\left(1+\frac{\lambda y}{2}\right)^{n-1} e^{-\chi y/\lambda\xi^2}. \tag{7.10}$$

We may now perform the integration over y_0, obtaining finally

$$\beta(l) = \bar{\alpha}_{\rm nl} + \frac{\lambda+\frac{2}{3}H}{4K(l)}\int_0^\infty dy\,[K(l-y)+K(l+y)-2K(y)]\left(1+\frac{\lambda y}{2}\right)^{n-1} e^{-\chi y/\lambda\xi^2}. \tag{7.11}$$

[Recall that for negative values of the argument, $K(y)$ is to be interpreted as $K(|y|)$.] It is interesting to note that at least in this approximation β is actually a function of l only, independent of $\mathbf{r}$.

Although we shall not require an explicit form, it is interesting to note that if K is assumed to have the two-scale exponential form (3.10), the integral (7.11) can be evaluated in terms of the incomplete Γ function. (Note that to do this, one must split the range of integration into separate ranges $0 < y < l$ and $l < y < \infty$.)

B. Evaluation of α

Now let us turn to the evaluation of α. The analogue of (7.5) is of course

$$\langle \mathbf{p}\cdot\mathbf{q}(y_0)\rangle \approx -\frac{\lambda+\frac{4}{3}H}{2}\int_{-\infty}^{y_0} dy_1\left(1+\frac{\lambda(y_0-y_1)}{2}\right)^{n-1} e^{-\chi y/\lambda\xi^2}\langle \mathbf{p}(y_0)\cdot\mathbf{p}(y_0-y)\rangle. \tag{7.12}$$

The factor of $\frac{2}{3}$ is replaced here by $\frac{4}{3}$ because, as in (5.33), we now have to include the effect of the rate of change of $\mathbf{p}(y_0)$.

Substituting from (3.48) and adding a nonlinear contribution, we then get as before

$$\langle \mathbf{p}\cdot\mathbf{q}(y_0)\rangle = -\bar{\alpha}_{\rm nl} - \frac{\lambda+\frac{4}{3}H}{2}\int_0^\infty dy\left\{\frac{1}{2}K''(y)+\frac{1}{4}[K'(y_0)+K'(l-y_0)] \times[K'(y_0-y)+K'(l-y_0+y)]\frac{\mathbf{r}^2-K(l)}{K(l)^2}\right\}\left(1+\frac{\lambda y}{2}\right)^{n-1} e^{-\chi y/\lambda\xi^2}. \tag{7.13}$$

First, let us examine the mean value $\bar{\alpha}$, given according to (4.19) by

$$\bar{\alpha} l = -\int_0^l dy_0\,\overline{\mathbf{p}\cdot\mathbf{q}}(y_0). \tag{7.14}$$

From (7.13), it is obvious that $\overline{\mathbf{p}\cdot\mathbf{q}}(y_0)$ (to which the second term in the braces does not contribute) is in fact independent of y_0. Hence we find

$$\begin{aligned}\bar{\alpha} = -\overline{\mathbf{p}\cdot\mathbf{q}} &= \bar{\alpha}_{\rm nl} + \frac{\lambda+\frac{4}{3}H}{4}\int_0^\infty dy\,K''(y)\left(1+\frac{\lambda y}{2}\right)^{n-1} e^{-\chi y/\lambda\xi^2}\\ &= \bar{\alpha}_{\rm nl} + \frac{\lambda+\frac{4}{3}H}{2}\int_0^\infty dy\,f(y)\left(1+\frac{\lambda y}{2}\right)^{n-1} e^{-\chi y/\lambda\xi^2},\end{aligned} \tag{7.15}$$

which may be compared with KC, Eq. (3.36). The important differences are the two extra factors in the integrand, especially the exponential representing the effect of intercommuting, and the fact that λ appears as well as H in the factor multiplying the integral.

It will be useful to define the dimensionless function F by

$$2\bar{\alpha}_{\rm lin} = F\left(\frac{\zeta}{\xi}, \frac{\bar{\xi}}{\xi}, \frac{t}{\xi}\right) = \frac{\lambda + \frac{4}{3}H}{2} \int_0^\infty dy\, K''(y) \left(1 + \frac{\lambda y}{2}\right)^{n-1} e^{-\chi y/\lambda \xi^2}. \tag{7.16}$$

If $f = \frac{1}{2}K''$ is assumed to have the two-scale form (3.9), and if the length scales of interest are large compared to $1/A$, then F is expressible in terms of incomplete Γ function:

$$F \approx 2w \left(1 + \frac{4H}{3\lambda}\right) x^{-n} e^{x} \Gamma(n, x), \tag{7.17}$$

with

$$x = \frac{2B}{\lambda} + n. \tag{7.18}$$

Returning to the remaining terms in α, we see from (4.19) and (7.13) that it may be written in the form (4.16). The function $\hat{\alpha}$ is given by

$$l\hat{\alpha}(l) = \frac{(\lambda + \frac{4}{3}H)}{8K(l)} \int_0^l dy_0 \,[K'(y_0) + K'(l - y_0)] \times \int_0^\infty dy\, [K'(y_0 - y) + K'(l - y_0 + y)] \left(1 + \frac{\lambda y}{2}\right)^{n-1} e^{-\chi y/\lambda \xi^2}. \tag{7.19}$$

Unfortunately, it is no longer possible to perform the y_0 integration explicitly without assuming the form of K.

C. Rates of change of length scales

It is now easy to compute the rates of change of the various scale lengths.

The rate of change of L is given by (4.15) [together with (7.15)]. From (6.15), we thus find

$$\frac{\dot{\xi}_{\rm str}}{\xi} = H\left(\frac{3}{2} - \frac{\bar{\alpha}_{\rm nl}}{2} - \frac{1}{4}F\right). \tag{7.20}$$

For the other length scales, we examine the rate of change of K, given by (4.22). To do this, we need to examine the large- and small-l behavior of the parameters α and β.

