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Perturbations of a topological defect as a theory of coupled scalar fields in curved space interacting with an external vector potential

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The evolution of small irregularities in a topological defect which propagates on a curved background spacetime is examined. These are described by a system of coupled scalar wave equations on the world sheet of the unperturbed defect which is not only manifestly covariant under world-sheet diffeomorphisms but also under local normal frame rotations. The scalars couple both through the surface torsion of the background world sheet geometry which acts as a vector potential and through an effective mass matrix which is a sum of a quadratic in the extrinsic curvature and a linear term in the spacetime curvature. The coupling simplifies enormously for many physically interesting geometries. This introduces a framework for examining the stability of topological defects generalizing both our earlier work on the perturbations of domain walls and the work of Garriga and Vilenkin on perturbations about a class of spherically symmetric defects in de Sitter space.

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I. INTRODUCTION

Topological defects of one form or another are expected to appear as by-products of phase transitions that occurred in the early Universe. Their cosmological implications, however, appear to depend sensitively on their stability with respect to perturbations. For example, an instability in the geometry of a closed cosmic string could disrupt its collapse to form a black hole [1]. Recently, Garriga and Vilenkin undertook an examination of the stability of spherically symmetric topological defects nucleating in de Sitter space [2]. The approximation they use is to model the defect as a membrane propagating on a curved background spacetime.

In this paper, we examine the evolution of small irregularities on a topological defect moving in a general curved background spacetime in the same approximation without any restrictions on the symmetry of the defect.

In an earlier paper, we treated the perturbations of domain walls [3]. The relevant covariant measure of the perturbation then is its projection onto the normal to the world sheet. We were able to show that this scalar satisfies a Klein-Gordon equation on the geometry of the unperturbed world sheet, coupling in a universal manner through an effective mass both to the world-sheet scalar curvature and the traced projection of the spacetime Ricci curvature onto the world sheet. This provided a generalization of the wave equation derived in Ref. [2] describing perturbations of domain walls in Minkowski space.

In the case of a lower-dimensional defect there will be one scalar corresponding to the projection of the perturbation in the world-sheet onto each normal direction. A new geometrical structure which is antisymmetric in its normal indices also appears. The geometrical role it plays in perturbative theory is that of a vector potential ensuring that the scalar field equations transform covariantly under local normal frame rotations. These scalars will generally satisfy a system of wave equations which are coupled not only through an effective mass matrix but also through this vector potential. In particular, this introduces a derivative coupling between the fields.

We begin in Sec. II with a derivation of the exact equations of motion for the defect. Our approach to perturbation theory in Sec. III will be to expand the action describing the evolution of the defect in a manifestly covariant way out to second order in the perturbation about a given classical solution. We model this on the treatment of Hawking and Ellis of the second variation of the arclength about a geodesic curve [4]. Geodesics, however, can be a poor guide to the behavior of higher-dimensional surfaces. The proper length along a curve has no higher dimensional analogue; the curvature of a connection has no one-dimensional analogue. It is therefore extremely gratifying that the formal expression one obtains is strikingly similar to the geodesic result when the parametrization along the latter is not affine. We exploit the classical theory of surfaces to bridge the gap between formal mathematics and a tractable system of equations with which one can begin to do physics [5].

In Sec. IV we discuss the equations of motion describing perturbations on various background geometries. In practice, one is interested in perturbations about defects possessing some level of symmetry. It is then, of course, sufficient to develop perturbation theory in a manner which is tailored to the symmetry. In Ref. [2], doing just this, it was shown that on a spherically symmetric string of maximum radius in de Sitter space these equations not

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only decouple completely but each component tends to mimic the single scalar characterizing the perturbation on a domain wall in a de Sitter space of one lower dimension. It is probably fair to say, however, that in the absence of a more general framework to steer by one is at a loss to provide an entirely adequate interpretation of the physics. It is not clear what features of the underlying geometry are responsible for the simplification in perturbation theory discovered in Ref. [2]. Do we always expect the effective mass to be tachyonic? We attempt to provide sufficient criteria determining when the equations will decouple. In particular, we demonstrate that whenever the world sheet of the defect can be embedded as a hypersurface in some lower-dimensional geometry, eliminate the equations of motion completely decouple. As a special case we rederive Eq. (58) of the first paper in Ref. [2].

II. THE EQUATIONS OF MOTION

Let us consider an oriented surface m of dimension D described by the timelike surface

$$x^{\mu}=X^{\mu}(\xi^{a}) , \tag{2.1}$$

with $\mu=0,\dots,N-1$, $a=0,\dots,D-1$, embedded in an N-dimensional spacetime M described by the metric $g_{\mu\nu}$. The D vectors

$$e_a=X^{\mu}_{,a}\partial_{\mu}$$

form a basis of tangent vectors to m at each point of m. The metric induced on the world sheet is then given by

$$\gamma_{ab}=X^{\mu}_{,a}X^{\nu}_{,b}g_{\mu\nu}=g(e_a,e_b) . \tag{2.2}$$

The action which describes the dynamics of our system is the most simple generally covariant action one can associate with the surface, proportional to the area swept out by the world sheet of the surface as it evolves:

$$S[X^{\mu},X^{\mu}_{,a}]=-\sigma\int_m d^D\xi\sqrt{-\gamma} . \tag{2.3}$$

The constant of proportionality σ represents the constant (positive) energy density in the surface in its rest frame. If the area is infinite the associated action will be infinite. However, the change in area corresponding to a variation in the embedding of compact support will always be finite.

