Analytical model of a resonant gravitational wave antenna

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We report on the development of a model for a resonant gravitational wave antenna. This model considers a resonant bar coupled with a resonant capacitive transducer connected to a dc SQUID amplifier, i.e., a system with three coupled oscillators. The model is used to derive an analytical expression of the apparatus sensitivity in terms of an effective temperature for "burst" signals. This leads to new and more general mechanical and electrical matching conditions for improving the sensitivity: when they are satisfied the Giffard limit is obtained.

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I. INTRODUCTION

The fundamental problem of gravitational wave research is to measure the metric tensor variations with a magnitude of less than 10^{-21} m; this is, in fact, the sensitivity necessary to detect gravitational waves coming from the Virgo cluster, where we expect a rate of several tens of star collapses per year [1].

It is possible, classically, to relate the energy ϵ deposited into a resonant bar detector by a gravitational wave burst to the component of the Fourier transform of the metric tensor perturbation at the resonance frequency ν_0 of the bar [2]

$$H(2\pi\nu_0)=\frac{L}{v^2}\left[\frac{\epsilon}{M}\right]^{1/2}, \quad (1)$$

where L is the length of the cylindrical bar with mass M, and $v=5400$ m/s represents the sound velocity in the bar. The minimum measurable energy ϵ_0, with $S/N=1$, is usually expressed in terms of effective temperature T_{eff}:

$$\epsilon_0=kT_{\text{eff}} \quad (2)$$

(k is the Boltzmann constant). The value of the effective temperature depends on the detector features [3] and on the algorithms used for filtering the data.

The analysis of a two-mode detector shows that the minimum value of T_{eff} is obtained only if certain matching conditions are satisfied [3,4]. However, when a detector using a resonant capacitive transducer is coupled to a dc superconducting quantum interference device (SQUID) amplifier, there is an additional mode due to the electrical circuit [5] and the results derived for a two-mode detector are no longer valid [6].

We propose here a three-mode model for a gravitational wave detector, similar to that used in Ref. [5], to analyze a number of issues related to detector sensitivity. From this we derive, using a modal approach, the expressions both of the effective noise temperature for short burst detection and of the matching conditions required for obtaining $T_{\text{eff}}=2T_n$, where T_n is the noise temperature of the amplifier [7]. This analytical approach, that complements the numerical analysis of the detectors [8–10], has the advantage of providing a better insight as regards the influence of the various parameters of a detector on its performance.

For simplicity, in our analysis we shall consider signals due to a gravitational wave burst, ignoring its shape, and we shall assume only that the duration of the burst be smaller than the smallest time constant of the detector [6].

II. THE MODEL

The operation of the cryogenic gravitational wave antennas, equipped with a capacitive transducer and SQUID amplifier, such as the detectors of the Rome group [11–14], is based on the interaction of three oscillators.

The first resonator is the cylindrical bar whose mass determines the energy cross section for an incoming gravitational wave. Taking into account its distributed nature, the bar is described by a partial differential equation, but near its fundamental longitudinal vibration mode it can be represented as an equivalent harmonic oscillator, with angular resonance frequency $\omega_b=\pi v/L$, merit factor Q_b, and equivalent mass $m_b=M/2$.

The displacement $x(t)$ of the bar face is sensed by means of a resonant capacitive transducer [15] that here we consider ideal, i.e., without electrical losses and with parallel displacement $y(t)$ of the vibrating plate [16] (Fig. 1). The resonant capacitive transducer is the second mechanical oscillator, with angular resonance frequency ω_t (usually close to ω_b for improving the energy transfer from the first oscillator), merit factor Q_t and equivalent mass m_t.

The transducer is connected to a superconducting transformer that provides the required impedance matching, whose secondary is connected to the input circuit of the d.c. SQUID amplifier [5], as shown in Fig. 1. The resonant electrical circuit, i.e., the third oscillator, arises from the inductance of the transformer and the capacity of the transducer.