First, we consider the limit $l \to \infty$, where $K(l) \approx 2\bar{\xi}l +$ constant. Substituting this form in the expression (7.11) for β, we find

$$\beta(\infty) = \bar{\alpha}_{\rm nl} + \frac{1}{2}G, \tag{7.21}$$

where the dimensionless function G is given by

$$G\left(\frac{\zeta}{\xi}, \frac{\bar{\xi}}{\xi}, \frac{t}{\xi}\right) = \left(\lambda + \frac{2}{3}H\right) \int_0^\infty dy \left(1 + \frac{\lambda y}{2}\right)^{n-1} \times e^{-\chi y/\lambda \xi^2}. \tag{7.22}$$

This function can be explicitly evaluated in terms of the incomplete Γ function as

$$G = 2\left(1 + \frac{2H}{3\lambda}\right) n^{-n} e^{n} \Gamma(n, n). \tag{7.23}$$

If we use the same large-l approximation in the expression (7.19) for $\hat{\alpha}l$ we see that the y_0 integral is of order $\bar{\xi}^2 l$. Clearly therefore

$$\hat{\alpha}(\infty) = 0. \tag{7.24}$$

These two results together yield

$$\frac{\dot{\bar{\xi}}_{\rm str}}{\bar{\xi}} = H[2\beta(\infty) - \bar{\alpha}] = H\left(\bar{\alpha}_{\rm nl} + G - \frac{1}{2}F\right). \tag{7.25}$$

Next, we turn to the small-l limit, using the approximation $K(l) \approx l^2 - l^3/3\zeta$. Now in (7.11), the expression in square brackets in the integrand is clearly an even function of l, equal to $l^2 K''(y) + O(l^4)$. We shall need the terms in β up to order l. Since the denominator is of order l^2, the l^4 term in the integrand is irrelevant. The integral that appears here is then exactly the same as the one in (7.16).

It is no accident that $\beta(0)$ is almost exactly equal to $\bar{\alpha}$. In fact, it is physically obvious that for short segments, the extension $\mathbf{r}$ and the length l must expand by identical factors. There is an apparent difference between the two, namely the replacement of $(\lambda + \frac{4}{3}H)$ by $(\lambda + \frac{2}{3}H)$. However, this difference is spurious. It arose because in the one case we included the effect of the rate of change of $\mathbf{p}(y_0)$, while in the other we ignored the change of $\mathbf{r}$. But for very short segments, $\mathbf{r}$ and $\mathbf{p}$ have of course essentially the same direction, and it is no longer reasonable to neglect $\dot{\mathbf{r}}$. Therefore in the small-l limit, we ought to replace $(\lambda + \frac{2}{3}H)$ by $(\lambda + \frac{4}{3}H)$.

In this way, we find

$$\beta(l) \approx \beta_0 + \beta_1 \frac{l}{\zeta} = \bar{\alpha}_{\rm nl} + \frac{1}{2}F\left(1 + \frac{l}{3\zeta}\right) \quad (l \ll \zeta). \tag{7.26}$$

The l/ζ term arises from expanding the denominator.

Next we turn to $\hat{\alpha}$. For small l, the leading term in the y_0 integral is

$$\int_0^l dy_0\,[2l][lK''(y)] = 2l^3K''(y). \tag{7.27}$$

Again, therefore, we find the integral F appearing:

$$\hat{\alpha}(l) = \hat{\alpha}_0 + O(l) = \frac{1}{2}F + O(l) \quad (l \to 0). \tag{7.28}$$

In this case, we shall not need the $O(l)$ term.

To evaluate the l^3 term in (4.22), we need to examine the limiting behavior of the function $K_{(2)}(l)$. As we saw in Sec. III F, as $l \to 0$, $K_{(2)} \propto l^5/\zeta$; to be specific let us assume that

$$K_{(2)} \approx C\frac{l^5}{\zeta}, \qquad l \to 0. \tag{7.29}$$

Equivalently, C is given by

$$\frac{K_{(2)}(l)}{K^2(l)} \approx \frac{Cl}{\zeta}, \qquad l \to 0. \tag{7.30}$$

Now from (4.22), we find, using (7.26) and (7.28),

$$\begin{aligned}\frac{\dot{\zeta}_{\rm str}}{\zeta} &= H(3\bar{\alpha} - 2\beta_0 + 6\beta_1 - 12C\hat{\alpha}_0)\\ &= H\left(\bar{\alpha}_{\rm nl} + \frac{3-12C}{2}F\right).\end{aligned} \tag{7.31}$$

VIII. GRAVITATIONAL RADIATION

There is one final effect that we have not so far considered, but which is in fact of great importance in the long-term evolution of the string network: gravitational backreaction. We have been able to ignore it so far because it operates on a very different length scale from most of the other effects.

Consider the gravitational radiation from a large length L of string. Essentially the only scale that can have any relevance here is the smallest scale ζ, which can roughly be identified with a mean interkink distance. We expect the rate of loss of energy, or equivalently length, to be

$$\left(\frac{\partial L}{\partial t}\right)_{\rm GR} = -\Gamma G\mu\frac{L}{\zeta}, \tag{8.1}$$

where Γ is a constant of order 10^1 or 10^2. In other words, the lifetime of the small-scale structure would be of order $\zeta/\Gamma G\mu$.

It should be noted that numerically Γ here may be expected to differ somewhat from the values quoted in the literature, which have mostly been derived from studies of oscillating loops, because in those cases the length scale used was the length of the loop, whereas ζ is defined somewhat differently. In fact, since the typical loop size is probably a few times ζ, our Γ is probably somewhat smaller.

Another way of expressing (8.1) is to think of the gravitational radiation as being generated by each encounter between a pair of kinks, one left moving and one right moving. If we think of ζ, roughly speaking, as the mean interkink distance, we find that the number of such encounters on a left-moving segment of length L in a time interval dt is

$$\frac{2Ldt}{\zeta^2}. \tag{8.2}$$

Hence (8.1) is equivalent to saying that each kink-kink encounter generates the release of an amount of energy equal to

$$\tfrac{1}{2}\Gamma G\mu^2\zeta. \tag{8.3}$$

The gravitational radiation from infinite strings has been studied by several authors [33,34,27]. In particular, Hindmarsh has obtained a formula for the power emitted from encounters between left-moving and right-moving sequences of small-angle kinks. He finds [Ref. [27], Eq. (23)] that the power per unit length is

$$\frac{dP}{dz} = 2\zeta(2)G\mu^2\theta_u^2\theta_v^2\ln(d/r_K)d^{-1}, \tag{8.4}$$

where θ_u and θ_v are the kink angles of the left- and right-moving kinks, d is the interkink distance, and r_K is the width of the string. Since the logarithm is slowly varying, it is reasonable to replace it with a constant, of order 10 to 10^2. The length d is of course related to our ζ; by (3.53), $d \sim \theta^2\zeta$.

Now let us consider how to apply this formula to our problem, namely, how to estimate the rate of loss of energy or length from a chosen segment of left-moving string of length l and extension $\mathbf{r}$. Consider first a relatively short segment, containing a single small-angle kink of angle 2θ. Clearly, we have

$$\theta^2 \sim \frac{l^2 - \mathbf{r}^2}{l^2}. \tag{8.5}$$

Hence the formula (8.4) suggests that the rate of loss of length should be proportional to $(l^2 - \mathbf{r}^2)/l^2$.