We confine our attention to closed defects without physical boundaries. The only boundary of the world-sheet is then the spacelike boundary, ∂m_t, we introduce to implement the variational principle, marking the temporal limits of the world-sheet on which the initial and final configurations are fixed.

The equations of motion of the defect are given by the extrema of S subject to variations

$$X^{\mu}(\xi)\to X^{\mu}(\xi)+\delta X^{\mu}(\xi) ,$$

which vanish on ∂m_t:

$$-\frac{\delta S}{\delta X^{\mu}}\equiv\sigma[\Delta X^{\mu}+\Gamma^{\mu}_{\alpha\beta}(X^{\nu})\gamma^{ab}X^{\alpha}_{,a}X^{\beta}_{,b}]=0 , \tag{2.4}$$

where Δ is the scalar Laplacian,

$$\Delta=\frac{1}{\sqrt{\gamma}}\partial_a(\sqrt{\gamma}\gamma^{ab}\partial_b) ,$$

and $\Gamma^{\mu}_{\alpha\beta}$ are the spacetime Christoffel symbols evaluated on m. We return to the derivation at the end of this section. Equation (2.4) is clearly a higher-dimensional generalization of the geodesic equation describing the motion of a point defect. Even in Minkowski space, however, this equation is highly nonlinear. The feature of string theory which makes it tractable is the fact that the world-sheet is two dimensional and any two-dimensional metric is conformally flat [6].

Despite the nice analogy, this form of the equations of motion is not very useful in practice. This is because all but $N-D$ linear combinations of these equations are identically satisfied. To see this, we note, both on shell and off, Gauss's equation [see Eq. (4.8a) below] can be rewritten in the form

$$\nabla_b X^{\mu}_{,a}+\Gamma^{\mu}_{\alpha\beta}(X^{\nu})X^{\alpha}_{,a}X^{\beta}_{,b}=K^{(i)}_{ab}n^{(i)\mu} ,$$

where ∇_a is the world-sheet covariant derivative compatible with γ_{ab}, $n^{(i)}$ is the ith unit normal to the world sheet, $i=1,\dots,N-D$, and the corresponding ith extrinsic curvature tensor $K^{(i)}_{ab}$ is defined by [5] (we introduce the notation $\mathcal{D}_a=X^{\mu}_{,a}D_{\mu}$)

$$K^{(i)}_{ab}=-X^{\nu}_{,a}D_{\nu}n^{(i)}_{\mu}=-g(e_a,\mathcal{D}_b n^{(i)}) . \tag{2.5}$$

The tangential projections of the Euler-Lagrange derivatives of S therefore vanish identically:

$$\frac{\delta S}{\delta X^{\mu}}X^{\mu}_{,b}=0 . \tag{2.6}$$

As we have remarked in Ref. [3], the geometrical reason for this redundancy is the invariance of the action with respect to world-sheet diffeomorphisms.

It is now clear that the equations describing the world sheet are entirely equivalent to the $N-D$ equations

$$K^{(i)}=0 . \tag{2.7}$$

These are just the equations describing an extremal surface and are well known in the mathematical literature [7]. They provide an obvious generalization of the more familiar notion of extremal hypersurface.

To derive Eq. (2.4), we note that

$$\delta S=-\frac{1}{2}\sigma\int_m d^D\xi\sqrt{-\gamma}\gamma^{ab}\mathcal{D}_{\delta}g(e_a,e_b) , \tag{2.8}$$

where we introduce the spacetime vector field

$$\delta=\delta X^{\mu}\partial_{\mu} , \tag{2.9}$$

to characterize the deformation in the world sheet. We also set $\mathcal{D}_{\delta}=\delta X^{\mu}D_{\mu}$. The key observation in the derivation is that the Lie derivative of the vector field δ with respect to e_a vanishes (the proof is sketched in Chap. 4 of Ref. [4]):

$$\mathcal{D}_\delta e_a = \mathcal{D}_a \delta \ . \quad (2.10)$$

Now

$$\gamma^{ab}\mathcal{D}_\delta g(e_a,e_b) = 2\gamma^{ab} g(\mathcal{D}_\delta e_a, e_b) = 2\gamma^{ab} g(\mathcal{D}_a \delta, e_b) = 2\gamma^{ab}[\mathcal{D}_a g(\delta,e_b) - g(\delta,\mathcal{D}_a e_b)] \ .$$

The first term on the last line can be reorganized as

$$\sqrt{-\gamma}\gamma^{ab}\mathcal{D}_a g(\delta,e_b) = \mathcal{D}_a[\sqrt{-\gamma}\gamma^{ab} g(\delta,e_b)] - \mathcal{D}_a(\sqrt{-\gamma}\gamma^{ab}) g(\delta,e_b)$$

to extract a divergence. Because δ vanishes on ∂m_t this term will also vanish there. We are left with the simple formal expression

$$\delta S = \sigma \int d^D\xi\, g[\delta, \mathcal{D}_a(\sqrt{-\gamma}\gamma^{ab} e_b)] \ . \quad (2.11)$$

The equations of motion are therefore

$$\mathcal{D}_a(\sqrt{-\gamma}\gamma^{ag} e_b) = 0 \ , \quad (2.12)$$

the spacetime components of which reproduce Eq. (2.4).

