Cosmological rotation of quantum-mechanical origin and anisotropy of the microwave background

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It is shown that rotational cosmological perturbations can be generated in the early Universe, similar to gravitational waves. The generating mechanism is quantum-mechanical in its nature, and the created perturbations should now be placed in squeezed vacuum quantum states. The physical conditions under which the phenomenon can occur are formulated. The generated perturbations can contribute to the large-angular-scale anisotropy of the cosmic microwave background radiation. An exact formula is derived for the angular correlation function of the temperature variations caused by the quantum-mechanically generated rotational perturbations. The multipole expansion begins from the dipole component. The comparison with the case of gravitational waves is made.

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I. INTRODUCTION

The recent discovery of the large-angular-scale anisotropy of the cosmic microwave background radiation (CMBR) [1] is leading to important implications. It shows that the deviations of our Universe from homogeneity and isotropy extend to the very large linear scales, of order and longer than the present day Hubble radius l_H. The main contribution to the large-angular-scale anisotropy is usually provided by perturbations with wavelengths λ of order of l_H. In other words, the lower index multipole components of $\delta T/T$, such as quadrupole ($l = 2$), octupole ($l = 3$), etc., are usually dominated by such long wavelength perturbations. In general, all possible wavelengths contribute to every single multipole. One can devise a spectrum of cosmological perturbations in such a way that the contributions to, say, the quadrupole component will be dominated by perturbations with $\lambda \gg l_H$ or, instead, by perturbations with $\lambda \ll l_H$. But, if the spectrum is not exceedingly "red" or "blue," the quadrupole component will be dominated by perturbations with wavelenghts λ of order of l_H. This means that the existence of the very long wavelength cosmological perturbations can now be considered as an observational fact. (We assume, of course, that the measured $\delta T/T$ is a genuine cosmological effect and not, say, a result of the poor data processing or unaccounted sources of different nature.) It is very likely that perturbations with such long wavelengths are "primordial," survived from the epochs when the Universe was much younger. It is hard to arrange for these perturbations to have been generated or contaminated by the local physical processes within the present day Hubble radius. If so, what is the origin and nature of these perturbations?

Following Lifshitz [2], it is customary to identify the perturbations superimposed on homogeneous isotropic cosmological models [Friedmann-Lemaitre-Robertson-Walker (FLRW), models] as density perturbations, rotational perturbations, and gravitational waves. This division is dictated by the classification of the 3×3 symmetric matrix whose elements describe all the independent components of the perturbed gravitational field. Two transverse-transverse components correspond to gravitational waves, two transverse-longitudinal components correspond to rotational perturbations, and two remaining components, one of which is scalar and another is longitudinal-longitudinal, correspond to density perturbations. The associated gravitational field of all of these perturbations is capable of affecting the propagating photons of CMBR and producing the angular anisotropy $\delta T/T$ [3]. So, in principle, one of these perturbations or all of them together could be responsible for the observed $\delta T/T$. However, in the framework of classical cosmology, there is not much to say about the expected types, amplitudes, and spectra of the perturbations. In classical cosmology, perturbations are put in game "by hand." If there is no initial perturbations, there is nothing to discuss.

It is remarkable that the situation changes dramatically if one takes into account the principles of quantum mechanics. It turns out that there may be no need anymore for initial perturbations, there may be no need for complicated generating mechanisms. It may be sufficient to have just what we already have: the variable gravitational field of the nonstationary homogeneous isotropic universe and the zero-point quantum fluctuations of cosmological perturbations listed above. The strong variable gravitational field of the early Universe plays the role of the "pump" field. It can supply energy to the zero-point quantum fluctuations and amplify them. It was first demonstrated for gravitational waves [4], and we will do here the same for rotational perturbations. In a more strict and precise language, the perturbations are parametrically coupled to the variable gravitational pump field and their interaction Hamiltonian is time dependent. The initial vacuum state of the quantized perturbations transforms, as a result of the quantum-mechanical Schrödinger evolution, into a multiparticle state known as a squeezed vacuum state. This applies to gravitational waves and density perturbations [5], and to rotational perturbations too, as will be shown below. The quantum mechanically generated perturbations can be useful for many purposes. Among other things, they are capable of producing $\delta T/T$ (for gravitational waves, see [6]).

It is important, however, to stress from the very beginning that there is a significant qualitative difference between gravitational waves on one side and rotational and density perturbations on the other side. Gravitational waves exist in empty space, they do not require any material medium for their definition. One or another sort of matter in FLRW models is only needed to govern the variable gravitational pump field but has nothing to do with gravitational waves themselves. Moreover, there is no ambiguity as for how gravitational waves couple to the pump field. The form of coupling follows from the Einstein equations and is such that the waves can be amplified indeed. In contrast, rotational and density perturbations are only defined if there is a cosmological medium, they are perturbations in the medium. The solutions to Einstein equations give zero for all non-transverse-transverse components of the perturbed gravitational field, if there is no matter. The difference between gravitational waves and other cosmological perturbations is about the same as the difference between photons, which exist in empty space, and phonons and various "excitons" which exist only in a condensed matter. It is necessary for the primeval cosmological medium, which has been filling the Universe in the very distant past, to be capable of supporting free oscillations of density and rotation regardless of the presence or absence of the pump field. In other words, a deformed element of the primeval medium should have been capable of experiencing the necessary restoring forces even if the element had been experimented with in our laboratory on Earth. Moreover, the coupling of these "harmonic oscillators" to the gravitational pump field should be appropriate if one is intended to amplify their zero-point quantum fluctuations. Thus, every attempt to apply the quantum-mechanical generation mechanism to density and rotational cosmological perturbations will always require the satisfaction of additional physical hypotheses, as compared with the case of gravitational waves, and we will not be able to avoid making such additional hypotheses in what follows.

It is very interesting that the behavior of particles and fields in the nonstationary Universe bothered Schrödinger as long ago as in 1939 [7]. His paper is remarkable in several aspects. It is instructive to follow his argumentation in order to understand better the physics involved and what we will be doing below.

Schrödinger considers "the familiar wave equation of the second order

$$-\Delta\psi + \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \mu^2\psi = 0 \tag{1}$$

($\mu = 0$ for light, $\mu = 2\pi mc/h$ for material particles)" which "is to be regarded as the covariant equation"

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x_\alpha}\left(g^{\alpha\beta}\sqrt{-g}\frac{\partial\psi}{\partial x_\beta}\right) + \mu^2\psi = 0 \tag{2}$$

specialized for the line element of the nonstatic universe,

$$ds^2 = c^2dt^2 - R^2(t)dl^2, \tag{3}$$

where dl^2 corresponds, in his paper, to a three-sphere. He discusses "the decomposition of an arbitrary wave function into proper vibrations" and notices that the positive and negative frequencies of proper vibrations "cannot be rigorously separated in the expanding universe." He says: "Generally speaking this is a phenomenon of outstanding importance. With particles it would mean production or annihilation of matter, merely by the expansion, whereas with light there would be a production of light traveling in the opposite direction, thus a sort of reflection of light in homogeneous space." He also describes this as a "mutual adulteration of positive and negative frequency terms in the course of time" giving rise to what he calls "the alarming phenomena." Schrödinger concludes: "They are certainly very slight, though, in two cases: viz., (1) when R varies slowly, (2) when it is a linear function of time," and he also says, with a sigh of relief: "in this case the alarming phenomena do not occur, even within arbitrarily long periods of time," "there is nothing like a secularly accumulated pair production."

From the position of current knowledge, we can make some comments on Schrödinger's paper. (It is easy to be critical of a great physicist's paper if you are doing this more than half a century later.)

First, Eq. (2) with $\mu = 0$ is not the equation for light, it is the equation for a hypothetical "massless scalar particle." As for light, Schrödinger would, perhaps, be pleased to know that electromagnetic waves in the nonstatic universe (3) are totally "immune," there is no "mutual adulteration" and "reflection" of electromagnetic waves at all. The covariant Maxwell equations

$$\frac{\partial F_{\mu\nu}}{\partial x^\alpha} + \frac{\partial F_{\nu\alpha}}{\partial x^\mu} + \frac{\partial F_{\alpha\mu}}{\partial x^\nu} = 0\,,$$

$$F_{\mu\nu;}{}^{\nu} = 0$$

specialized for the line element of the nonstatic universe (3) have solutions $F_{\mu\nu}$ exactly the same as solutions $F_{\mu\nu}$ to the Maxwell equations in the Minkowski world. The scale factor $R(t)$ participates in the electromagnetic energy-momentum tensor and makes the energy density vary as R^{-4}, which is consistent, of course, with photons being redshifted or blueshifted. But, if one sends light at $t = t_1$ and receives at $t = t_2$, the result is totally independent of the variability rate of $R(t)$ in the interval of time between t_1 and t_2 (in sharp contrast to what takes place for gravitational waves).