As in the case of stretching there is an important consistency condition. The long string can be divided up conceptually into segments of any chosen length, and the proportional rate of loss of length must be independent of the choice, and must agree with (8.1), i.e.,

$$\bar{\dot{l}}_{\rm GR} = -\frac{\Gamma G\mu}{\zeta}l. \tag{8.6}$$

Putting these two requirements together, we find that the expression for $\dot{l}_{\rm GR}$ must be

$$\dot{l}_{\rm GR} = -\frac{\Gamma G\mu}{\zeta}\,l\,\frac{l^2 - \mathbf{r}^2}{l^2 - K(l)}. \tag{8.7}$$

So far, we have concentrated on the change in the length l of our segment, but we should also ask whether there will be any change in the extension $\mathbf{r}$. At first sight, it might seem that the answer should be no. Certainly in the case where l is large, gravitational radiation will change the extension at most by an insignificant amount. However, it is also clear that for small l, particularly if the segment is chosen to end near a kink, there could be a significant reduction in $\mathbf{r}$. Indeed, as we shall see, it turns out that this is essential for consistency.

If there is a reduction in $\mathbf{r}$ it seems reasonable to suppose that, for segments of a given length l, it too is proportional to $(l^2 - \mathbf{r}^2)$; certainly we would expect it to vanish in the extreme case of a straight segment, with $\mathbf{r}^2 = l^2$. Let us therefore assume that it takes the form

$$\dot{\mathbf{r}}_{\mathrm{GR}} = -\frac{\Gamma G\mu}{\zeta} h(l) \frac{l^2 - \mathbf{r}^2}{l^2 - K(l)} \mathbf{r}, \tag{8.8}$$

where $h(l)$ is an as yet unknown function.

It also seems plausible to assume that for very large segments gravitational radiation has no significant effect on the overall extension. This would imply that

$$h(l) \to 0 \quad \text{as } l \to \infty. \tag{8.9}$$

We are now in a position to evaluate the change due to emission of gravitational radiation in the probability distribution $p[\mathbf{r}(l)]$. It is obtained in exactly the same way as in the case of stretching, from the analogue of (4.8): namely,

$$\left(\frac{\partial p}{\partial t}\right)_{\mathrm{GR}} = -\frac{\partial}{\partial l}(\dot{l}_{\mathrm{GR}} p) - \frac{\partial}{\partial \mathbf{r}}(\dot{\mathbf{r}}_{\mathrm{GR}} p) + \frac{\dot{L}_{\mathrm{GR}}}{L} p. \tag{8.10}$$

From this, we easily find

$$\left(\frac{\partial K}{\partial t}\right)_{\mathrm{GR}} = -\frac{\partial}{\partial l}\left(\overline{\dot{l}_{\mathrm{GR}} \mathbf{r}^2}\right) + 2\overline{\mathbf{r}\cdot\dot{\mathbf{r}}_{\mathrm{GR}}} + \frac{\dot{L}_{\mathrm{GR}}}{L} K. \tag{8.11}$$

Substituting from (8.1), (8.7), and (8.8), and using the identity

$$\overline{\mathbf{r}^2 \frac{l^2 - \mathbf{r}^2}{l^2 - K}} = \overline{\mathbf{r}^2} - \overline{\mathbf{r}^2 \frac{\mathbf{r}^2 - K}{l^2 - K}} = K - \frac{K_{(2)}}{l^2 - K}, \tag{8.12}$$

we obtain

$$\left(\frac{\partial K}{\partial t}\right)_{\mathrm{GR}} = \frac{\Gamma G\mu}{\zeta}\left\{ lK' - \frac{\partial}{\partial l}\left(\frac{lK_{(2)}}{l^2 - K}\right) - 2hK + 2h\frac{K_{(2)}}{l^2 - K}\right\}. \tag{8.13}$$

As before, we can now find the rates of change of the various length scales. From (6.15) and (8.1), we get

$$\frac{\dot{\xi}_{\mathrm{GR}}}{\xi} = \frac{\Gamma G\mu}{2\zeta}. \tag{8.14}$$

Similarly, from (4.40) and (8.13), we find

$$\frac{\dot{\bar{\xi}}_{\mathrm{GR}}}{\bar{\xi}} = \frac{\Gamma G\mu}{\zeta}. \tag{8.15}$$

When we come to examine the third length scale, ζ, we can see why the presence of h is essential for consistency. Consider the limit $l \to 0$. For small l, the leading terms in $K(l)$ are given by (3.50), so $l^2 - K \approx l^3/3\zeta$. Moreover, as we saw earlier, $K_{(2)} \approx Cl^5/\zeta$. Thus we find that the expression in the curly brackets in (8.13) has a term that behaves like l^2 for small l. This is inconsistent with the assumed form of K. Hence for consistency of our approximation we must assume that the coefficient of l^2 vanishes, which requires that

$$h(0) = \frac{1 - \frac{9}{2}C}{1 - 3C}. \tag{8.16}$$

The value of $h'(0)$ turns out also to be important, but is not constrained by any consistency requirement.

Then, by (4.42), the l^3 term in (8.13) yields

$$\frac{\dot{\zeta}_{\mathrm{GR}}}{\zeta} = \hat{C}\frac{\Gamma G\mu}{\zeta}, \tag{8.17}$$

where $\hat{C}$ is another constant, related in a somewhat complicated way to the leading terms in the power-series expansions of K, $K_{(2)}$, and h. Specifically, if

$$\begin{aligned} K(l) &= l^2 - \frac{l^3}{3\zeta} + \frac{k_4 l^4}{12\zeta^2} + \cdots, \\ K_{(2)}(l) &= \frac{Cl^5}{\zeta} - \frac{C_6 l^6}{\zeta^2} + \cdots, \\ h(l) &= h(0) - \frac{h_1 l}{\zeta} + \cdots, \end{aligned} \tag{8.18}$$

then

$$\hat{C} = -3(1 - 12C_6 + 3k_4 C) + 2h(0)(1 - 9C_6 + \tfrac{9}{4}k_4 C) + 6h_1(1 - 3C). \tag{8.19}$$

From (3.56) we see that if the model of Gaussian-distributed small-angle kinks is correct, then

$$C = \frac{\overline{\theta^2}}{15}. \tag{8.20}$$

In any event, it seems likely that $C \ll 1$.

As a good first approximation, we may set $C = 0$ and $C_6 = 0$, which means that $h(0) = 1$ and the value of k_4 becomes irrelevant. Then (8.17) becomes

$$\frac{\dot{\zeta}_{\mathrm{GR}}}{\zeta} = (6h_1 - 1)\frac{\Gamma G\mu}{\zeta}. \tag{8.21}$$

Note, however, that the value of h_1 is clearly important. The sign of this term is crucial, as we shall see, in determining the nature of the solution. In our rough approximation, $\hat{C} > 0$ requires $h_1 > \frac{1}{6}$.