III. THE QUADRATIC ACTION

At lowest order, the dynamics of the irregularities in the defect is still expected to be accurately described by the action Eq. (2.3). The approach we will follow will be to expand the action out to quadratic order about the classical solution satisfying Eq. (2.7). When this is done, it will be a relatively straightforward matter to write down the corresponding equations of motion.

As we have seen, variations along tangential directions correspond to world-sheet diffeomorphisms. We can, however, provide a diffeomorphism invariant description of the perturbation δX^μ in the wall by constructing the $N-D$ scalars

$$\Phi^{(i)} \equiv n^{(i)}_\mu \delta X^\mu \quad (3.1)$$

representing the independent projections of the spacetime vector δX^μ onto the different normal directions. The choice of the $\Phi^{(i)}$ is not unique reflecting the fact that the defining relations for the normal vectors,

$$g(e_a, n^{(i)}) = 0, \quad g(n^{(i)}, n^{(j)}) = \delta^{(i)(j)} \ ,$$

only determine these vectors up to $(N-D)$-dimensional frame rotations. If the geometry possesses some symmetry, it is very convenient to choose the normal vectors so that they reflect the symmetry.

In our earlier treatment of domain walls it was possible to exploit Gaussian normal coordinates based on the world-sheet hypersurface to facilitate the calculation. In general, there is no simple analogue of Gaussian coordinates corresponding to a lower-dimensional embedding [8]. It is fortunate, therefore, that the covariant formalism we have been pursuing is tractable.

We now evaluate the second variation of the action at its stationary points. This is given by

$$\delta^2 S = \sigma \int d^D\xi\, g[\delta, \mathcal{D}_\delta \mathcal{D}_a(\sqrt{-\gamma}\gamma^{ab} e_b)] \ . \quad (3.2)$$

Thus, the relevant equation is

$$\mathcal{D}_\delta \mathcal{D}_a(\sqrt{-\gamma}\gamma^{ab} e_b) = 0 \ . \quad (3.3)$$

While formally Eq. (3.3) describes small perturbations, it is not very useful in its present form. What we need to do is to cast Eq. (3.3) explicitly as a linear system of coupled scalar wave equations

$$\mathcal{L}_{(i)(j)}\Phi^{(j)} = 0 \ , \quad (3.4)$$

Our task is to find the linear hyperbolic partial differential operator $\mathcal{L}$. Such an equation can be derived from an action of the form

$$S = \frac{1}{2}\delta^2 S = \frac{1}{2}\int d^D\xi \sqrt{-\gamma}\, \Phi^{(i)} \tilde{\mathcal{L}}_{(i)(j)} \Phi^{(j)} \ ,$$

where $\tilde{\mathcal{L}}_{(i)(j)}$ is some other linear hyperbolic operator. As we will see $\tilde{\mathcal{L}}_{(i)(j)}$ is linear in first derivatives of the fields $\Phi^{(i)}$. As a consequence it will not coincide with $\mathcal{L}_{(i)(j)}$. We are always free to symmetrize $\tilde{\mathcal{L}}_{(i)(j)}$ (but not $\mathcal{L}_{(i)(j)}$) with respect to the indices i and j.

Let us examine projection of the left-hand side (LHS) of Eq. (3.3) on $n^{(i)}$:

$$g[n^{(i)}, \mathcal{D}_\delta \mathcal{D}_a(\sqrt{-\gamma}\gamma^{ab} e_b)] = 0 \ .$$

We need only consider vector fields δ which are normal to m. The idea is to push $\mathcal{D}_\delta$ to the right through $\mathcal{D}_a$ and e_a. Let us begin then by exploiting the spacetime Ricci identity on the spacetime vector field $v = \sqrt{-\gamma}\gamma^{ab} e_b$:

$$\mathcal{D}_\delta \mathcal{D}_a v = \mathcal{D}_a \mathcal{D}_\delta v + {}^N\! R(e_a,\delta) v \ . \quad (3.5)$$

We note that by exploiting the projection tensor,

$$h^{\mu\nu} = \gamma^{ab} X^\mu_{,a} X^\nu_{,b} = g^{\mu\nu} - n^{(k)\mu} n^{(k)\nu} \ , \quad (3.6)$$

we can express

$$\gamma^{ab} g(\delta, {}^N\! R(e_a,\delta) e_b) = ({}^N\! R_{\mu\nu} n^{(i)\mu} n^{(j)\nu} - {}^N\! R_{\mu\alpha\nu\beta} n^{(i)\mu} n^{(k)\alpha} n^{(j)\nu} n^{(k)\beta}) \Phi^{(i)} \Phi^{(j)} \ .$$

We now decompose $\mathcal{D}_a \mathcal{D}_\delta v$ into three terms as follows:

$$g(n^{(i)}, \mathcal{D}_a \mathcal{D}_\delta[\sqrt{-\gamma}\gamma^{ab} e_b]) = \mathcal{D}_a[\sqrt{-\gamma}\gamma^{ab} g(n^{(i)}, \mathcal{D}_\delta e_b)] - \mathcal{D}_\delta(\sqrt{-\gamma}\gamma^{ab}) g(\mathcal{D}_a n^{(i)}, e_b) - (\sqrt{-\gamma}\gamma^{ab}) g(\mathcal{D}_a n^{(i)}, \mathcal{D}_b \delta) \ . \quad (3.7)$$