Second, Schrödinger's conclusion about the extreme weakness of the "alarming phenomena" when $R(t)$ is a linear function of time, for the kind of the wave equation with $\mu = 0$ he studies, is premature. This statement requires us to go into some detail. Schrödinger takes the function ψ as a product of the time-dependent amplitude

$f(t)$ and a spatial eigenfunction with the wave number n. For $f(t)$ he derives the equation

$$R^{-3}\frac{d}{dt}\left(R^3\frac{df}{dt}\right)+\frac{c^2n(n+2)}{R^2}f=0\,. \tag{4}$$

We may note that the same equation in the spatially flat universe (3) would read

$$R^{-3}\frac{d}{dt}\left(R^3\frac{df}{dt}\right)+\frac{c^2n^2}{R^2}f=0 \tag{5}$$

and

$$u''+\left(n^2-\frac{R''}{R}\right)u=0 \tag{6}$$

if one makes the substitutions $u = fR$, $R(\eta)d\eta = cdt$, $' = d/d\eta$.

Schrödinger investigates the special case

$$R=a+bt\quad . \tag{7}$$

He writes: "Specialising for light ($\mu = 0$) and putting for the moment

$$\frac{c^2n(n+2)}{b^2}=k^2+1 \tag{8}$$

we have $\cdots f = \frac{1}{R}R^{\pm ik}$." He comes to the conclusion that the solution $f(t)$ is oscillatory, with the amplitude being proportional to R^{-1}, and that the waves traveling in opposite directions keep rigorously separated for any length of time. This conclusion is a consequence of his following statement: "Since n is very large, k is real, for b is certainly not much larger than c."

It is here where the core of the problem lies. Presumably, it is hard for Schrödinger to accept that the velocity of change of the scale factor can be larger than the velocity of light, this is why "b is certainly not much larger than c." If he relaxed this assumption, the left-hand side of Eq. (8) could be made less than 1 and, hence, k could be made imaginary, at least, for some values of n. This could be done even in the closed universe, where the set of n is discrete and begins from some lowest wave number labeling the longest wavelength which can still fit into the Universe. In the spatially flat universe, the situation is even better. Equation (8) now reads

$$\frac{c^2n^2}{b^2}=k^2+1\,, \tag{9}$$

where n is continuous and can go to zero. In this case, one can even accept $b < c$, as there will always be some n's for which k is imaginary.

The factor (cn/b) plays a crucial role. This factor is the ratio of the characteristic time scale on which $R(t)$ varies and the period of wave with the wave number n. According to Schrödinger's assumptions this ratio is much larger than 1. But this is precisely the condition which allows us to say that "$R(t)$ varies slowly," adiabatically. If this condition is satisfied, the second Schrödinger case belongs, in fact, to the same category as the first case. There is no wonder that the effect is "very slight." Schrödinger's assumptions exclude the possibility of the "alarming phenomena" from the very beginning. It is precisely when this ratio is less than 1 that the character of the solution for $f(t)$ changes, it ceases to be oscillatory, and the final effect becomes significant. It is not a particular functional dependence $R(t)$ itself that is important, but comparison of the time scale of variation of $R(t)$ with the period of a given wave.

This can be conveniently illustrated with the help of Eq. (6). In the interval of time where $R(t)$ obeys Eq. (7), the height of the "potential barrier" $U(\eta) \equiv R''/R$ is just a constant: $R''/R = b^2/c^2$. One can arrange for the $R(\eta)$ to be such that $U(\eta)$ would go to zero, more or less gradually, outside this interval of time. Then, for the high-frequency waves, i.e., for $n^2 \gg b^2/c^2$, the solution is $u = A\sin(n\eta + \varphi)$, where A is almost a constant, the "alarming phenomena" do not occur. However, for the low-frequency waves, for which $n^2 \ll b^2/c^2$, and which, therefore, hit the barrier, the solution is $u = A\sin(n\eta + \varphi_1)$ initially, that is far to the left of the barrier, and $u = B\sin(n\eta + \varphi_2)$ finally, that is far to the right of the barrier. The remarkable fact is that one always gets $\bar{B}^2 \gg A^2$ (the overbar denotes averaging over the arbitrary initial phase φ_1), and the wave amplification takes place. It is not even important whether the Universe was expanding or contracting. With the spatial dependence taken into account, the effect means amplification of the traveling wave and simultaneous generation of the "reflected" wave, exactly as Schrödinger envisaged.

Equations (5) and (6) are precisely the equations for the time-dependent amplitudes of gravitational waves (not electromagnetic waves). They have been investigated in [4] and the scale factor of the form (7) has been considered as an example (unfortunately, I did not know about Schrödinger's paper at that time and became aware of it much later). The quantum-mechanical version of the process considered above leads to the generation of gravitons (gravitational waves). The initial vacuum state evolves into a strongly squeezed vacuum state with the expectation number of gravitons much larger than one. One can also think of this process, in classical terms, as of generation of standing gravitational waves. It is a pity that we will never know whether Schrödinger would qualify the production of gravitational waves as an "outstanding" or "alarming" phenomenon.

It is necessary to say that a systematic study of the quantized version of Eq. (2) was first undertaken by Parker [8]. He investigated the general principles of the field quantization in spatially flat FLRW models and formulated a number of important theorems. In particular, Parker emphasizes the important role of conformal invariance and exhibits the ambiguity in the choice of the equation for scalar particles. He recognizes that the zero or nonzero production of the massless scalar particles depends on the chosen form of the wave equation (form of coupling of particles to the external gravitational field). However, one of his conclusions, "we show that massless particles of arbitrary nonzero spin, such as photons or gravitons, are not created by the expansion, regardless of its form," turned out to be incorrect in part of the gravitons.

The conviction that massless particles cannot be created in FLRW gravitational fields was predominant in late 1960s through early 1970s. Since the creation of massive particles (with masses less than the Planckian) was shown to be negligibly small, it has led to the study of anisotropic cosmologies. It was believed that only in this case the effects are nonvanishing and interesting. For instance, Zeldovich and Starobinsky, in an influential paper [9], have laid a foundation for quantization of fields in the spatially flat anisotropic models. One of their conclusions is "It is established that vacuum polarization and production of particles of a zero mass field are absent only in the isotropic case," and they call the isotropic case "degenerate."

The interest toward quantized fields and quantized cosmological perturbations has increased in the context of the inflationary hypothesis [10]. The inflationary models belong to the class of FLRW homogeneous isotropic models and have a specific form of the scale factor $R(t)$. The authors of papers on quantized perturbations often associate the generating mechanism with such things as "inflation," "rolling the scalar field down the inflationary potential," "horizons," etc. The reader can find a review of the extensive literature on the subject in [11]. The important early references on the quantization of density perturbations are [12,13].

Our goal is to study the quantum-mechanical generation of rotational perturbations in FLRW cosmological models. The notion of a gravitational pump field acting on a harmonic oscillator, often used above, may seem to be too vague and overly mechanistic. In my view, however, this notion accurately and precisely describes the physics involved, if one is willing to use the "field-theoretical" formulation of Einstein's general relativity (see, for instance, [14]). In this approach, the FLRW cosmological models get represented as gravitational fields in Minkowski space-time and the scale factor $R(t)$ manifests itself, literally, as the gravitational pump field. The coupling of the gravitational pump field to the quantized gravitational perturbations is provided by the nonlinear character of the total gravitational field. The form of coupling follows automatically (at least, in the case of gravitational waves) from the Einstein equations which acquire now the form of nonlinear wave equations in Minkowski space-time. Along this route, one can easily demonstrate the far-reaching analogy between the quantum-mechanical generation of cosmological gravitational waves and the generation of squeezed light in the laboratory settings of quantum nonlinear optics [15]. However, the "field-theoretical" approach is not familiar to, or not accepted by, most researchers on the subject, and, therefore, we will be mostly using here the usual geometrical formulation.