IX. OVERALL EVOLUTION EQUATIONS

We can now combine all the various terms together to give composite rate equations for L, for $p[\mathbf{r}(l)]$, and for $K(l)$. We hope to return to these equations at a later date. For the moment, however, we shall not write them down explicitly, but concentrate instead on the equations for the three length scales.

A. Rates of change of length scales

We begin with ξ. From (7.20), (6.16), and (8.14), we obtain

$$\frac{\dot{\xi}}{\xi} = H\left(\frac{3}{2} - \frac{\bar{\alpha}_{\rm nl}}{2} - \frac{1}{4}F\right) + \frac{1}{2\bar{\xi}}c + \frac{\Gamma G\mu}{2\zeta}, \qquad (9.1)$$

where the dimensionless functions F and c were defined in (7.16) and (6.17).

For the other large length scale $\bar{\xi}$, we have, from (7.25), (4.41), (6.26), and (8.15),

$$\frac{\dot{\bar{\xi}}}{\bar{\xi}} = H\left(\bar{\alpha}_{\rm nl} + G - \frac{1}{2}F\right) - \frac{\chi}{w}\frac{\bar{\xi}}{\xi^2} + \frac{1}{2\bar{\xi}}I + \frac{\Gamma G\mu}{\zeta}, \quad (9.2)$$

where I was defined in (6.27).

Finally, we turn to the small-distance scale length ζ. Using (7.31), (4.43), (6.29), and (8.17), we find

$$\frac{\dot{\zeta}}{\zeta} = H\left(\bar{\alpha}_{\rm nl} + \frac{3-12C}{2}F\right) - \frac{\chi\zeta}{\xi^2} + \frac{k}{\bar{\xi}}c + \frac{\Gamma G\mu}{\zeta}\hat{C}, \quad (9.3)$$

where $\hat{C}$ was defined in (8.19).

B. Estimation of parameters

To determine the outcome of the evolutionary process, we first have to estimate how the unknown functions c, F, G, I depend on the ratios of length scales, and the likely magnitude of the additional parameters $\bar{\alpha}_{\rm nl}, w, C, \hat{C}$, and k.

In some ways the most basic parameter is c, which governs the rate of loop formation λ via the relation $\lambda = c/\bar{\xi}$. As we argued in Sec. VI B, if initially all the length scales are of comparable magnitude, then λ will be very small because almost all loops formed reconnect. In that situation, $c \ll 1$. Only when the upper cutoff ξ^2/t has grown to be significantly larger than ζ do many loops start to survive. Once that happens, we expect c to become of order 0.1 to 1. At present, we are not able to calculate the value very precisely, because it is strongly influenced by the intermediate-scale angular correlation effect discussed in Sec. VI B, for which we have only a very qualitative treatment. Fortunately, the precise value of c is not critical, because it appears as a common factor in several of the important terms.

We note that, according to (5.53),

$$\bar{\alpha}_{\rm nl} \approx \frac{\lambda\zeta}{2} = \frac{c\zeta}{2\bar{\xi}}. \qquad (9.4)$$

This parameter will be very small, $\bar{\alpha}_{\rm nl} \ll 1$, throughout most of the relevant region of parameter space, initially because $c \ll 1$ and later because $\zeta \ll \bar{\xi}$. There might be just a short period in which it becomes non-negligible, when ξ has grown large compared to $\bar{\xi}$, but $\bar{\xi}$ and ζ are still comparable in magnitude, but for the most part it can be safely neglected.

Now let us turn to the function G which, in (7.23), was expressed in terms of the incomplete Γ function. It depends primarily on the value of $\lambda\xi$. According to (5.34), when $\lambda\xi$ is small, $n \gg 1$. Then we can use the fact that asymptotically $\Gamma(n,n) \approx \frac{1}{2}\Gamma(n)$ to get

$$G \approx \left(1 + \frac{2H}{3\lambda}\right)\sqrt{\frac{2\pi}{n}} \approx \sqrt{\frac{\pi}{\chi}}(\lambda + \tfrac{2}{3}H)\xi \qquad (\text{small } \lambda\xi). \qquad (9.5)$$

In the opposite (unrealistic) limit of large $\lambda\xi$, where $H\xi$ is certainly negligible, we have

$$G \approx -2\mathrm{Ei}(-n) \approx -2\ln n \approx 4\ln(\lambda\xi) \qquad (\text{large } \lambda\xi). \qquad (9.6)$$

Next, let us consider the function F, given by (7.16). There are two large scales involved here (quite apart from the small scale ζ). The function F depends primarily on the two variables $\lambda\xi$ and $\lambda\bar{\xi}$, though it also has a weak dependence on the small length scale. Note that, according to (7.18), the argument x of the incomplete γ function in (7.17) is

$$x = \frac{2B}{\lambda} + n = \frac{2w}{\lambda\bar{\xi}} + \frac{2\chi}{\lambda^2\xi^2}. \qquad (9.7)$$

Consider first the case where $\lambda\bar{\xi} \gg (\lambda\xi)^2$. Then $x \approx n$ and so (if $H \ll \lambda$)

$$F \approx wG \qquad [\lambda\bar{\xi} \gg (\lambda\xi)^2]. \qquad (9.8)$$

Note that, directly from the integral definitions (7.16) and (7.22), it follows that, at least when H is negligible, we always have $F < G$.

In the opposite limit, where $\lambda\bar{\xi} \ll (\lambda\xi)^2$, we have $x \approx 2w/\lambda\bar{\xi} \gg n$, and consequently $F \ll G$. If $\lambda\bar{\xi} \ll 1$, we can use the large-x form of the incomplete Γ function to get

$$F \approx (\lambda + \tfrac{4}{3}H)\bar{\xi} = \left(1 + \frac{4H}{3\lambda}\right)c \qquad (\text{small } \lambda\bar{\xi}). \quad (9.9)$$

On the other hand, if $\lambda\bar{\xi}$ ever became large, we could use the small-x approximation, obtaining

$$F \approx 2w\ln(\lambda\bar{\xi}) \qquad [(\lambda\xi)^2 \gg \lambda\bar{\xi} \gg 1]. \qquad (9.10)$$

In the intermediate region, where $\lambda\bar{\xi}$ and $\lambda\xi$ are of order unity, or somewhat less, F and G are both likely to be, very roughly, of order unity, with $F < G$.

Strictly speaking, the expressions for F and G are both based on a linear approximation and cease to be valid for large $\lambda\xi$ or $\lambda\bar{\xi}$. Recall that $F/2$ is the linear contribution to the value of $\bar{\alpha}$. From the simulations, we know that $\bar{\alpha} \sim 0.2$, so F is almost certainly significantly less than unity.