From the fundamental equations of motion (see later), using the Maxwell analogy, it is possible to obtain an electrical model of the apparatus [3] where the mechani-

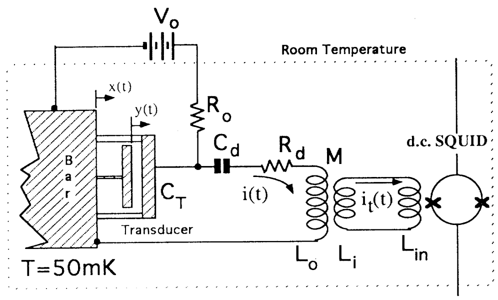

FIG. 1. Scheme of the experimental apparatus.

cal oscillators (bar and transducer) are represented by two electrical loops as reported on the left side of the scheme of Fig. 2. The bias electrical field E_0 (V/m) of the transducer is the parameter that converts the mechanical signal into the electrical signal. The decoupling capacity C_d, the superconducting transformer and the input network of the d.c. SQUID in Fig. 1 are reported in Fig. 2. In the scheme we have also represented the noise sources of the d.c. SQUID [17,18]: the voltage noise generator e_n and the current noise generator i_n, which we assume uncorrelated; with the gravitational signal source F_A.

We list below the frequently used symbols:

m_b = equivalent mass of the bar ,

β_b = dissipation coefficient of the bar ,

ω_b = angular resonance frequency of the bar ,

C_t = capacity of the transducer ,

m_t = equivalent mass of the transducer ,

β_t = dissipation coefficient of the transducer ,

ω_t = angular resonance frequency of the transducer ,

C_d = decoupling capacity ,

R_d = resistance of electrical circuit ,

L_0 = primary inductance of the transformer ,

M = mutual inductance of the transformer ,

L_i = secondary inductance of the transformer ,

E_0 = bias electrical field of the transducer ,

$L_{\rm in}$ = input inductance of the d.c. SQUID amplifier,

F_A = force acting on the bar ,

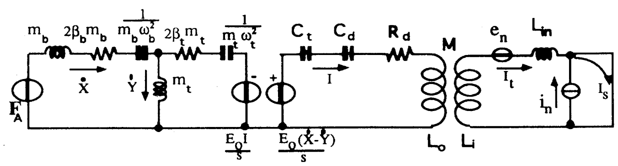

FIG. 2. Electrical model of the detector.

e_n = voltage noise generator of the amplifier ,

i_n = current noise generator of the amplifier ,

$\dot{X}$ = displacement velocity of the bar ,

$\dot{Y}$ = displacement velocity of the transducer ,

I = electrical circuit current .

The fundamental equations of motion of the system are

$$
\begin{aligned}
&-F_A+Z_1\dot{X}-Z_7(\dot{X}-\dot{Y})=0 ,\\
&-Z_7\dot{X}+Z_2(\dot{X}-\dot{Y})-Z_8 I=0 ,\\
&-Z_8(\dot{X}-\dot{Y})+Z_3 I+Z_4 I-Z_9 I_t=0 ,\\
&-Z_9 I+(Z_5+Z_6)I_t+e_n=0 ,\\
&i_n+I_t=I_s ,
\end{aligned} \tag{3}
$$

where the expressions of the parameters Z are given by

$$
\begin{aligned}
&Z_1=2\beta_b m_b+sm_b\left[1+\mu+\frac{\omega_b^2}{s^2}\right] ,\\
&Z_2=2\beta_t m_t+sm_t\left[1+\frac{\omega_t^2}{s^2}\right] ,\\
&Z_3=\frac{1}{S}\left[\frac{1}{C_t}+\frac{1}{C_d}\right], \quad Z_4=R_d+sL_0 ,\\
&Z_5=sL_i, \quad Z_6=sL_{\rm in} ,\\
&Z_7=sm_t, \quad Z_8=E_0/s ,\\
&Z_9=sM, \quad \mu=m_t/m_b .
\end{aligned} \tag{4}
$$

The basic dynamic properties of the detector can be derived from the determinant of the system matrix. If we assume $e_n=i_n=0$, as when we are interested only in the signal, by substituting the fourth in the third one of Eqs. (3) we get

$$
I_t=I_s, \quad I_t=\frac{Z_9}{Z_5+Z_6}I \tag{5}
$$

and the equations of motion result:

$$
\begin{aligned}
&Z_1\dot{X}-Z_7(\dot{X}-\dot{Y})=F_A ,\\
&-Z_7\dot{X}+Z_2(\dot{X}-\dot{Y})-Z_8 I=0 ,\\
&-Z_8(\dot{X}-\dot{Y})+\left[Z_3+Z_4-\frac{Z_9^2}{Z_5+Z_6}\right]I=0 .
\end{aligned} \tag{6}
$$

We observe that Eqs. (3) and (6) contain the same information on the detector response to the gravitational wave signal.