The basic equations for rotational perturbations in the spatially flat FLRW cosmological models are presented in Sec. II. As one of our physical hypotheses, we assume that the perturbations could sustain as torque oscillations in the primeval cosmological medium. In other words, we assume that the torsional velocity of sound was not equal to zero and, in fact, it could be as big as the velocity of light. We also assume that the coupling of the torque oscillations to the curvature was "minimal," that is, the same as for gravitational waves. Under these conditions, the time-dependent amplitudes of the rotational perturbations satisfy Eq. (6) and the most of the results previously derived for gravitational waves can be used here. The quantization of rotational perturbations is considered in Sec. III. The important difference with the case of gravitational waves is only in the form of polarization tensors. All conclusions with regard to squeezing and standing waves are essentially the same. In Sec. IV, as an application to realistic cosmological models, we consider the scale factors which are power-law dependent on the η time. They include the scale factors of the inflationary models. We assume that the torsional velocity of sound drops to zero at the beginning of the matter-dominated stage. The torque oscillations get converted into usual rotational perturbations governed by the conservation law for the angular momentum. In Sec. V, we consider the angular anisotropy of the CMBR produced by rotational perturbations. Rotational perturbations generate all multipoles of $\delta T/T$ beginning from the dipole component ($l = 1$). An exact formula is derived for the expected angular correlation function of $\delta T/T$ caused by rotational perturbations of quantum-mechanical origin. The results are compared with those for gravitational waves. A brief summary is given in Sec. VI.

II. BASIC EQUATIONS FOR ROTATIONAL PERTURBATIONS

We write the unperturbed metric in the form

$$ds^2 = a^2(\eta)(d\eta^2 - dl^2) \,, \tag{10}$$

where

$$dl^2 = dx^{1^2} + dx^{2^2} + dx^{3^2} \,. \tag{11}$$

The scale factor $a(\eta)$ is governed by matter with the energy-momentum tensor $T_{\mu\nu}$. The nonvanishing components of the unperturbed $T_{\mu\nu}$ must have the form $T_0^0 = \epsilon(\eta)$, $T_i^k = -p(\eta)\delta_i^k$. The perturbed metric can be written as

$$ds^2 = a^2(\eta)(\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \,. \tag{12}$$

Without loss of generality, one can choose a synchronous coordinate system, so that $h_{00} = 0$, $h_{0i} = 0$. The nonvanishing components of $h_{\mu\nu}$ form a symmetric 3×3 matrix h_{ij}.

The construction of rotational perturbations is based on the three-vector fields $Q_i(x^1, x^2, x^3)$ with vanishing divergence [2]. They are eigenfunctions of the Laplace operator in the three-space (11) and satisfy the equations

$$Q_{i,k,}{}^k + n^2 Q_i = 0 \quad Q^i{}_{,i} = 0 \,. \tag{13}$$

All the manipulations with Q_i are to be performed with the help of the metric tensor defined by Eq. (11). Using Q_i, one can construct a symmetric tensor Q_{ij} and an antisymmetric tensor W_{ij}:

$$Q_{ij} = -\frac{1}{2n}(Q_{i,j} + Q_{j,i}), \quad W_{ij} = -\frac{1}{2n}(Q_{i,j} - Q_{j,i}).$$

In Sec. II we need only to know the general properties of rotational perturbations, which are listed above, but we will also discuss their explicit functional dependence later on.

With a given eigenfunction Q_i is associated the contribution to h_{ij} which can be written as

$$h_{ij} = 2h(\eta)Q_{ij}. \tag{14}$$

The nonvanishing components of perturbations to the energy-momentum tensor $T_{\mu\nu}$ may only have the general form

$$\begin{aligned} T_i^0 &= a^{-2}\omega(\eta)Q_i, \\ T_0^i &= -a^{-2}\omega(\eta)Q_i, \\ T_i^k &= 2a^{-2}\chi(\eta)n^{-1}Q_i^k. \end{aligned} \tag{15}$$

The perturbations in the matter distribution and the accompanying perturbations in the gravitational field are governed together by the perturbed Einstein equations. The functions $h(\eta)$, $\omega(\eta)$, $\chi(\eta)$ are to be determined from the equations

$$h'' + 2\frac{a'}{a}h' = 2\kappa\chi(\eta)n^{-1}, \tag{16}$$

$$-\frac{n}{2}h' = \kappa\omega(\eta), \tag{17}$$

where $\kappa = 8\pi G/c^4$. A consequence of Eqs. (16) and (17) is the equation

$$\omega' + 2\frac{a'}{a}\omega + \chi = 0. \tag{18}$$

One can also consider Eq. (16) as a consequence of Eqs. (17) and (18). [Equations (16)–(18) are consistent with those in Bardeen's paper [16] if one corrects the following misprints: the factor $k/5$ in Eq. (A2c) should read k and the factor k in Eq. (A5) should read $k/2$.]

Let us first study Eq. (18). This equation is the linearized version of the covariant equations $T_{i\ ;\alpha}^{\alpha} = 0$ which have the meaning of the differential conservation laws in curved space-time, that is, in the presence of gravitational field. They have, of course, a definite physical meaning even when the gravitational field can be neglected.

In case of a material medium placed in flat space-time, we write

$$T_{i,0}^0 + T_{i,k}^k = 0. \tag{19}$$

The components T_i^0 describe fluxes of energy (momenta), the components T_i^k describe fluxes of momentum (stresses). Taking derivatives of Eq. (19), one arrives at the equation

$$\left(T_{i,j}^0 - T_{j,i}^0\right)_{,0} + \left(T_{i,k,j}^k - T_{j,k,i}^k\right) = 0. \tag{20}$$

The antisymmetric tensor $\omega_{ij} = T_{i,j}^0 - T_{j,i}^0$ is the rotation tensor defined locally, in every point. The antisymmetric tensor $f_{ij} = T_{i,k,j}^k - T_{j,k,i}^k$ describes the torque forces acting on an element of the deformed medium. In flat space-time, the components of the energy-momentum tensor can be taken in the same form as in Eq. (15) assuming that $a(\eta) = 1$ [see also Eq. (10)]. Then, one gets $\omega_{ij} = -2n\omega(\eta)W_{ij}$ and $f_{ij} = -2n\chi(\eta)W_{ij}$. The function $\omega(\eta)$ plays the role of the angular velocity. The nonzero $\omega(\eta)$ is a confirmation of the fact that we are dealing with the rotational motion, as opposed to the potential motion. (We will not be going here into further detail such as the presence or absence of vortex lines, etc.) By substituting ω_{ij} and f_{ij} in Eq. (20) one derives the equation

$$\omega' + \chi = 0 \tag{21}$$

which basically expresses a familiar law: the rate of change of the angular momentum is equal to the torque. This equation is precisely the limiting form of Eq. (18) in the limit of $a'/a \to 0$.

Not every deformed medium experiences torque forces. But if it does, the medium can support torque oscillations. They arise if a given element of the medium is displaced from the equilibrium position at a small angle $\theta(\eta)$. The restoring force is usually proportional to $\theta(\eta)$: $\chi(\eta) \sim b^2\theta(\eta)$, where $b^2 = v_t^2/c^2$ and v_t is the torsional velocity of sound. One can write

$$\chi(\eta) = n^2b^2\theta(\eta). \tag{22}$$

Since $\omega(\eta) = \theta'(\eta)$, Eq. (21) takes the form of the equation for a harmonic oscillator with frequency nb:

$$\theta'' + n^2b^2\theta = 0. \tag{23}$$

Equation (22) provides us with the simplest dispersion law known in the theory of vibrations: frequency of oscillations is linearly proportional to the wave number. In the limit of $v_t \to 0$, the frequency goes to zero and the angular velocity $\omega(\eta)$ takes a constant value (free rotation).

In cosmological problems, one often considers an ideal fluid with the energy-momentum tensor $T_{\mu\nu} = (\epsilon + p)u_\mu u_\nu - pg_{\mu\nu}$. A deformed element of the ideal fluid does not experience torque forces. The perturbed components of T_μ^ν are $T_i^0 = (\epsilon + p)u^0\delta u_i$, $T_i^k = -\delta p\delta_i^k$, and the tensor f_{ij} vanishes. This means that the $\chi(\eta)$ should be put equal to zero. The angular momentum conservation law in the nonstationary Universe, Eq. (18), leads to $\omega(\eta) = \text{const} \times a^{-2}$. In other words, rotation in the expanding Universe filled with the ideal fluid can only decay. (This was recognized by Zeldovich long ago [17].) Accordingly, the function $h'(\eta)$ accompanying rotational perturbations does also decay, as follows from Eq. (17). Moreover, if $\omega(\eta) = 0$, Eqs. (16) and (17) require $h(\eta) = \text{const}$ which means that there are no perturbations at all, as $h_{ij} = \text{const} \times Q_{ij}$ can be reduced to zero by a simple transformation of spatial coordinates.