Now we turn to the function I defined by (6.27). Initially, if all the length scales are comparable, I, like the other functions, will be small, because the loop-formation probability is small. Consider, however, a later time at which loop formation has started and $\bar{\xi}$ has grown to be $\gg \zeta$. The integral is then dominated by the region where l_1 is a few times ζ. In that region, $K_1 \ll 4\bar{\xi}l_1$. Hence we have, to a good approximation,

$$I \approx 2\bar{\xi} \int_0^\infty dl\, l\Lambda(l) \frac{K(l) - Q(l)}{K(l) + Q(l)}. \tag{9.11}$$

Clearly, the ratio Q/K plays a very important role here. If it is approximately constant over the relevant range, then we get a very simple expression for I:

$$I \approx 2\frac{K - Q}{K + Q}\lambda\bar{\xi} = 2\frac{K - Q}{K + Q}c. \tag{9.12}$$

Now, how large is Q likely to be? If the original expression (4.59) for the loop-formation probability function $\Theta(\mathbf{r}, l)$ were correct, we should expect

$$\frac{1}{Q} = \frac{1}{K} + \frac{2a}{3l^2}, \tag{9.13}$$

where the second term arises from the volume factor Δ. But in addition to this, we saw in Sec. VI B that the angular correlation effect would be expected to contribute another term to $1/Q$. Although we have at present no means of estimating this contribution with any precision, it seems reasonable to expect that it too would be proportional to $1/l^2$ and of a similar order of magnitude. Even from (9.13), we see that $1/Q$ should be significantly larger than $1/K$. The additional angular correlation term enhances this effect, suggesting that $1/Q$ should be very substantially larger. Consequently, $Q \ll K$ and it may be a reasonable first approximation to set $Q/K = 0$, which would mean that $I \approx 2c$. In any event, we expect that the ratio I/c will be not too far below 2. We shall see later that I/c is closely related to the parameter q introduced by CKA [21]; in fact $I/c \approx q - 1$.

There are several other unknown parameters that enter our evolution equations. The parameter w is defined in terms of the large-l behavior of the variance function K: for large l, $K \approx 2\bar{\xi}l - 2\bar{\xi}^2/w$. It was originally introduced in terms of the illustrative two-exponential model of Sec. III A, which suggests that it is limited to the range $0 < w < 1$; a typical value might be $\frac{1}{2}$. So long as it is not *very* small compared to 1, the precise value of w is not critical.

The parameter C is defined by (7.29) in terms of the small-l behavior of the function $K_{(2)}(l)$. Simple models of kinks suggest, e.g., as in (8.20) that C is small compared to unity, say of order 0.1 or less. Within this general range, the precise value is probably not significant.

The effect of gravitational radiation on the small length scale ζ is governed, according to (8.17), by the parameter $\hat{C}$, defined by the rather complicated relation (8.19). In view of the number of independent parameters that contribute to this relation, it seems to be very difficult to estimate $\hat{C}$ from first principles. However, on physical grounds it seems clear that $\hat{C}$ should be positive, and presumably of order unity. The effect of gravitational backreaction *must* be to smooth out the small-scale kinks on the string, and the expected time-scale for this process must be $\zeta/\Gamma G\mu$.

Finally, we have to consider the parameter k. Like I/c, k is related to the parameter $q - 1$ defined by CKA [21], except that it is not concerned with the large-scale wiggles described by $\bar{\xi}$ but with the small-scale kinkiness described by ζ.

One indication of its magnitude can be obtained by considering a related but slightly different parameter. Consider loops of length l and extension $\mathbf{r}$. The probability of finding such a loop is proportional to $p[\mathbf{r}(l)]\Theta(\mathbf{r}, l)$. Hence if we choose a loop of length l at random, the variance of its extension is

$$\left(\frac{1}{K} + \frac{1}{Q}\right)^{-1} = \frac{KQ}{K + Q}. \tag{9.14}$$

Since we know that Q is small compared to K, this variance is $\ll K$. This statement applies to the whole loop, not to a segment on the loop, but of course the two are related. If a loop typically has an extension much less than that of a similar piece of long string, the same must be true for a small segment on the loop, though the effect is probably less dramatic. So we must expect $\overline{\mathbf{r}^2_{\text{loop}}} < K(l)$, which implies $k > 0$. It is not so easy to estimate its magnitude, but as we shall see there are reasons for believing that it cannot be large.

C. Equations for scaling variables

In our previous work [21], to discuss the possibility of scaling, we expressed both our length scales as fractions of the horizon distance $R\tau$. It turns out, however, that we can obtain slightly simpler equations if instead of the horizon we use the expansion time, $1/H$. We define the three dimensionless ratios

$$\gamma = \frac{1}{H\xi}, \quad \bar{\gamma} = \frac{1}{H\bar{\xi}}, \quad \epsilon = \frac{1}{H\zeta}. \tag{9.15}$$

In the radiation era, γ and $\bar{\gamma}$ are identical to the variables used by KC and CKA, but in the matter-dominated era they are half as large.

We also define p so that $R \propto t^{1/p}$, i.e., $H = 1/pt$. Thus $p = 2$ in the radiation era and $p = \frac{3}{2}$ in the matter era.

The dimensionless functions F, G, c, I may now be regarded as functions of the ratios of these variables, e.g.,

$$c = c\left(\frac{\gamma}{\epsilon}, \frac{\gamma}{\bar{\gamma}}, \frac{\gamma}{p}\right). \tag{9.16}$$

Then, substituting into the evolution equations and dividing by H, we obtain

$$\begin{aligned}
-pt\frac{\dot{\gamma}}{\gamma} &= -p + \left(\frac{3}{2} - \frac{\bar{\alpha}_{\text{nl}}}{2} - \frac{F}{4}\right) + \frac{c}{2}\bar{\gamma} + \frac{\Gamma G\mu}{2}\epsilon, \\
-pt\frac{\dot{\bar{\gamma}}}{\bar{\gamma}} &= -p + \left(\bar{\alpha}_{\text{nl}} + G - \frac{F}{2}\right) - \frac{\chi\gamma^2}{w\bar{\gamma}} + \frac{I}{2}\bar{\gamma} + \Gamma G\mu\epsilon, \\
-pt\frac{\dot{\epsilon}}{\epsilon} &= -p + \left(\bar{\alpha}_{\text{nl}} + \frac{3 - 12C}{2}F\right) - \frac{\chi\gamma^2}{\epsilon} + kc\bar{\gamma} \\
&\quad + \Gamma G\mu\hat{C}\epsilon.
\end{aligned} \tag{9.17}$$

It is trivial to write down equations for the rates of change of the *ratios* of length scales. In particular, we find

$$p\frac{d\ln(\bar{\xi}/\zeta)}{d\ln t} = G - 2(1-3C)F - \chi\left(\frac{\gamma^2}{w\bar{\gamma}} - \frac{\gamma^2}{\epsilon}\right) + \left(\frac{I}{2} - kc\right)\bar{\gamma} - \Gamma G\mu(\hat{C}-1)\epsilon. \quad (9.18)$$

This will be useful later.