From Eqs. (6) we derive the system matrix M:

$$
M=\begin{vmatrix} Z_1 & -Z_7 & 0 \\ -Z_7 & Z_2 & -Z_8 \\ 0 & -Z_8 & Z_3+\hat{Z}_4 \end{vmatrix} , \tag{7}
$$

where

$$\hat{Z}_4 = Z_4 - \frac{Z_9^2}{Z_5+Z_6} = R_d + sL_0 - \frac{s^2M^2}{s(L_i+L_{\rm in})}$$

$$= R_d + s\xi L_0 = R_d + s\hat{L}_0 \; , \qquad (8)$$

with

$$\xi = 1 + k^2\frac{L_i}{L_i+L_{\rm in}}, \quad \hat{L}_0 = \xi L_0 \; .$$

III. MODES, TRANSFER FUNCTION, AND ENERGY COUPLING FACTORS

We compute in what follows the modes of the detector neglecting the losses of the oscillators.

We derive the analytical expression both of the resonance frequencies of modes and of the energy coupling factors. The determinant of the system matrix M (7) is

$$|M| = Z_1[Z_2(Z_3+\hat{Z}_4) - Z_8^2] - Z_7^2(Z_3+\hat{Z}_4) \; . \qquad (9)$$

By neglecting the terms with the dissipation coefficients in the parameter expressions (4) we have

$$|M| = \frac{m_b m_t \hat{L}_0}{s^3}\left[[(1+\mu)s^2+\omega_b^2]\left[(s^2+\omega_t^2)(s^2+\omega_e^2) - \frac{E_0^2}{m_t\hat{L}_0}\right] - s^4\mu(s^2+\omega_e^2)\right] ,$$

where $\omega_e = 1/\sqrt{\hat{L}_0 C_t}$ is the angular electrical resonance frequency (we neglect the term with C_d since, in the cases of interest, $C_d \gg C_t$). We notice that the expression of the system matrix determinant can be rewritten as

$$|M| = \frac{m_b m_t \hat{L}_0 P(s)}{s^3} \; , \qquad (10)$$

where

$$P(s) = s^6 + As^4 + Bs^2 + C$$
$$= (s^2+\omega_-^2)(s^2+\omega_+^2)(s^2+\omega_0^2)$$

and ω_-, ω_+, and ω_0 are the angular resonance frequencies of the modes: minus, plus, and zero mode, respectively. They will be indicated as ω_i using the symbolic index $i = \{+,-,0\}$. From (10) results

$$A = \omega_e^2 + \omega_b^2 + (1+\mu)\omega_t^2 \; ,$$

$$B = \omega_t^2\omega_e^2(1+\mu) + \omega_b^2(\omega_t^2+\omega_e^2) - \frac{(1+\mu)E_0^2}{m_t\hat{L}_0} \; , \qquad (11)$$

$$C = \omega_b^2\omega_t^2\omega_e^2 - \frac{\omega_b^2 E_0^2}{m_t\hat{L}_0} \; .$$

The expressions of the angular resonance frequencies of the modes as a function of the coefficients (11) can be derived from the determinant of matrix M: i.e.,

$$-\omega^6 + A\omega^4 - B\omega^4 + C = 0 \; . \qquad (12)$$

The roots of this polynomial are obtained as follows:

$$\omega_k^2 = 2\sqrt{-\sigma}\cos\left[\frac{\theta+2k\pi}{3}\right] + \frac{A}{3}, \quad k = 0,1,2 \; , \qquad (13)$$

with

$$\sigma = \frac{3B - A^2}{9}, \quad \tau = \frac{2A^3+27C-9AB}{54} \; ,$$

$$\cos\theta = \frac{\tau}{\sqrt{-\sigma^3}} \; .$$

In Eq. (13) for the angular resonance frequencies of modes we have used the numerical index k instead of the symbolic index i, so it is possible to write their analytical expressions in a much closer form.

Unlike the case of two mechanical oscillators, here there is an important difference concerning the role of the electrical field E_0 [Eq. (11)]. While with two resonators and with the transducer connected to a field effect transistor (FET) amplifier, the electrical field E_0 modifies only the uncoupled resonance frequency of the transducer (in accordance with the relation $\omega_t'^2 = \omega_t^2 - E_0^2 C_t/m_t$) [3,16],

We consider now a specific example with the numerical values of the parameters given in Table I.

In Fig. 3 we report the frequencies of modes $\nu_i = \omega_i/2\pi$, $i = \{+,-,0\}$, versus the ratio ν_e/ν_b, and in Fig. 4 the frequencies of the modes as a function of the electrical field E_0 in the case of "tuning," i.e., for $\nu_e = \nu_b = \nu_t = 915$ Hz.