Our main physical hypothesis is that the primeval cosmological medium could have supported torque oscillations. We assume that Eq. (22) was valid in the very early Universe. Then, Eq. (18) reads

$$\theta'' + 2\frac{a'}{a}\theta' + n^2 b^2 \theta = 0 .$$

Also, with the help of Eq. (17), one can write Eq. (16) in the form

$$h'' + 2\frac{a'}{a}h' + n^2 b^2 h = 0 \qquad (24)$$

and, by introducing $\mu(\eta) = ah(\eta)$, in the form

$$\mu'' + (n^2 b^2 - a''/a)\mu = 0. \qquad (25)$$

Equations (24) and (25) govern the time-dependent amplitude of components h_{ij} of the perturbed gravitational field accompanying torque oscillations in the medium. These equations have the same form as equations for the time-dependent amplitude of gravitational waves. The only difference is that the constant b is not strictly 1, as in the case of gravitational waves, but can now lie in the interval from 0 to 1. We will not hesitate to assume that b can be close to 1 or equal 1. In general, the properties of the primeval cosmological medium (presently unknown) could require the relationship (22) to be more complicated and could make the torsional velocity of sound a function of time. Nevertheless, it is sufficient for our purposes to ignore these possible complications and to consider the simplest assumptions formulated above.

Having reduced the problem to the problem of a parametrically excited oscillator, we can expect that torque oscillations in the primeval cosmological medium could have been generated quantum mechanically (see Sec. III), very much similar to gravitational waves. They would evolve as torque oscillations until the primeval medium became an ideal fluid. At this time, the torsional velocity of sound drops to zero and the oscillations convert into free rotational perturbations. Without having support of restoring torque forces, they will gradually decay in the course of expansion, as was described above. However, according to this hypothesis, the amplitude of rotational perturbations in the postrecombination Universe is not necessarily zero and might have been determined by laws of quantum mechanics, not by a voluntary choice of initial conditions.

In the contemporary inflationary scenarios, the favorite model of the primeval medium is one or another modification of a scalar field. Scalar fields can support neither torque oscillations nor rotation. There is no use in scalar fields or ideal fluids in terms of quantum-mechanical generation of rotational perturbations. However, whether or not the inflationary stage of expansion (if it took place at all) was governed by a scalar field or by something else is totally unknown (in this context, see, for instance, [18] about the superstring-motivated cosmologies). One may hope that there will never be shortage in the "microphysical" models for the constituents of matter in the very early Universe, as long as they lead to interesting and verifiable consequences.

It is also worth mentioning that there may be some ambiguity in coupling of the perturbations to the external gravitational field. We intend to use the simplest formula (22). However, one could alternatively suggest

$$\chi(\eta) = n^2\theta(\eta)\left[b^2 + \frac{1}{n^2}\frac{a''}{a}\right] . \qquad (22')$$

Both formulas would agree in flat space-time ($a(\eta) =$ const) but the second one would be a different generalization to the case of the variable $a(\eta)$. Equation (22′) would lead to $\mu'' + n^2 b^2 \mu = 0$, instead of Eq. (25). As in the case of electromagnetic waves, the oscillations would not be superadiabatically amplified, regardless of the rate of variability of $a(\eta)$. (Similar ambiguity is also involved, but rarely acknowledged, in attempts to generate density perturbations through the amplification of the scalar field fluctuations.) It is not clear whether the hypothesis (22′) is fully consistent, it requires a separate study, but I do not think that it can be simply ignored.

III. QUANTIZATION OF ROTATIONAL PERTURBATIONS

In flat three-space (11), the vector eigenfunctions $Q_i(x^1, x^2, x^3)$ can be conveniently written as $Q_i \equiv q_i e^{i\mathbf{n}\cdot\mathbf{x}} + q_i^* e^{-i\mathbf{n}\cdot\mathbf{x}}$. The second of Eqs. (13) requires the complex vector q_i to be orthogonal to n^i: $q_i n^i = 0$. There are two independent real unit vectors orthogonal to the unit vector

$$\mathbf{n}/n = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

and to each other. We take them in the form

$$\overset{1}{q}_i = l_i = (\sin\varphi, -\cos\varphi, 0) ,$$

$$\overset{2}{q}_i = m_i = \pm(\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) .$$

In the definition of the vector m_i, the sign + is valid for $\theta < \pi/2$ and the sign − is valid for $\theta > \pi/2$. The vectors l_i, m_i are the basis for the construction of the polarization tensors $\overset{s}{P}_{ij}(\mathbf{n})$, $s = 1, 2$:

$$\overset{1}{P}_{ij}(\mathbf{n}) = \frac{1}{n}(l_i n_j + l_j n_i) , \quad \overset{2}{P}_{ij}(\mathbf{n}) = \frac{1}{n}(m_i n_j + m_j n_i) .$$

They describe the transverse-longitudinal degrees of freedom since

$$\overset{1}{P}_{ij}\frac{n^j}{n} = l_i , \quad \overset{2}{P}_{ij}\frac{n^j}{n} = m_i ,$$

and $\overset{s}{P}_{ij} n^i n^j = 0$. Other important relations are

$$\overset{s}{P}_{ij}\delta^{ij} = 0 , \quad \overset{s}{P}_{ij}(-\mathbf{n}) = \overset{s}{P}_{ij}(\mathbf{n}) , \quad \overset{s}{P}_{ij}\overset{s'}{P}{}^{ij} = 2\delta^{ss'} .$$

In flat space-time, the general expression for the components h_{ij} can be written as

$$h_{ij}(t,\mathbf{y}) = C\frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty} d^3k \sum_{s=1}^{2} \overset{s}{P}_{ij}(\mathbf{k})\frac{1}{\sqrt{2\omega_k}}\left[\overset{s}{a}_{\mathbf{k}} e^{-i\omega_k t}e^{i\mathbf{k}\cdot\mathbf{y}} + \overset{s\dagger}{a}_{\mathbf{k}} e^{i\omega_k t}e^{-i\mathbf{k}\cdot\mathbf{y}}\right].$$

For the classical h_{ij} field, the quantities $\overset{s}{a}_{\mathbf{k}}$, $\overset{s\dagger}{a}_{\mathbf{k}}$ are complex conjugate numbers. In the quantized version of the field, $\overset{s}{a}_{\mathbf{k}}$ and $\overset{s\dagger}{a}_{\mathbf{k}}$ are the creation and annihilation operators for a rotation wave traveling in the direction $\mathbf{k}$. The frequency of the wave is $\omega_k = v_t|\mathbf{k}|$.

As usual, the normalization constant C is determined from the requirement of having energy $\frac{1}{2}\hbar\omega_k$ in each $\mathbf{k}$ mode:

$$\langle 0|\int_{-\infty}^{\infty} t_{00}d^3y|0\rangle = \frac{1}{2}\hbar\int_{-\infty}^{\infty} d^3k\,\omega_k\sum_{s=1}^{2}\langle 0|\overset{s}{a}_{\mathbf{k}}\overset{s\dagger}{a}_{\mathbf{k}}+\overset{s\dagger}{a}_{\mathbf{k}}\overset{s}{a}_{\mathbf{k}}|0\rangle.$$

The component t_{00} of the energy-momentum tensor $t_{\mu\nu}$ for the field $h_{\mu\nu}$ has been defined in [14]. The symbol $h_{\mu\nu}$ denotes different objects here and in Ref. [14] but in the approximation that we are working in they coincide. By using Eq. (47) from Ref. [14] specified to $h_{00} = 0$, $h_{0i} = 0$, and $h_{\mu\nu}\eta^{\mu\nu} = 0$, one obtains

$$\kappa t_{00} = \tfrac{1}{4}h^{ij}{}_{,0}h_{ij,0} - \tfrac{1}{8}h^{ij,0}h_{ij,0} - \tfrac{1}{8}h^{ij,k}h_{ij,k} + \tfrac{1}{4}h^{ij,k}h_{jk,i}.$$

The second and third terms cancel each other if $b = 1$, which we will assume for simplicity in what follows. The last term gives the factor $(-1/2)$ contribution of the first term. (In the case of gravitational waves, all terms, except the first one, vanish.) As a result, we find the constant C: $C = 4\sqrt{2\pi c}l_{\rm Pl}$. Then, we make the rescaling

$$\mathbf{k} = \frac{\mathbf{n}}{a},\quad \mathbf{y} = a\mathbf{x},\quad \omega_k = \frac{cn}{a},\quad \overset{s}{a}_{\mathbf{k}}(t) = a^{3/2}\overset{s}{c}_{\mathbf{n}}(\eta).$$

This allows us to write the normalized expression for $h_{ij}(\eta,\mathbf{x})$ in curved space-time (10) as

$$h_{ij}(\eta,\mathbf{x}) = 4(2\pi)^{1/2}\frac{l_{\rm Pl}}{a}\frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty} d^3n\sum_{s=1}^{2}\overset{s}{P}_{ij}(\mathbf{n})\frac{1}{\sqrt{2n}}\left[\overset{s}{c}_{\mathbf{n}}(\eta)e^{i\mathbf{n}\cdot\mathbf{x}} + \overset{s\dagger}{c}_{\mathbf{n}}(\eta)e^{-i\mathbf{n}\cdot\mathbf{x}}\right]. \tag{26}$$

In this expression, $a(\eta)$ has the dimension of length, $l_{\rm Pl} = (G\hbar/c^3)^{1/2}$ is the Planck length, all other quantities are dimensionless.