D. Scaling of ξ and $\bar{\xi}$

We now return to the main issue: what do the evolution equations tell us about how the length scales evolve?

Let us suppose, as in Sec. VI B, that we start with all three length scales comparable in magnitude, and at least somewhat smaller than the horizon size. Initially, most loops that form will reconnect, so c will be small, as will the other dimensionless functions F, G, and I. It is then clear that ξ will start to grow, because of the stretching effect, while initially $\bar{\xi}$ and ζ will not. This will continue until loop production starts to be significant, when the upper cutoff ξ^2/t exceeds a few times ζ. In that region, we have $\zeta \sim \bar{\xi} \ll \xi$.

As λ grows, it will first reach the point where $\lambda\xi \sim 1$, while $\lambda\bar{\xi}$ is still small. In that region, c, F, and I remain small, but G becomes of order unity. From (9.18) it is clear that the ratio $\bar{\xi}/\zeta$ will then start to grow.

Some time later, we may reach the point where $\lambda\bar{\xi}$ is approaching unity, while $\lambda\xi \gg 1$. If so, c, F, and I would be not far below order unity but G would be even larger, so $\bar{\xi}$ would grow rapidly until it caught up with ξ.

Assuming the gravitational radiation terms are still negligible, do we then reach a regime where ξ and $\bar{\xi}$ at least approximately scale?

For this to happen, the right-hand sides of the first two equations in (9.17) must vanish. If we suppose for a moment that the values of c, F, G, I are known, then we can solve for γ and $\bar{\gamma}$. The first equation always yields a positive value of $\bar{\gamma}$:

$$\bar{\gamma} = \frac{1}{c}\left(2p - 3 + \bar{\alpha}_{\rm nl} + \frac{F}{2}\right). \quad (9.19)$$

The condition that the second equation yields a positive value of $\gamma^2/\bar{\gamma}$ is

$$(-2p + 2\bar{\alpha}_{\rm nl} + 2G - F) + \frac{I}{c}\left(2p - 3 + \bar{\alpha}_{\rm nl} + \frac{F}{2}\right) > 0. \quad (9.20)$$

The ratio I/c plays the same role as $q-1$ in CKA. Scaling requires a sufficiently large value of this parameter.

If the condition (9.20) is *not* satisfied, scaling cannot be achieved at the given values of c, F, G, I. But of course these are not constants. If $\bar{\gamma}$ satisfies (9.19) but (9.20) is violated, then clearly $\bar{\gamma}$ will start to grow. The right-hand side of the first of Eqs. (9.17) then becomes positive, so γ begins to fall. In other words, ξ grows faster than $\bar{\xi}$. This of course affects the values of the various functions.

If $c = \lambda\bar{\xi}$ remains of order 0.1 to 1, then $\lambda\xi$ must grow, leading eventually to an increase in G (and a smaller increase in F). This in turn decreases the right-hand side of the first equation in (9.17) and increases that of the second, leading us back towards scaling. It seems likely therefore that the parameters will adjust to reach the point where ξ and $\bar{\xi}$ do indeed scale.

We must also consider what happens to the third length scale in this partial scaling regime. To find out, we examine the third of Eqs. (9.17).

Let us assume that $\gamma, \bar{\gamma} \ll \epsilon \ll (\Gamma G\mu)^{-1}$. Then both the intercommuting and gravitational radiation terms in the third equation are negligible. The essential question is: what is the sign of the right-hand side? Using the partial-scaling value of $\bar{\gamma}$, given by (9.19), we find that for ϵ to grow we need

$$k < \frac{p - \bar{\alpha}_{\rm nl} - \frac{3}{2}(1-4C)F}{2p - 3 + \bar{\alpha}_{\rm nl} + \frac{1}{2}F}. \quad (9.21)$$

For example, if we take $\bar{\alpha}_{\rm nl} = 0$, $C = 0$, $F = 0.4$, and $p = 2$, then we require $k < \frac{7}{6}$. (For a similar value of F, the condition is less restrictive in the matter-dominated era, where $p = \frac{3}{2}$.)

In passing, it is worth asking what would happen if the condition (9.21) were violated. In that case of course ϵ would decrease; ζ would start to catch up with ξ and $\bar{\xi}$. If the dimensionless functions c, F, G, I were simply constants, ζ would continue to grow faster than ξ and $\bar{\xi}$ until it actually exceeded them. This is obvious nonsense. What really happens is that eventually λ starts to drop, thus returning us to a situation closer to our starting point. In fact, it appears that we would then reach a complete scaling regime even without invoking the gravitational radiation term.

This should not be a surprising conclusion. We have already noted that k, like I/c, is analogous to the parameter $q-1$ of our earlier work; k represents the fractional excess small-scale kinkiness of a loop as compared to a long string. If k is large, this means that the loop-formation process very efficiently removes small-scale kinkiness, so that complete scaling of all three length scales is indeed possible.

What the simulations suggest, however, is a very different scenario, in which ξ and $\bar{\xi}$ at least approximately scale, but ζ does not. This is exactly what we expect if the inequality (9.21) *is* satisfied.

Our conclusion here is really a straightforward generalization of our earlier results. In Ref. [21], we showed that scaling should occur if the parameter $q-1$ exceeds a critical value. In our present work, we treat separately the large-scale and small-scale kinkiness.

The parameter that represents the excess large-scale kinkiness is I/c. Provided it is big enough, ξ and $\bar{\xi}$ will scale (and, as we have argued, that is not a restrictive condition, because the parameters can adjust themselves until it is satisfied). On the other hand, for ζ to scale as well, without $\bar{\xi}/\zeta$ becoming large, the excess small-scale kinkiness parameter k would have to exceed a critical value. This condition is almost certainly *not* satisfied in the real system.

E. Possibility of complete scaling

Let us assume then that (9.21) holds. Then while γ and $\bar{\gamma}$ settle down to scaling values, ϵ continues to grow.

This *transient scaling* regime continues until ϵ grows to be of order $(\Gamma G\mu)^{-1}$, so that the gravitational radiation terms become important. The question is: do we then reach a new scaling regime in which all three length scales grow in proportion to t? Equivalently, do we reach a fixed point of the three equations (9.17)?