We examine each term separately. For the first term, we rewrite the argument of $\mathcal{D}_a$:

$$\sqrt{-\gamma}\gamma^{ab} g(n^{(i)}, \mathcal{D}_\delta e_b) = \mathcal{D}_b[\sqrt{-\gamma}\gamma^{ab} g(n^{(i)},\delta)] - \sqrt{-\gamma}\gamma^{ab} g(\mathcal{D}_b n^{(i)},\delta) - \mathcal{D}_b(\sqrt{\gamma}\gamma^{ab})\Phi^{(i)} \ ,$$

so that

$$\mathcal{D}_a[\sqrt{-\gamma}\gamma^{ab}g(n^{(i)},\mathcal{D}_\delta e_b)] = \sqrt{-\gamma}[\Delta\Phi^{(i)} - \nabla_a(T^{a(i)(j)}\Phi^{(j)})] ,$$

where we have introduced the surface torsion (or normal form)

$$T_a^{(i)(j)} = g(\mathcal{D}_a n^{(i)}, n^{(j)}) = -T_a^{(j)(i)} . \quad (3.8)$$

The second term gives

$$\mathcal{D}_\delta(\sqrt{-\gamma}\gamma^{ab})g(\mathcal{D}_a n^{(i)},e_b) = -\mathcal{D}_\delta(\sqrt{-\gamma}\gamma^{ab})K_{ab}^{(i)} = \sqrt{-\gamma}\mathcal{D}_\delta(\gamma_{ab})K^{(i)ab} .$$

In the last line we use the background equations of motion Eq. (2.7) to justify dropping a term proportional to $K^{(i)}$. We can decompose

$$\mathcal{D}_\delta(\gamma_{ab}) = g(\mathcal{D}_\delta e_a, e_b) + g(e_a, \mathcal{D}_\delta e_b) = 2K_{ab}^{(j)}\Phi^{(j)} .$$

We could have used this result directly to derive the equations of motion, Eq. (2.7).

To evaluate the third term, we note that

$$g(\mathcal{D}_a n^{(i)},\mathcal{D}_b\delta) = T^{(i)(j)}{}_a\nabla_b\Phi^{(j)} + D_\mu n^{(i)\nu}D_\alpha n^{(j)\nu}\Phi^{(j)} ,$$

so that

$$(\sqrt{-\gamma}\gamma^{ab})g(\mathcal{D}_a n^{(i)},\mathcal{D}_b\delta) = \sqrt{-\gamma}(\gamma^{ab}T^{(i)(j)}{}_a\nabla_b\Phi^{(j)} + h^{\mu\nu}D_\mu n^{(i)\alpha}D_\nu n_\alpha^{(j)}\Phi^{(j)})$$
$$= \sqrt{-\gamma}(\gamma^{ab}T^{(i)(j)}{}_a\nabla_b\Phi^{(j)} + K_{ab}^{(i)}K^{(j)ab} - T^{(i)(k)a}T_a^{(k)(j)}) ,$$

using the definition (3.6) of the projection tensor.

We now add the three terms on the RHS of Eq. (3.7). The action is given by

$$S = \frac{1}{2}\int d^D\xi\sqrt{-\gamma}(\Phi^{(i)}\Delta\Phi^{(i)} - 2\Phi^{(i)}T^{(i)(j)a}\nabla_a\Phi^{(j)} + \Phi^{(i)}\nabla_a T^{(i)(j)a}\Phi^{(j)}$$
$$+\Phi^{(i)}[{}^N R_{\mu\nu}n^{(i)\mu}n^{(j)\nu} - {}^N R_{\mu\alpha\nu\beta}n^{(i)\mu}n^{(k)\alpha}n^{(j)\nu}n^{(k)\beta}]\Phi^{(j)} + \Phi^{(i)}[K_{ab}^{(i)}K^{(j)ab} + T^{(i)(k)a}T_a^{(k)(j)}]\Phi^{(j)}) . \quad (3.9)$$

This coincides with Eq. (A6) of the third paper in Ref. [2] when the background geometry is Minkowski space. The term involving the world-sheet divergence of $T_a^{(i)(j)}$ can be dropped because it involves a contraction on the normal indices of a term which is symmetric with a term which is antisymmetric in these indices. Such a term will however show up in the equations of motion. Let us now define

$$\tilde{\nabla}_a^{(i)(j)} = \nabla_a\delta^{(i)(j)} - T_a^{(i)(j)} . \quad (3.10)$$

Then

$$S = \frac{1}{2}\int d^D\xi\sqrt{-\gamma}[\Phi^{(i)}\tilde{\Delta}^{(i)(j)}\Phi^{(j)} - \Phi^{(i)}(M^2)_{(i)(j)}\Phi^{(j)}] , \quad (3.11)$$

where

$$\tilde{\Delta}^{(i)(j)} = \tilde{\nabla}^{a(i)(k)}\tilde{\nabla}_a^{(k)(j)} , \quad (3.12)$$

and

$$(M^2)_{(i)(j)} = {}^N R_{\mu\alpha\nu\beta}n^{(i)\mu}n^{(k)\alpha}n^{(j)\nu}n^{(k)\beta} - {}^N R_{\mu\nu}n^{(i)\mu}n^{(j)\nu} - K_{ab}^{(i)}K^{(j)ab} . \quad (3.13)$$

All three terms involving torsion get absorbed into the definition of $\tilde{\Delta}^{(i)(j)}$.