As in the case of gravitational waves [6], the time evolution of the operators $c_{\mathbf{n}}(\eta)$, $c^\dagger_{\mathbf{n}}(\eta)$ (for each s) is governed by the Heisenberg equations of motion $dc_{\mathbf{n}}/d\eta = -i[c_{\mathbf{n}},H]$, $dc^\dagger_{\mathbf{n}}/d\eta = -i[c^\dagger_{\mathbf{n}},H]$ with the Hamiltonian

$$H = nc^\dagger_{\mathbf{n}}c_{\mathbf{n}} + nc^\dagger_{-\mathbf{n}}c_{-\mathbf{n}} + 2\sigma(\eta)c^\dagger_{\mathbf{n}}c^\dagger_{-\mathbf{n}} + 2\sigma^*(\eta)c_{\mathbf{n}}c_{-\mathbf{n}},$$

where $\sigma(\eta) = ia'/2a$. The solution to the Heisenberg equations of motion is

$$c_{\mathbf{n}}(\eta) = u_n(\eta)c_{\mathbf{n}}(0) + v_n(\eta)c^\dagger_{-\mathbf{n}}(0),$$

$$c^\dagger_{\mathbf{n}}(\eta) = u^*_n(\eta)c^\dagger_{\mathbf{n}}(0) + v^*_n(\eta)c_{-\mathbf{n}}(0),$$

where $c_{\mathbf{n}}(0)$, $c^\dagger_{\mathbf{n}}(0)$ are the initial values of the operators taken at some initial time $\eta = \eta_0$. The complex functions $u_n(\eta)$, $v_n(\eta)$ satisfy the equations

$$iu'_n = nu_n + i(a'/a)v^*_n,\quad iv'_n = nv_n + i(a'/a)u^*_n,$$

where $|u_n|^2 - |v_n|^2 = 1$ and $u_n(0) = 1$, $v_n(0) = 0$. One can easily show that the function $\mu_n \equiv u_n + v^*_n$ should satisfy the equation $\mu''_n + (n^2 - a''/a)\mu_n = 0$ and the initial conditions

$$\mu(0) = 1,\quad \mu'(0) = -in + (a'/a)(0). \tag{27}$$

For both polarizations $s = 1,2$, the solutions are identically the same.

The operators $\overset{s}{c}_{\mathbf{n}}(0)$, $\overset{s\dagger}{c}_{\mathbf{n}}(0)$ obey the commutation relations $[\overset{s}{c}_{\mathbf{n}}(0),\overset{s'\dagger}{c}_{\mathbf{m}}(0)] = \delta^{ss'}\delta^3(\mathbf{n}-\mathbf{m})$ and the same is true for the evolved operators: $[\overset{s}{c}_{\mathbf{n}}(\eta),\overset{s'\dagger}{c}_{\mathbf{m}}(\eta)] = \delta^{ss'}\delta^3(\mathbf{n}-\mathbf{m})$.

In the Schrödinger picture, the initial vacuum state evolves into a strongly squeezed vacuum state. The squeeze parameters can be derived from the solutions for $u_n(\eta)$, $v_n(\eta)$ (see [6] and references therein). The total linear momentum and the total angular momentum of the field remain equal to zero. In the Heisenberg picture, which we will adopt for further calculations, the quantum state remains a vacuum while all the dynamical properties of the field are described by the time-dependent annihilation and creation operators. In Sec. V we will be calculating the expectation values with respect to the vacuum state defined by $\overset{s}{c}_{\mathbf{n}}(0)|0\rangle = 0$.

As is clear from the physical description of the phenomenon, there is nothing specifically "gravitational" or "cosmological," let alone "inflationary," in the amplification process. The kind of parametric interaction that is provided by $a(\eta)$ in cosmology can be realized by nongravitational means in the laboratory conditions. It is not excluded that the squeezed vacuum state for the analogue of torsional oscillations can be generated in the quantum optics experiments similar, for example, to those discussed in [15].

IV. APPLICATION OF THE QUANTUM-MECHANICAL GENERATING MECHANISM TO SIMPLE COSMOLOGICAL MODELS

The presently available observational data about our part of the Universe refer to the interval of evolution that took place since the epoch of the primordial nucleosynthesis. The major uncertainties in the functional form of the scale factor $a(\eta)$ pertain to the earlier times. This earlier part of evolution was governed by an unknown primeval cosmological medium with an unknown equation of state. The "microphysical" details of the primeval medium are quite irrelevant as long as we are only interested in the scale factor $a(\eta)$. The function $U(\eta) = a''/a$ is essentially all we need to know in order to find solutions to Eq. (25) and calculate the present-day characteristics of the quantum-mechanically generated perturbations. The origin of perturbations makes it clear that it is only $a(\eta)$ (the pump field) that is directly involved and responsible for the final result. For example, the spectrum of gravitational waves is in one-to-one correspondence with the variable Hubble parameter of the very early Universe and can be used for its reconstruction [19].

We will explore the η-time power-law scale factors: $a(\eta) = a_0\eta^{1+\beta}$, where β is a constant. This parametrization has been used before [4]. It provides a simple description of the two cases when the barrier $U(\eta) = a''/a$ vanishes: $\beta = -1, 0$. The value $\beta = -1$ gives the flat space-time; the value $\beta = 0$ gives the scale factor of the radiation-dominated universe. The matter-dominated universe is described by $\beta = 1$. We call evolution controlled by $\beta = 0$, $\beta = 1$ the e stage and m stage, respectively. Any value of β smaller than -1 describes a sort of inflationary (i stage) expansion. The expansion is inflationary in the sense that the length scale equal to the Hubble radius at some early time of expansion can grow at the consecutive i, e, m stages up to, at least, the size of the present-day Hubble radius l_H. The value $\beta = -2$ corresponds to the de Sitter case.

Specifically, we consider expanding models with the following scale factor (see also [20]).

i stage:

$$a(\eta) = l_0|\eta|^{1+\beta}\ , \quad \eta \le \eta_1\ , \quad \eta_1 < 0\ , \tag{28}$$

e stage:

$$a(\eta) = l_0 a_e(\eta - \eta_e)\ , \quad \eta_1 \le \eta \le \eta_2\ ,$$

m stage:

$$a(\eta) = l_0 a_m(\eta - \eta_m)^2\ , \quad \eta_2 \le \eta\ ,$$

where $a_e = -(1+\beta)|\eta_1|^\beta$, $\eta_e = \beta\eta_1/(1+\beta)$, $a_m = a_e/4(\eta_2 - \eta_e)$, $\eta_m = -\eta_2 + 2\eta_e$, and l_0 is a constant with the dimensionality of length. The scale factor $a(\eta)$ and its first derivative $a'(\eta)$ are continuous at $\eta = \eta_1$ and $\eta = \eta_2$. We will denote the present time by $\eta = \eta_R$. The scale factor at the beginning of the m stage $a(\eta_2)$ is related to its present-day value $a(\eta_R)$ by the relation $a(\eta_2)/a(\eta_R) \approx 10^{-4}$. The i stage is governed by "matter" with the effective equation of state $p = q(\beta)\epsilon$ where $q(\beta) = (1-\beta)/3(1+\beta)$.