If there is a complete scaling solution, it is in the region where $\epsilon \gg \gamma \sim \bar{\gamma}$. This means that the intercommuting term in the third equation, $\chi\gamma^2/\epsilon$, is very small. If we neglect it, then it is straightforward, for given values of c, F, G, I, to solve the first and third equations for $\bar{\gamma}$ and ϵ, and then to find γ from the second.

The solutions for $\bar{\gamma}$ and ϵ are given by

$$\bar{\gamma} = \frac{(2p-3)\hat{C} - p + (\hat{C}+1)\bar{\alpha}_{\rm nl} + \frac{1}{2}(\hat{C}+3-12C)F}{(\hat{C}-k)c},$$
$$\epsilon = \frac{p-(2p-3)k-(1+k)\bar{\alpha}_{\rm nl} - \frac{1}{2}(3-12C+k)F}{(\hat{C}-k)\Gamma G\mu}. \qquad (9.22)$$

A consistent solution requires that the numerators of both these expressions be positive. This places a lower bound on $\hat{C}$: namely,

$$\hat{C} > \frac{p-\bar{\alpha}_{\rm nl}-\frac{3}{2}(1-4C)F}{2p-3+\bar{\alpha}_{\rm nl}+\frac{1}{2}F} > k, \qquad (9.23)$$

where the second inequality is merely (9.21).

We recall that, according to (8.17), $\hat{C}$ is the parameter that defines the efficiency with which gravitational radiation removes small-scale kinkiness. It is not at all surprising to find that this parameter must exceed some critical value if scaling is to be achieved. In fact, the second inequality in (9.23) ensures that loop formation is *not* sufficient to induce scaling of ζ, while the first inequality ensures that gravitational radiation *is* sufficient.

The corresponding scaling value of γ is found from the second of Eqs. (9.17). As before, there is a consistency condition stemming from the requirement that $\gamma^2/\bar{\gamma}$ be positive. This condition may be written

$$(p-3+2\bar{\alpha}_{\rm nl}+G)(\hat{C}-k) > \left(1-\frac{I}{2c}\right)\{[2p-3+\bar{\alpha}_{\rm nl}+\tfrac{1}{2}F]\hat{C} - [p-\bar{\alpha}_{\rm nl}-\tfrac{3}{2}(1-4C)F]\}. \qquad (9.24)$$

Since both factors on the right-hand side are definitely positive [the second in virtue of (9.23)], this requires a minimum value of G:

$$G + 2\bar{\alpha}_{\rm nl} > 3-p. \qquad (9.25)$$

As in the earlier transient scaling regime, if this condition is not satisfied, then $\bar{\gamma}$ will start to grow and γ to fall until it is.

We conclude that the parameters $\hat{C}$ and k are the most important in determining whether complete scaling will be achieved. The essential condition is (9.23).

F. Stability

Let us assume that a scaling solution exists and ask whether it is stable.

To study this question, we consider small perturbations in the three scaling parameters, $\delta\gamma$, $\delta\bar{\gamma}$, and $\delta\epsilon$, looking for solutions proportional to t^λ. Strictly speaking, of course, the functions c, F, G, I are functions of these parameters, but we shall assume that they vary slowly over the relevant range, so that they may be treated for the purpose of the stability analysis as constants.

The condition for stability is that the three eigenvalues λ are all negative. In other words, if we linearize the set of equations (9.17) about the presumed scaling solution, the resulting 3×3 matrix must be positive definite. This matrix is

$$\begin{bmatrix} 0 & \frac{c}{2} & \frac{\Gamma G\mu}{2} \\ -\frac{2\chi\gamma}{w\bar{\gamma}} & \frac{\chi\gamma^2}{w\bar{\gamma}^2}+\frac{I}{2} & \Gamma G\mu \\ -\frac{2\chi\gamma}{\epsilon} & kc & \frac{\chi\gamma^2}{\epsilon^2}+\Gamma G\mu\hat{C} \end{bmatrix}. \qquad (9.26)$$

There are three required conditions, the positivity of the trace, of the sum of 2×2 principal minor determinants and of the determinant. The first is automatic. The other two are

$$\frac{\chi\gamma}{w\bar{\gamma}}c + \Gamma G\mu\left[\hat{C}\left(\frac{\chi\gamma^2}{w\bar{\gamma}^2}+\frac{I}{2}\right)-kc\right] + \chi\left[\frac{\gamma^2}{\epsilon^2}\left(\frac{\chi\gamma^2}{w\bar{\gamma}^2}+\frac{I}{2}\right)+\frac{\gamma}{\epsilon}\Gamma G\mu\right] > 0, \qquad (9.27)$$

and

$$(\hat{C}-k)\Gamma G\mu\frac{c\gamma}{w\bar{\gamma}}+\frac{\gamma}{\epsilon}\left[\Gamma G\mu\left(\frac{\chi\gamma^2}{w\bar{\gamma}^2}+\frac{I}{2}-c\right)+\frac{c\chi\gamma^2}{w\bar{\gamma}\epsilon}\right]>0. \tag{9.28}$$

In both cases, the terms are ordered according to the number of powers of the small quantities $\Gamma G\mu$ and γ/ϵ that they contain. We see that in both equations, the leading terms are automatically positive. Thus we conclude that if the scaling solution exists it is almost certainly stable. The essential conditions are the bounds (9.23) on k and $\hat{C}$ required for the solution to exist.

X. CONCLUSIONS

We have analyzed in considerable detail the various processes that affect the evolution of a network of cosmic strings—stretching, intercommuting, loop formation, and gravitational radiation. We have derived a set of coupled evolution equations for the three length scales that describe the configuration—ξ, the interstring distance, $\bar{\xi}$, the large-scale persistence length along the string, and ζ, which characterizes the small-scale kinkiness.

We showed that under reasonable assumptions ξ and $\bar{\xi}$ reach a scaling regime, growing in proportion to the horizon, while ζ grows slowly if at all. This continues until the ratio ξ/ζ becomes large enough for gravitational radiation effects to be significant. Thereafter, it is likely that a new scaling regime is reached, in which all three lengths scale, with $\xi \sim \bar{\xi}$ and $\zeta/\xi \sim \Gamma G\mu$.

These conclusions depend on estimates of various functions and parameters that are still somewhat unreliable. The most significant parameters are k, which describes the excess small-scale kinkiness of loops as compared with long pieces of string, and $\hat{C}$, which determines the rate at which gravitational backreaction smoothes the small-scale kinkiness. The fact that small-scale structure is seen to build up in the simulations strongly suggests that loop formation alone is not able to smooth the small-scale kinkiness, or in other words that k is less than the critical value, (9.21). The essential condition for a full scaling solution to be reached is that $\hat{C}$ exceeds the critical value, i.e., that gravitational radiation *is* effective in smoothing the small-scale kinks.