IV. THE LINEARIZED EQUATIONS

The variation of Eq. (3.9) with respect to $\Phi^{(i)}$ gives

$$\tilde{\Delta}^{(i)(j)}\Phi^{(j)} - (M^2)_{(i)(j)}\Phi^{(j)} = 0 . \quad (4.1)$$

This is a system of $N-D$ nontrivially coupled scalar wave equations for the $\Phi^{(i)}$ on the curved background geometry of the world sheet.

It is worthwhile at this point to pause a moment to comment on the geometric role played by torsion in Eq. (4.1). Under a local normal frame rotation,

$$n^{(i)} \to O^{(i)(j)}(\xi)n^{(j)} , \quad (4.2)$$

we note that $\tilde{\nabla}_a^{(i)(j)}\Phi^{(j)}$ transforms covariantly. This is because the torsion transforms like a vector potential:

$$T_a^{(i)(j)} \to O^{(i)(k)}T_a^{(k)(l)}(O^{-1})^{(l)(j)} + (\mathcal{D}_a O O^{-1})^{(i)(j)} . \quad (4.3)$$

The torsion is not itself gauge invariant. The gauge-invariant measure of the torsion is its curvature defined by

$$(\tilde{\nabla}_a^{(i)(k)}\tilde{\nabla}_b^{(k)(j)} - \tilde{\nabla}_b^{(i)(k)}\tilde{\nabla}_a^{(k)(j)})\Phi^{(j)} = T^{(i)(j)}{}_{ab}\Phi^{(j)} ,$$

so that

$$T^{(i)(j)}{}_{ab} = \nabla_b T_a^{(i)(j)} - T_a^{(i)(k)}T_b^{(k)(j)} - (a \leftrightarrow b) . \quad (4.4)$$

A choice of normals such that the torsion vanishes exists if and only if $T^{(i)(j)}{}_{ab} = 0$. However, if one decides to be perverse with one's choice of normals one can always introduce a torsion. Whatever the value of $T^{(i)(j)}{}_{ab}$, unless torsion appears explicitly in Eq. (4.1), its covariance under normal frame rotations will be lost.

It is always possible to orient the normals along a curve $\xi^a = \Xi^a(s)$ in m such that the torsion vanishes along that curve. This can be accomplished by re-orienting the normals at parameter s with the rotation matrix

$$O(s)^{(i)(j)}=O(0)^{(i)(k)} \times P\left[\exp\{-\int_0^s ds'\dot{\Xi}^a(s')T_a[\Xi(s')]\}\right]^{(k)(j)} ,$$

where P represents the path ordered product. This is an analogue of the well-known result that a coordinate system always exists in which the Riemmanian connection vanishes along any given curve.

There are two ways that the scalar fields can couple. One way is through the effective mass matrix $(M^2)_{(i)(j)}$ given by Eq. (3.13). If there is torsion, however, they can also couple through $\tilde{\Delta}$, which therefore acts like an external vector potential. Though $T^{(i)(k)a}T_a^{(k)(j)}$ couples the $\Phi^{(i)}$ like a mass term, it is more naturally grouped in the combination appearing in the definition of $\tilde{\nabla}$. $(M^2)_{(i)(j)}$ need not possess a global sign. It can be diagonalized at any point with its eigenvalues forming its diagonal entries. If the world sheet were Minkowski space, a negative eigenvalue of $M^2_{(i)(j)}$ would signal an instability. However, in general, there is no simple correlation between tachyonic masses and instabilities. An explicit counterexample is provided by perturbation theory about a class of defects in de Sitter space discussed in Ref. [2] and which we will examine below.

In the case of a domain wall with a single normal vector, both the torsion and the total projection of the spacetime Riemann curvature onto the normal vanish. Equation (4.1) then reduces to the form

$$\Delta\Phi+(R_{\mu\nu}n^\mu n^\nu+K^{ab}K_{ab})\Phi=0 . \qquad (4.1')$$

In this case, the quadratic in the extrinsic curvature can be eliminated in favor of intrinsic geometric scalars using the contracted Gauss-Codazzi equations. We reproduce Eq. (4.1) of Ref. [2] with $\rho=0$. Henceforth, we will assume that the codimension of the defect exceeds one.