The general solution to Eq. (25), for a given mode n, is

$$\mu_n = (nb\eta)^{1/2}\left[A_1 J_{\beta+1/2}(nb\eta) + A_2 J_{-\beta-1/2}(nb\eta)\right]\ . \tag{29}$$

In order to work solely with the Bessel functions, as two linearly independent solutions to Eq. (25), we exclude (temporarily) the half-integer β's. The form of the solution (29) shows explicitly, once again, that the most important parameter involved is the ratio of the frequency of the wave to the frequency of variations of the scale factor (frequency of the pump). In terms of η time, those frequencies are, respectively, $\omega_w = nb$ and $\omega_p = 2\pi a'/a$. Their ratio, which can be called the parameter of adiabaticity, is $nb\eta/2\pi(1+\beta)$. For periodic parametric couplings, the equality $\omega_p = 2\omega_w$ is the condition of the parametric resonance. The solution (29) ceases to oscillate and loses its adiabatic character when the ω_w/ω_p falls down to a number of order 1. This is when the parametric amplification becomes significant. If $b \ll 1$, this happens long before the wavelength of the oscillation, $\lambda = 2\pi a/n$, gets comparable with the length of the Hubble radius, $l = a^2/a'$.

To simplify calculations and make them identical to the case of gravitational waves, we will assume that the torsional velocity of sound at the i stage was equal to the velocity of light, $b_i = 1$. We will also assume that the value of b at the e stage was not zero (at least, in one unspecified component of the prerecombination matter) and could be close to 1, $b_e \approx 1$. And, finally, we assume that b has fallen down to zero at the m stage, $b_m = 0$. We will be working directly with the functions $h(\eta)$, $h'(\eta)$ where $\mu(\eta) = ah(\eta)$. It is $h'(\eta)$ at the m stage that we will eventually need for calculations of $\delta T/T$. Then, the solution to Eq. (25) can be written at the three stages as follows:

i stage:

$$h(\eta) = \frac{1}{a(\eta)}(n\eta)^{1/2}\left[A_1 J_{\beta+1/2}(n\eta) + A_2 J_{-\beta-1/2}(n\eta)\right]\ , \tag{30}$$

e stage:

$$h(\eta) = \frac{1}{a(\eta)}\left[B_1 e^{-inb_e(\eta-\eta_e)} + B_2 e^{+inb_e(\eta-\eta_e)}\right]\ , \tag{31}$$

m stage:

$$h(\eta) = \tfrac{1}{a(\eta)}\left[C_1 n^2(\eta-\eta_m)^2 + \tfrac{C_2}{n(\eta-\eta_m)}\right]\ ,$$

$$h'(\eta) = -\tfrac{1}{a(\eta)}\tfrac{3nC_2}{n^2(\eta-\eta_m)^2}. \tag{32}$$

The initial conditions (27) taken at $\eta = \eta_0$ lead to

$$A_1 = -\frac{i}{\cos\beta\pi}\sqrt{\frac{\pi}{2}}e^{i(n\eta_0+\pi\beta/2)}, \quad A_2 = iA_1 e^{-i\pi\beta},$$

where $n|\eta_0| \gg 2\pi|1+\beta|$. It is worth noting that had we assumed $b_i = 0$, the solution to Eq. (25) at the i stage would have been

$$h(\eta) = A_1 + A_2 \int a^{-2} d\eta$$

and the initial conditions $\mu(\eta_0) = 1$, $\mu'(\eta_0) = a'/a(\eta_0)$ would have required $A_2 = 0$, $h(\eta) = A_1$. This is another way of saying that the perturbations could not have been generated quantum mechanically in a medium with zero torsional velocity of sound.

The coefficients B_1, B_2, C_1, C_2 are determined by the continuous matching of solutions (30)–(32) at $\eta = \eta_1$ and $\eta = \eta_2$. We are interested in waves that have interacted with the barrier $U(\eta)$ at the i stage. Their wave numbers obey the equation $n|\eta_1| \ll 2\pi|1+\beta|$. For the coefficients B_1, B_2 with these wave numbers, one derives

$$B_1 \approx -B_2 \approx \tfrac{1}{2}e^{in\eta_0}(\beta+1)\psi(\beta)(n\eta_1)^\beta \equiv B\,, \tag{33}$$

where $\psi(\beta) = \sqrt{\frac{\pi}{2}}e^{i\pi\beta/2}[2^{\beta+1/2}\Gamma(\beta+3/2)\cos\beta\pi]^{-1}$, $|\psi(\beta)| = 1$ for $\beta = -2$. The exact formulas for C_1, C_2 are

$$C_1 = \frac{1}{12y_2^2}[\bar{B}_1(\sin b_e y_2 + 2b_e y_2 \cos b_2 y_2) + \bar{B}_2(\cos b_e y_2 - 2b_e y_2 \sin b_e y_2)]\,,$$

$$C_2 = \tfrac{4}{3}y_2[\bar{B}_1(\sin b_e y_2 - b_e y_2 \cos b_e y_2) + \bar{B}_2(\cos b_e y_2 + b_e y_2 \sin b_e y_2)]\,,$$

where $\bar{B}_1 = i(B_2 - B_1)$, $\bar{B}_2 = B_1 + B_2$, and $y_2 = n(\eta_2 - \eta_e)$. The coefficient C_1 represents the constant part of $h(\eta)$ which plays no role in our calculations in Sec. V and can, in fact, be eliminated by a coordinate transformation, but we will discuss C_1 together with C_2 for the completeness of our analysis. The formulas for C_1, C_2, simplify due to Eq. (33) and their approximate expressions can be written as

$$\begin{aligned} C_1 &\approx -\tfrac{1}{6}iBy_2^{-2}(\sin b_e y_2 + 2b_e y_2 \cos b_e y_2)\,, \\ C_2 &\approx -\tfrac{8}{3}iBy_2(\sin b_e y_2 - b_e y_2 \cos b_e y_2)\,. \end{aligned} \tag{34}$$

The wave number $n_r = 1/b_e(\eta_2 - \eta_e)$ defined by the condition $b_e y_2 = 1$ separates the functional dependence of C_1, C_2, on n into two different regimes. For $n \ll n_r$ one has

$$C_1 \approx -\frac{1}{2}iBb_e^2\left(\frac{n_r}{n}\right), \quad C_2 = -\frac{8}{9}iB\frac{1}{b_e}\left(\frac{n}{n_r}\right)^4,$$

and the dependence of C_1, C_2 on n is smooth. For $n \gg n_r$, one has

$$C_1 \approx -\frac{1}{3}iBb_e^2\left(\frac{n_r}{n}\right)\cos\frac{n}{n_r}\,,$$

$$C_2 \approx \frac{8}{3}iB\frac{1}{b_e}\left(\frac{n}{n_r}\right)^2\cos\frac{n}{n_r}\,,$$

and the dependence of C_1, C_2 on n is oscillatory. There exists a series of frequencies n at which both, $h(\eta)$ and $h'(\eta)$, go to zero for all $\eta > \eta_2$. These frequencies are defined by the requirement that the factor $\cos(n/n_r)$ vanishes. In the framework of quantum mechanics, one can say that the n modes with those frequencies are having been "desqueezed," stripped off the energy accumulated at the i stage by the very late times of their evolution at the m stage [21].

The n-dependent spectrum of the field $h'(\eta)$, which we will need in our further calculations, is smooth for $n \ll n_r$ and is oscillatory for $n \gg n_r$. For a qualitative description of the spectrum we introduce $h'(n) = l_{\rm Pl}(|\mu_n|/a)'$. Then we can write

$$\begin{aligned} h'(n) &\approx \frac{1}{3}\frac{l_{\rm Pl}}{l_0}|\psi(\beta)|b_e^2\,, \frac{a^2(\eta_2)}{a^2(\eta)}\frac{n^{\beta+3}}{n_r}\,, \quad n \ll n_r\,, \\ h'(n) &\approx \frac{l_{\rm Pl}}{l_0}|\psi(\beta)|b_e^2\,\frac{a^2(\eta_2)}{a^2(\eta)}n_r n^{\beta+1}\left|\cos\frac{n}{n_r}\right|\,, \quad n \gg n_r\,. \end{aligned} \tag{35}$$

The absolute values of $h'(n)$ are primarily determined by b_e and by the parameters l_0, β of the i stage.

In the limiting case $b_e = 0$, the form of solution for $h(\eta)$ is the limiting form of Eq. (31):

$$h(\eta) = \frac{1}{a(\eta)}\left[\bar{\bar{B}}_1 n(\eta - \eta_e) + \bar{B}_2\right]\,,$$

where $\bar{\bar{B}}_1$ and $\bar{B}_2$ are constants. The $h'(\eta)$ at $\eta = \eta_R$ is not zero, as would follow from the approximate formulas (35) in the limit $b_e = 0$, but is really very small, its value is determined by small terms omitted in the course of derivation of Eq. (35). In the case of $b_e = 0$, rotation is still being generated at the i stage but decays since the beginning of the e stage and its small amplitude now makes it probably useless for astrophysical applications.