It would obviously be desirable to be able to estimate the parameters k and $\hat{C}$ from first principles. We hope to do this at a later date.

Another key part of the discussion concerns the buildup of angular correlations between the left and right movers due primarily to loop formation. We have given a fairly precise account of the small-scale effect, leading to correlations between the **p** and **q** vectors, but the treatment of the intermediate-scale correlations involving segments of length a few times ζ is still rather sketchy. It is therefore very important to try to improve this part of the discussion. We hope to return to this question shortly.

Once some of the remaining uncertainties have been resolved, we should be in a position to give a more accurate description of the scaling regime. It is very important to realize that the simulations performed so far, which neglect gravitational radiation, may have given misleading information about the typical scales involved. From the evolution equations, (9.17), it is clear that when the gravitational radiation terms come into play, the effect will be to make $\bar{\gamma}$ decrease. [It is easy to verify that the value of $\bar{\gamma}$ in (9.22) is less than (9.19) provided that (9.21) is satisfied.] In other words, the typical persistence length in the final scaling regime is probably a larger fraction of the horizon distance than it was in the temporary scaling regime where gravitational radiation was negligible — the only one so far accessible to simulations. Whether γ also decreases is less obvious. That seems to depend primarily on the magnitude of the ratio I/c. We hope to provide a more detailed analysis of this behavior in a future publication.

There are many interesting questions that we should be able to address. With an analytic model it should be possible to estimate the typical density fluctuations, and hence microwave anisotropies, generated by a network of strings. Such an approach has previously been adopted in [14] using the one-scale model, but it is essential that we understand the role the small-scale structure plays in determining the form and amplitude of the fluctuations. For example, do we see any evidence of the effects from the string tension renormalization as indicated in the models of Albrecht and Stebbins [5]?

Another interesting aspect which we hope to address concerns the transition period from radiation to matter domination. What happens to the scaling solutions in the two regimes? Do they smoothly evolve from one scaling value to the other? A more detailed analysis of the evolution equations will enable a determination to be made of the approach to scaling. We have established the conditions required for scaling but have not discussed the time scales over which such conditions could be met. In particular, it would be interesting to examine the way in which the duration of the transient scaling regime and the way in which it is goes over to full scaling. This is important because the numerical simulations only directly give us information about evolution over a few expansion times. It might be possible to tackle this problem numerically in a simulation incorporating the effects of gravitational radiation.

There are still many issues to be resolved, but we believe we have provided a firm foundation for future analytic work on the evolution of a network of cosmic strings.

ACKNOWLEDGMENTS

We have benefited from conversations with many colleagues, in particular Andreas Albrecht, Bruce Allen, David Bennett, Franz Embacher, Mark Hindmarsh, Paul

Shellard, Albert Stebbins, Neil Turok, Tanmay Vachaspati, and Alex Vilenkin. We are grateful for the hospitality of the Instititute for Theoretical Physics, University of California at Santa Barbara, where part of this work was performed. This research was supported in part by the National Science Foundation under Grant No. PHY89-04035. D.A. and E.C. are indebted to the Science and Engineering Research Council for support.

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[1] T.W.B. Kibble, J. Phys. A **9**, 1387 (1976).
[2] Ya.B. Zel'dovich, Mon. Not. R. Astron. Soc. **192**, 663 (1980).
[3] A. Vilenkin, Phys. Rev. Lett. **46**, 1169 (1981); **46**, 1496(E) (1981).
[4] A. Vilenkin and T. Vachaspati, Phys. Rev. Lett. **67**, 1057 (1991).
[5] A. Albrecht and A. Stebbins, Phys. Rev. Lett. **68**, 2121 (1992); **69**, 2615 (1992).
[6] R.H. Brandenberger, Phys. Scr. **T36**, 114 (1991).
[7] C.J. Hogan and M.J. Rees, Nature (London) **311**, 109 (1984).
[8] D.R. Stinebring, M.F. Ryba, J.H. Taylor, and R.W. Romani, Phys. Rev. Lett. **65**, 285 (1990).
[9] D.P. Bennett and F.R. Bouchet, Phys. Rev. D **41**, 720 (1990).
[10] A. Vilenkin, Phys. Rev. D **24**, 2802 (1981).
[11] A. Vilenkin, Astrophys. J. **282**, L51 (1984).
[12] J.R. Gott, Astrophys. J. **288**, 422 (1985).
[13] N. Kaiser and A. Stebbins, Nature (London) **310**, 391 (1984).
[14] T.W.B. Kibble, Nucl. Phys. **B252**, 227 (1985).
[15] D.P. Bennett, Phys. Rev. D **33**, 872 (1986).
[16] D.P. Bennett, Phys. Rev. D **34**, 3592 (1986).
[17] A. Albrecht and N. Turok, Phys. Rev. D **40**, 973 (1989).
[18] D.P. Bennett and F.R. Bouchet, Phys. Rev. Lett. **60**, 257 (1988); **63**, 2776 (1989).
[19] B. Allen and E.P.S. Shellard, Phys. Rev. Lett. **64**, 119 (1990); E.P.S. Shellard and B. Allen, in *The Formation and Evolution of Cosmic Strings*, edited by G.W. Gibbons, S.W. Hawking, and T. Vachaspati (Cambridge University Press, Cambridge, England, 1990), pp. 421–448.
[20] T.W.B. Kibble and E. Copeland, Phys. Scr. **T36**, 153 (1991).
[21] E. Copeland, T.W.B. Kibble, and D. Austin, Phys. Rev. D **45**, R1000 (1992).
[22] F. Embacher, Nucl. Phys. **B387**, 129 (1992); **B387**, 163 (1992).
[23] A. Vilenkin, Phys. Rev. D **23**, 852 (1981).
[24] N. Turok, Nucl. Phys. **B242**, 520 (1984).
[25] C.J. Burden, Phys. Lett. **164B**, 277 (1985).
[26] T. Vachaspati and A. Vilenkin, Phys. Rev. D **31**, 3052 (1985).
[27] M.B. Hindmarsh, Phys. Lett. B **251**, 28 (1990).
[28] J. Quashnock and D. Spergel, Phys. Rev. D **42**, 2505 (1990).
[29] B. Allen and R. Caldwell, Phys. Rev. D **43**, 2457 (1991).
[30] J. Quashnock and T. Piran, Phys. Rev. D **43**, 3785 (1991).
[31] D. Austin, Phys. Rev. D **48**, R3422 (1993).
[32] A. Albrecht, in *The Formation and Evolution of Cosmic Strings* [19], pp. 403–419.
[33] T. Vachaspati, Nucl. Phys. **B277**, 593 (1986).
[34] D. Garfinkle, Phys. Rev. D **41**, 1112 (1990).