Even when the background geometry is flat so that ${}^NR^\mu{}_{\nu\alpha\beta}=0$, Eq. (4.1) is extremely complicated, involving scalars in the extrinsic geometry ($K_{ab}^{(i)}$ and $T_a^{(i)(j)}$) in combinations which, it appears, cannot be eliminated in favor of intrinsic geometric scalars. To see this, let us recall the complete set of consistency conditions for the embedding [5]. These are

$$g({}^NR(e_a,e_b)e_c,e_d)={}^DR_{abcd}+K_{ac}^{(i)}K_{bd}^{(i)}-K_{ad}^{(i)}K_{bc}^{(i)} \qquad \text{(Gauss-Codazzi)} ,$$

$$g({}^NR(e_a,e_b)e_c,n^{(i)})=\nabla_aK_c^{(i)}+T_b^{(i)(j)}K_{ac}^{(j)}-(a\leftrightarrow b) \qquad \text{(Codazzi-Mainardi)} ,$$

and

$$g({}^NR(e_a,e_b)n^{(i)},n^{(j)}) =T^{(i)(j)}{}_{ab}-[K_{ac}^{(i)}K_b^{(j)c}-(a\leftrightarrow b)], \quad \text{(Ricci)} ,$$

where $T^{(i)(j)}{}_{ab}$ is given by Eq. (4.4). Note that the torsion only occurs in gauge-invariant combinations in these equations.

Unlike the case of a hypersurface, we cannot exploit the Gauss-Codazzi equations to eliminate the quadratic in $K_{ab}^{(i)}$ in favor of spacetime and world-sheet curvature scalars. This is because it is the traced product over the normal indices, $K_{ab}^{(i)}K_{cd}^{(i)}$ which appears in these equations. The quadratics in $T_a^{(i)(j)}$ which appear in the Ricci equations are antisymmetric in both world-sheet and normal indices. These equations therefore do not help us to eliminate the quadratic in $T_a^{(i)(j)}$ appearing in Eq. (4.1).

If, however, we can choose our normal vectors such that all but one of them, for example, $n^{(1)}$, are parallel transported along any curve on the world sheet,

$$\mathcal{D}_a n^{(i)}=0 , \qquad (4.5)$$

then $T_a^{(i)(j)}=0$ for all i and j. The vanishing of $T_a^{(1)(j)}$ is assured by the antisymmetry of $T_a^{(i)(j)}$ with respect to its normal indices. We thus identify a sufficient set of conditions on the embedding under which the surface torsion vanishes.

In addition, the conditions Eq. (4.5) imply that the only linear combination of extrinsic curvature tensors which is nonvanishing is the one that corresponds to the exceptional normal direction:

$$K_{ab}^{(i)}=0, \quad i=2,\ldots,N-D . \qquad (4.6)$$

Only one of the background equations of motion will be nontrivial. The marvelous thing about Eq. (4.5) for our present purposes is that the quadratic in $K_{ab}^{(i)}$ appearing in Eq. (4.1) can be replaced by its trace:

$$K_{ab}^{(i)}K^{(j)ab}=\delta^{(i)(1)}\delta^{(j)(1)}K_{ab}^{(k)}K^{(k)ab} ,$$

so that the contracted Gauss-Codazzi equation can be used exactly as it was in the case of a hypersurface to eliminate it in terms of curvatures:

$$\begin{aligned} K^{ab(k)}K_{ab}^{(k)} &= {}^NR_{\mu\nu\alpha\beta}h^{\mu\alpha}h^{\nu\beta}-{}^DR \\ &= {}^NR-2\,{}^NR_{\mu\nu}n^{(k)\mu}n^{(k)\nu} \\ &\quad +2\,{}^NR_{\mu\nu\alpha\beta}n^{(k)\mu}n^{(k)\alpha}n^{(l)\nu}n^{(l)\beta}-{}^DR , \end{aligned} \qquad (4.7)$$

where we have exploited the definition of the projection tensor, Eq. (3.8). The normal projections of the spacetime Ricci and Riemann tensors are of a kind already encountered in Eq. (4.1). However, here they do not imply any coupling between different $\Phi^{(i)}$'s.

To provide a geometrical picture for what Eq. (4.5) implies, it is useful to recall the form of the Gauss-Weingarten equations [5] which describe the change in the basis vectors as one moves about the surface:

$$\mathcal{D}_a e_b=\gamma^c_{ab}e_c+K_{ab}^{(i)}n^{(i)} , \qquad (4.8a)$$

$$\mathcal{D}_a n^{(i)}=-K_{ab}^{(i)}e^b+T_a^{(i)(j)}n^{(j)} , \qquad (4.8b)$$

where the γ^c_{ab} are the world-sheet connection coefficients. The consistency conditions we wrote down earlier are the integrability conditions on these equations. When Eq. (4.5) is satisfied, only $n^{(1)}$ changes as we move about the world sheet and Eqs. (4.8) reduce to the form

$$\begin{aligned} \mathcal{D}_a e_b &= \gamma^c_{ab}e_c+K_{ab}^{(1)}n^{(1)} , \\ \mathcal{D}_a n^{(1)} &= -K_{ab}^{(1)}e^b , \end{aligned} \qquad (4.8')$$

and

$$\mathcal{D}_a n^{(I)}=0, \quad I=2,\ldots,N-D \ .$$

The first two equations are simply the hypersurface form of the Gauss-Weingarten equations. The world sheet can be embedded as a hypersurface in a $(D+1)$ one-dimensional submanifold of M, let us say $\mathcal{M}$.