V. ANGULAR ANISOTROPY OF THE CMBR CAUSED BY ROTATIONAL PERTURBATIONS

Our major goal is to derive the angular correlation function for the anisotropy $\delta T/T$ produced by rotational perturbations of quantum-mechanical origin. However, we will start from the analysis of the problem at the classical level.

The solution (32) for rotational perturbations is given in a synchronous coordinate system. It is convenient, first, to go over to a comoving coordinate system where, by definition, the components T_0^i of the energy-momentum tensor $T_{\mu\nu}$ vanish. This allows us to describe the world line of the observer by simple equations $x^i = 0$. It is assumed that the observer's world line is one of the world lines of the matter. If the observer has a peculiar velocity, the observed $\delta T/T$ will have an additional dipole component which is known how to deal with. The spatial hypersurfaces of constant time should be retained the same as in the synchronous coordinate system. The energy density of matter is constant over these hypersurfaces and the radiation temperature was everywhere the same at the time of decoupling of the CMBR. This allows us to define the emission time of the photons of CMBR as $\eta = \eta_E$ regardless of the direction of observations.

The transition to the comoving coordinate system is achieved with the help of a small coordinate transformation

$$\bar{\eta} = \eta, \quad \bar{x}^i = x^i - \frac{C_2}{8}\frac{1}{a(\eta)}(\eta - \eta_m)Q^i , \tag{36}$$

where $\bar{\eta}$, $\bar{x}^i$, are the coordinates of the comoving system. The transformed metric tensor is

$$ds^2 = a^2(\eta)(\eta_{\mu\nu} + \bar{h}_{\mu\nu})dx^\mu dx^\nu , \tag{37}$$

where

$$\bar{h}_{00} = 0, \quad \bar{h}_{0i} = \frac{C_2}{8}\frac{1}{a}Q_i = \bar{g}(\eta)Q_i,$$

$$\bar{h}_{ij} = 2\left[h(\eta) + \frac{C_2 n(\eta - \eta_m)}{8a}\right] Q_{ij} = 2\bar{h}(\eta)Q_{ij},$$

and

$$\bar{h}'(\eta) = -\frac{1}{a(\eta)}\frac{3nC_2}{n^2(\eta - \eta_m)^2}\left[1 + \frac{1}{24}n^2(\eta - \eta_m)^2\right],$$

$$\bar{g}'(\eta) = -\frac{1}{a(\eta)}\frac{nC_2}{4n(\eta - \eta_m)}. \tag{38}$$

After performing the transformation, we do not write the overbar over the coordinates x^μ which are now supposed to be the comoving coordinates. The transformed components of the perturbed energy-momentum tensor are

$$\bar{T}_0^i = 0 , \quad \bar{T}_i^0 = a^{-2}\omega Q_i , \quad \bar{T}_i^k = 2n^{-1}a^{-2}\chi Q_i^k . \tag{39}$$

The transition from Eqs. (12) and (15) to Eqs. (37) and (39) has been done with the help of a usual coordinate transformation. I prefer to reserve the notions of gauge transformations and gauge invariance to the "field-theoretical" formulation of general relativity where they have their genuine meaning: distinct from coordinate transformations, independent of any approximation scheme, unrelated to any prescribed form of the participating functions, etc. However, if one wishes to use a different name for coordinate transformations of the type of Eq. (36), one can say that the transition from Eqs. (12) and (15) to Eqs. (37) and (39) has been performed with the help of a "gauge transformation."

The calculation of the CMBR temperature variations caused by the gravitational field of cosmological perturbations was first undertaken by Sachs and Wolfe [3]. The authors work in the comoving coordinate system and derive the formula

$$\frac{\delta T}{T} = \frac{1}{2}\int_0^{\eta_R - \eta_E} \left(\frac{\partial \bar{h}_{ij}}{\partial \eta}e^i e^j - 2\frac{\partial \bar{h}_{0j}}{\partial \eta}e^j\right)_{(0)} dy , \tag{40}$$

where e^k is a unit vector in the direction of observations. This formula is valid for all types of cosmological perturbations.

The expression under the integral in (40) depends on $\bar{h}'(\eta)$, $\bar{g}'(\eta)$, Eq. (38). The spectrum of perturbations in comoving coordinates is different from the one in synchronous coordinates. In our case, the approximate expressions for the spectrum $h'(n)$ in synchronous coordinates are given by Eq. (35). To derive the spectrum $\bar{h}'(n)$ in comoving coordinates one should combine Eqs. (38) and (35). For the spectrum $\bar{h}'(n)$ at the present epoch $\eta = \eta_R$, one finds $\bar{h}'(n) \sim n^{\beta+3}$ for $n \ll n_H$, $\bar{h}'(n) \sim n^{\beta+5}$ for $n_H \ll n \ll n_r$, $\bar{h}'(n) \sim n^{\beta+3}|\cos(n/n_r)|$ for $n \gg n_r$.

We will now turn to the calculation of the angular correlation function for $\delta T/T$ caused by rotational perturbations of quantum-mechanical origin. For quantized perturbations, the $\delta T/T$ becomes a quantum-mechanical operator. In our case, by using Eqs. (40), (26), (38), we can write

$$\frac{\delta T}{T}(e^k) = \frac{1}{\pi}l_{\text{Pl}}\int_0^{w_1} dw \int_{-\infty}^{\infty} d^3n \sum_{s=1}^{2}\left[\overset{s}{r}_{\mathbf{n}}(\eta_R - w)\overset{s}{c}_{\mathbf{n}}(0)e^{in_k e^k w} + \overset{s*}{r}_{\mathbf{n}}(\eta_R - w)\overset{s\dagger}{c}_{\mathbf{n}}(0)e^{-in_k e^k w}\right],$$

where $w = \eta_R - \eta$, $x^k = e^k w$, $w_1 = \eta_R - \eta_E$, and

$$\overset{s}{r}_{\mathbf{n}}(\eta_R - w) = \overset{s}{P}_{ij}(\mathbf{n})e^i e^j f_n - 2i\overset{s}{q}_j(\mathbf{n})e^j\phi_n, \quad f_n = \frac{1}{\sqrt{2n}}\bar{h}'_n, \quad \phi_n = \frac{1}{\sqrt{2n}}\bar{g}'_n.$$

The mean value of $\delta T/T$ is obviously equal to zero: $\langle 0|\delta T/T|0\rangle = 0$. The expectation value of the angular correlation function is defined as

$$K = \langle 0|\frac{\delta T}{T}(e_1^k)\frac{\delta T}{T}(e_2^k)|0\rangle ,$$

where e_1^k and e_2^k are two different unit vectors with the angle δ between them, $\cos\delta = e_1^i e_2^j \delta_{ij}$. It is easy to show that

$$K = \frac{1}{\pi^2} l_{\rm Pl}^2 \int_0^{w_1} dw \int_0^{w_1} d\bar{w} \int_{-\infty}^{\infty} d^3n \sum_{s=1}^{2} \overset{s}{r}_n(\eta_R - w) \overset{s}{r}{}^*_n(\eta_R - \bar{w}) e^{in_k\zeta^k} , \tag{41}$$

where $\zeta^k = e_1^k w - e_2^k \bar{w}$. One can also show that

$$\begin{aligned}\sum_{s=1}^{2} \overset{s}{r}_n(\eta_R - w) \overset{s}{r}{}^*_n(\eta_R - \bar{w}) = {} & f_n(\eta_R - w) f_n^*(\eta_R - \bar{w}) R^{11}(\mathbf{n}) \\ & +2if_n(\eta_R - w)\phi_n^*(\eta_R - \bar{w}) R^{12}(\mathbf{n}) - 2if_n^*(\eta_R - \bar{w})\phi_n(\eta_R - w) R^{21}(\mathbf{n}) \\ & +4\phi_n(\eta_R - w)\phi_n^*(\eta_R - \bar{w}) R^{22}(\mathbf{n}) ,\end{aligned} \tag{42}$$

where

$$R^{11} = \sum_{s=1}^{2} \overset{s}{P}_{ij} e_1^i e_1^j \overset{s}{P}_{ij} e_2^i e_2^j, \quad R^{12} = \sum_{s=1}^{2} \overset{s}{P}_{ij} e_1^i e_1^j \overset{s}{q}_j e_2^j, \quad R^{21} = \sum_{s=1}^{2} \overset{s}{P}_{ij} e_2^i e_2^j \overset{s}{q}_j e_1^j, \quad R^{22} = \sum_{s=1}^{2} \overset{s}{q}_i e_1^i \overset{s}{q}_j e_2^j.$$