Let us now look for conditions on the geometry in the neighborhood of the world sheet making it consistent with Eq. (4.5). To do this, we construct a coordinate system for M adapted to $\mathcal{M}$ in the neighborhood of m. Let y^A, $A=0,\ldots,D$ be coordinates for $\mathcal{M}$ in this neighborhood. We now complete the coordinate system for M by complementing the coordinates on $\mathcal{M}$ with $N-D-1$ coordinates $\{z^I\}$, $I=2,\ldots,N-D$ such that $\mathcal{M}$ is given by $z^I=0$. The normals to $m, n^{(I)}$, $I=2,\ldots,N-D$ are then linear combinations of the gradients of the z^I evaluated on m. With respect to these coordinates, Eqs. (4.5) can be replaced by the following conditions on the spacetime metric evaluated on m:

$$g_{AB,I}=0 \ , \qquad g_{AI}=0 \ . \tag{4.9}$$

Any defect in Minkowski space which lies in a D-dimensional plane will satisfy these conditions.

Let us describe de Sitter space by a Friedmann-Robertson-Walker (FRW) closed line element

$$ds^2=-dt^2+H^{-2}\cosh^2(Ht)d\Omega_{N-1}^2 \ ,$$

where $d\Omega_N^2$ is the line element on a round $N-1$ sphere and H is the Hubble parameter. The subspace consisting of any number of fixed azimuthal angles is also a de Sitter space with the same Hubble parameter. A D-dimensional defect in de Sitter space with $N-D-1$ fixed meridians will also satisfy Eq. (4.9).

In the two cases considered above, spacetime is homogeneous and isotropic. A less trivial example satisfying Eq. (4.9) is a string in Schwarzschild space on a fixed meridian. Thus we see that Eqs. (4.9) is consistent with a reasonably large class of geometries.

Let us suppose, in addition, that the geometry of the world sheet is spherically symmetric, i.e., invariant under the rotation group, $O(D-1)$. The world sheet of the defect is then a D-dimensional FRW homogeneous and isotropic closed universe described by the line element

$$ds^2=-d\tau^2+a^2(\tau)d\Omega_{D-1}^2 \ ,$$

where τ is the proper time registered on a comoving clock. The function $a(\tau)$ is the proper circumferential radius r on the $D-1$ sphere at proper time τ. Consistency then demands that the spacetime metric be invariant under some $O(d-1)$ with $d\geq D$ with $D-1$ common axes of symmetry. The only nontrivial dynamics now takes place in a $(1+1)$-dimensional subspace of $\mathcal{M}$. The Gauss-Weingarten equations mimic the Frenet-Serret equations describing the motion of a particle in this two-dimensional spacetime:

$$\mathcal{D}_\tau e_\tau=K^{(1)}_{\tau\tau}n^{(1)} \ , \qquad \mathcal{D}_\tau n^{(1)}=-K^{(1)}_{\tau\tau}e^\tau \ . \tag{4.10}$$

The condition $\gamma^a_{\tau\tau}=0$ is the analogue of the statement the acceleration along a timelike curve is orthogonal to the velocity when the curve is parametrized by proper time.

There is another simplification which occurs whenever $D=N-2$, an example of which is provided by a string in any four-dimensional manifold. For then, the coupling between the two scalar field components $\Phi^{(1)}$ and $\Phi^{(2)}$ which is mediated by the terms of the form

$${}^N R_{\mu\alpha\nu\beta}n^{(i)\mu}n^{(k)\alpha}n^{(j)\nu}n^{(k)\beta}\Phi^{(j)} \tag{4.11}$$

in Eq. (4.1) vanishes. Let $i=1$. Then, the only surviving term in Eq. (3.13) is

$${}^N R_{\mu\alpha\nu\beta}n^{(1)\mu}n^{(2)\alpha}n^{(1)\nu}n^{(2)\beta}\Phi^{(2)} \ ;$$

the fields decouple. This condition may not be independent of Eqs. (4.5).

If the background spacetime is Einstein, with cosmological constant Λ,

$$R_{\alpha\beta}=\frac{2\Lambda}{N-2}g_{\alpha\beta} \ ,$$

the Ricci curvature coupling between different scalar fields disappears:

$$R_{\mu\nu}n^{(i)\mu}n^{(j)\nu}=\frac{2\Lambda}{N-2}\delta^{(i)(j)} \ .$$

If the background is de Sitter space, the Riemann curvature coupling also disappears independent of the dimension of the defect. For then

$$R_{\mu\alpha\nu\beta}=H^2(g_{\mu\nu}g_{\alpha\beta}-g_{\mu\beta}g_{\nu\alpha}) \ , \tag{4.12}$$

so that

$$R_{\mu\alpha\nu\beta}n^{(i)\mu}n^{(k)\alpha}n^{(j)\nu}n^{(k)\beta}=H^2(N-D-1)\delta^{(i)(j)} \ .$$

In particular, in the case of any defect lying on the subspace with $N-D-1$ fixed meridans (it need not itself be spherically symmetric) in de Sitter space, the best of all worlds is realized. The scalar fields completely decouple. Without any essential loss of generality, we will consider codimension two. Now

$$-(M^2)_{(1)(1)}={}^N R+{}^N R_{\mu\nu}n^{(1)\mu}n^{(1)\nu}-{}^N R_{\mu\nu}n^{(k)\mu}n^{(k)\nu} +{}^N R_{\mu\alpha\nu\beta}n^{(1)\mu}n^{(2)\alpha}n^{(1)\nu}n^{(2)\beta}-{}^{N-2}R \tag{4.13a}$$

and

$$-(M^2)_{(2)(2)}={}^N R_{\mu\nu}n^{(2)\mu}n^{(2)\nu} -{}^N R_{\mu\alpha\nu\beta}n^{(1)\mu}n^{(2)\alpha}n^{(1)\nu}n^{(2)\beta} \ . \tag{4.13b}$$