The next step is integration of the functions $R^{11}, R^{12}, R^{21}, R^{22}$ multiplied by $e^{in_k\zeta^k}$ over the angular variables φ, θ. Our intention is to derive the formula in the most general form, applicable to arbitrary functions $f_n(\eta_R - w)$, $\phi_n(\eta_R - w)$, so we leave the integration over $w, \bar{w}$, and n to the very end. A lengthy calculation gives the following results. For R^{11},

$$\begin{aligned}\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \cos(n_k\zeta^k) R^{11}(\mathbf{n}) = 16\pi\sqrt{\frac{\pi}{2}} \{ & (3\cos^2\delta - 1)(n\zeta)^{-3/2} J_{3/2}(n\zeta) + [\cos\delta(\cos^2\delta - 1) nw\, n\bar{w} \\ & -4(3\cos^2\delta - 1)](n\zeta)^{-5/2} J_{5/2}(n\zeta) - 8\cos\delta(\cos^2\delta - 1) nw\, n\bar{w} (n\zeta)^{-7/2} J_{7/2}(n\zeta) \\ & -(\cos^2\delta - 1)^2 (nw)^2 (n\bar{w})^2 (n\zeta)^{-9/2} J_{9/2}(n\zeta) \} .\end{aligned}$$

For R^{12},

$$\begin{aligned}\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \sin(n_k\zeta^k) R^{12}(\mathbf{n}) = 8\pi\sqrt{\frac{\pi}{2}} \{ & [2nw\cos\delta - n\bar{w}(3\cos^2\delta - 1)](n\zeta)^{-5/2} J_{5/2}(n\zeta) \\ & +(\cos^2\delta - 1) nw\, n\bar{w} [nw - n\bar{w}\cos\delta](n\zeta)^{-7/2} J_{7/2}(n\zeta) \} .\end{aligned}$$

For R^{22},

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \cos(n_k\zeta^k) R^{22}(\mathbf{n}) = 4\pi\sqrt{\frac{\pi}{2}} \{ 2\cos\delta (n\zeta)^{-3/2} J_{3/2}(n\zeta) + (\cos^2\delta - 1) nw\, n\bar{w} (n\zeta)^{-5/2} J_{5/2}(n\zeta) \} ,$$

where $\zeta = (w^2 - 2w\bar{w}\cos\delta + \bar{w}^2)^{1/2}$. The integration of $R^{21}(\mathbf{n})$ gives the result which differs from the above one for $R^{12}(\mathbf{n})$ by the replacement $w \longleftrightarrow \bar{w}$ and by the opposite total sign.

One should now use the "summation theorem" [23] and the relations between the Gegenbauer polynomials and the associated Legendre polynomials [24] which can be combined together into the formula (for half-integer ν)

$$(n\zeta)^{-\nu} J_\nu(n\zeta) = \sqrt{2\pi} \sum_{k=0}^{\infty} (\nu + k) \frac{J_{\nu+k}(nw)}{(nw)^\nu} \frac{J_{\nu+k}(n\bar{w})}{(n\bar{w})^\nu} \frac{d^{\nu-1/2}}{dx^{\nu-1/2}} P_{k+\nu-1/2}(x) , \tag{43}$$

where $x = \cos\delta$ and $P_l(x)$ are the Legendre polynomials.

In agreement with Eq. (42), the expression for K can be presented in the form

$$K = K^1 + K^2 + K^3 ,$$

where the K^1 involves products $f_n f_n^*$, the K^2 involves products $f_n\phi_n^*$ and $f_n^*\phi_n$, and the K^3 involves products $\phi_n\phi_n^*$. By using Eq. (43) one obtains, for K^1,

$$K^1 = l_{\rm Pl}^2 \sum_{l=1}^{\infty} K_l^1 P_l(x) ,$$

where

$$K_l^1 = \frac{8l(l+1)}{2l+1} \int_0^\infty n^2 \left| \int_0^{w_1} \frac{dw}{(nw)^{3/2}} \left[(l-1)J_{l-1/2}(nw) - (l+2)J_{l+3/2}(nw) \right] f_n(\eta_R - w) \right|^2 dn \,. \tag{44}$$

The expression for K^2 is more complicated:

$$\begin{aligned} K^2 = -16 l_{\rm Pl}^2 \int_0^{w_1} dw \int_0^{w_1} d\bar{w} \int_0^\infty n^2 dn \Bigg\{ & \sum_{k=0}^\infty (5/2+k) \frac{J_{5/2+k}(nw)}{(nw)^{5/2}} \frac{J_{5/2+k}(n\bar{w})}{(n\bar{w})^{5/2}} \frac{d^2}{dx^2} P_{k+2} \\ & \times \{ f_n(\eta_R - w)\phi_n^*(\eta_R - \bar{w})[nwx + n\bar{w}(1-3x^2)] + f_n(\eta_R - \bar{w})\phi_n^*(\eta_R - w)[n\bar{w}x + nw(1-3x^2)] \} \\ & -(1-x^2) \sum_{k=0}^\infty (7/2+k) \frac{J_{7/2+k}(nw)}{(nw)^{5/2}} \frac{J_{7/2+k}(n\bar{w})}{(n\bar{w})^{5/2}} \frac{d^3}{dx^3} P_{k+3} \\ & \times [f_n(\eta_R - w)\phi_n^*(\eta_R - \bar{w})(nw - n\bar{w}x) + f_n(\eta_R - \bar{w})\phi_n^*(\eta_R - w)(n\bar{w} - nwx)] \Bigg\} \,. \end{aligned} \tag{45}$$

Finally, the expression for K^3 reads

$$K^3 = l_{\rm Pl}^2 \sum_{l=1}^\infty K_l^3 P_l(x) \,,$$

where

$$K_l^3 = 8(2l+1)l(l+1) \int_0^\infty n^2 \left| \int_0^{w_1} \frac{dw}{(nw)^{3/2}} J_{l+1/2}(nw)\phi_n(\eta_R - w) \right|^2 dn \,. \tag{46}$$

The total angular correlation function K is rotationally symmetric (depends only on the angle δ between the directions of observations) and its multipole expansion begins from the dipole term ($l = 1$). The numerical values of the multipole components are different for different cosmological models. The free parameters b_e, l_0, β of the models considered in Sec. IV can be chosen in such a way that the level of the predicted variations would be consistent with what is actually observed by the Cosmic Background Explorer (COBE) [1]. The derivation of the detailed multipole distributions following from Eqs. (44), (45), and (46) and the construction of the resulting correlation function as a function of the separation angle δ require numerical calculations. This will be a subject of a further discussion.

VI. CONCLUSIONS

We have shown that rotational cosmological perturbations with a very broad spectrum might have been generated quantum mechanically in the very early Universe. We have formulated conditions under which the phenomenon could take place. The main emphasis has been on long wavelength perturbations which are probably responsible for the observed large-angular-scale anisotropy of CMBR. The angular correlation function was derived, and it was shown that the multipole expansion begins from the dipole term. (In the limit of the wavelengths exceeding the present-day Hubble radius the dipole component is suppressed [22].) The numerical values of the expected variations $\delta T/T$ depend on the parameters of the cosmological models (essentially, parameters of the cosmological pump "machine"). In principle, rotational perturbations alone could account for the observed anisotropy. The comparison with the case of gravitational waves [20] shows, however, that the contribution of gravitational waves to the large-angular-scale anisotropy is likely to be much larger than that of rotational perturbations since gravitational waves do not decay in the course of time as fast as free rotational perturbations do. Moreover, quantum-mechanical generation of gravitational waves does not require any additional physical hypotheses to be satisfied, while the rotational perturbations (and density perturbations) do. However, it is important to realize that the "primordial" cosmological rotation, although small, could have been generated quantum mechanically. The role of the "primordial" rotation at smaller linear scales was out of the scope of the present paper. Nevertheless, one may speculate that, if the generating mechanism did really work, the "seeds" of rotation of quantum-mechanical origin may have also played a role at the smaller linear scales characteristic of galaxies and their clusters (for instance, it is hard to avoid the questions whether the flat rotation curves in spiral galaxies are an evidence for dark matter or may be the remnants of the "primordial" rotation and whether, or not, the "primordial" rotation could have played a role in the generation of cosmic magnetic fields).

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