We substitute Eq. (4.12) into Eqs. (4.13) to get

$$-(M^2)_{(1)(1)}=(N-2)H^2+{}^D R-(N-2)(N-3)H^2 \ , \qquad -(M^2)_{(2)(2)}=(N-2)H^2 \ . \tag{4.14}$$

$M_{(2)(2)}$ is independent of the motion of the defect and is always tachyonic. Both wave equations depend only on the intrinsic geometry of the world sheet.

Let us now specialize to spherically symmetric defects of codimension two. In four dimensions these are circular strings. A circular string can follow two qualitatively different trajectories $a = a(\tau)$ in de Sitter space [1]. One of these consists of trajectories which begin with $a = 0$ grow to a maximum value before recollapsing to $a = 0$. The other is the bounce which consists of a trajectory originating on the equator contracting to a minimum value and then bouncing back to the equator. The description in terms of $a(\tau)$ is qualitatively identical to that for spherical domain walls.

In particular, there is a bounce which does not really bounce at all representing a circular string which spans the equator. This solution can be interpreted as a string which tunnels from nothing due to quantum mechanical processes [1]. These are the strings about which perturbation theory was examined in Ref. [2].

The world sheet is now an embedded $(N-2)$-dimensional de Sitter space with

$$ {}^{N-2}R = (N-2)(N-3)H^2 \ . \tag{4.15} $$

We note that now, not only do Eqs. (4.5) hold, but in addition $\mathcal{D}_a n^{(1)} = 0$. The normal directions $n^{(1)}$ and $n^{(2)}$ are now entirely equivalent. Our construction which did not exploit the extra symmetry of a defect which spans the equator degenerates. Geometrically, $K_{ab}^{(i)} = 0$ for all i. In mathematical parlance, the world sheet is totally geodesic [5].

It should not be surprising that perturbation theory simplifies dramatically in this case. The two effective mass eigenvalues now coincide and are tachyonic. The equations of motion for $\Phi^{(1)}$ and $\Phi^{(2)}$ are therefore identical,

$$ \Delta\Phi^{(1),(2)} + (N-2)H^2\Phi^{(1),(2)} = 0 \ , \tag{4.16} $$

reproducing the expression obtained in Ref. [2]. The wave equation for each component mimics the equation for an equatorial domain wall in an $(N-1)$-dimensional de Sitter spacetime. We note that the technique used in Ref. [2] to derive Eq. (4.14) depended sensitively on the fact that the embedded domain wall spanned the equator. Now, however, we possess a general framework which not only has permitted us to predict that decoupling would occur but also explains why the effective masses coincide in this geometry.

V. CONCLUSIONS

We have provided a framework for the examination of perturbations on topological defects on a given spacetime background which generalizes our earlier work on domain walls to lower-dimensional topological defects. In either case, however, the coupling of the perturbation to extrinsic geometry makes the theory very different from the scalar field theories we are familiar with. When the codimension of the world sheet is r, there will be r scalar fields describing the perturbation. There is a coupling through an effective mass matrix involving quadratics in the extrinsic curvature as well as appropriate projections of the spacetime Riemann curvature. In addition, however, on a lower-dimensional defect there will be a coupling between the scalar fields mediated by the torsion of the embedding. We have examined the geometrical role played by torsion in the formalism. It is this coupling which ensures that the equations of motion transform covariantly under local normal frame rotations. As such, it plays the role of a vector potential coupling to the scalar field. The only invariant measure of the torsion is its curvature. If the curvature vanishes the torsion can be gauged away by an appropriate local rotation of the normal vectors.

If the geometry under consideration possesses a symmetry some simplification is always likely. We focused on the identification of a sufficient set of simplifying conditions without any attempt to be rigorous. We showed, however, that these conditions are realized under geometrical conditions which are sufficiently general to be useful. When, in particular, the background is de Sitter space and the defect is oriented along any number of fixed azimuthal angles, the scalar fields completely decouple.

A challenge is to formulate a consistent quantum field theory of perturbations. The renormalization of the theory will require the addition of counterterms involving extrinsic geometry.

The formalism should also prove useful for the examination of fluctuations about instantons in the semiclassical approximation to tunneling. Now the signature of both the background spacetime and the world sheet is Euclidean. One is then interested in the eigenmodes of the corresponding Euclidean operator

$$ (\tilde{\Delta}\Phi)^{(i)} - (M^2)_{(i)(j)}\Phi^{(j)} = \lambda\Phi^{(i)} \ , $$

in particular those which correspond to negative or zero eigenvalues.

The weak point in our analysis is that it fails to treat the topological defect as a source for gravity. We are currently addressing this problem in the context of domain walls. The treatment of the perturbation is, however, likely to be problematical for codimensions higher than one. As such, it would probably be more rewarding to examine perturbation in the context of a field theoretical model for the topological defect.

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