Let us analyze now the detector response when a gravitational wave pulse $f(t) = f_0\delta(t)$ [$\delta(t)$ is the Dirac δ function], i.e., $F_A = f_0$, impinges onto the bar. According to Eqs. (6), neglecting the losses of the oscillators, we derive the transfer function $W(s)$ between the gravitational wave force F_A, and the input current I_s of the d.c. SQUID:

$$W(s) = I_s/F_A \; .$$

If we take into account Eq. (5) we have

TABLE I. Numerical values for parameters used in Figs. 3 and 4.

$m_b = 1160$ kg
$\nu_b = 915$ Hz
$m_t = 0.35$ kg
$\nu_t = 915$ Hz
$C_t = 4$ nF
$L_{\rm in} = 1.5\ \mu$H
$L_i = 1.5\ \mu$H
$k = 0.7$

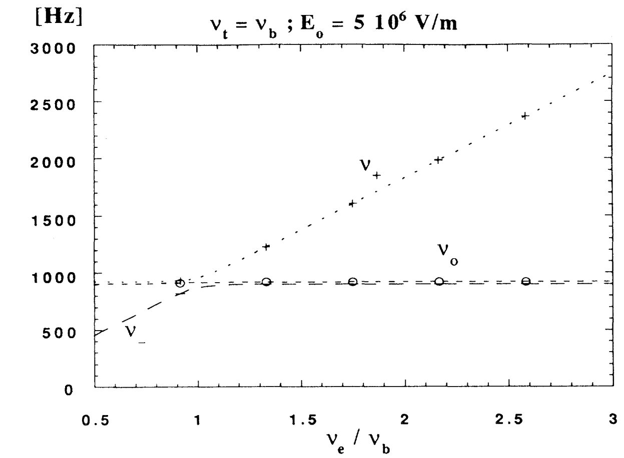

FIG. 3. Resonance frequencies of modes versus ν_e/ν_b.

$$I_s=\frac{Z_9}{Z_5+Z_6}I=\frac{M}{L_i+L_{\rm in}}I$$

and the expression of the current I results:

$$I=\frac{F_A Z_7 Z_8}{|M|} .$$

By using the expression of the system matrix determinant (10) we have

$$W(s)=\frac{I_s}{F_A}=\frac{I_s}{I}\frac{I}{F_A}=\frac{E_0}{m_b\hat{L}_0}\frac{M}{L_i+L_{\rm in}}\frac{s^3}{P(s)} . \quad (14)$$

By performing the inverse Laplace transform we obtain [6]

$$i_s(t)=-\frac{E_0 f_0}{m_b\hat{L}_0 D}\frac{M}{L_i+L_{\rm in}}[\omega_0^2(\omega_+^2-\omega_-^2)\cos(\omega_0 t) +\omega_+^2(\omega_-^2-\omega_0^2)\cos(\omega_+ t) +\omega_-^2(\omega_0^2-\omega_+^2)\cos(\omega_- t)] , \quad (15)$$

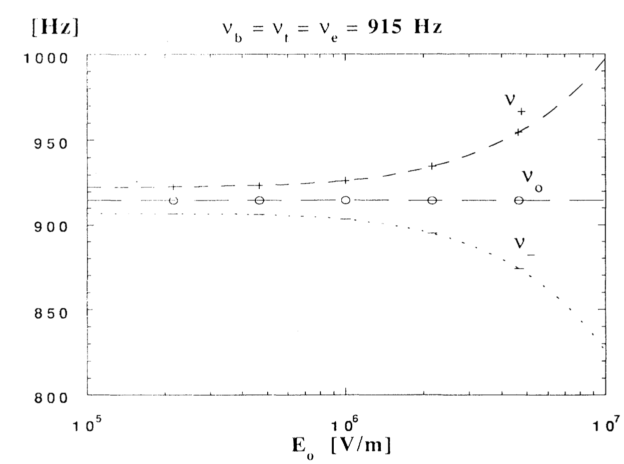

FIG. 4. Resonance frequencies of modes versus E_0.

where

$$D=\omega_0^4(\omega_+^2-\omega_-^2)+\omega_+^4(\omega_-^4-\omega_0^2)+\omega_-^4(\omega_0^2-\omega_+^2) . \quad (16)$$

We notice that a gravitational wave pulse excites the modes whose current intensity are given by

$$I_{s_i}=-\frac{E_0 f_0}{m_b\xi L_0}\frac{M}{L_i+L_{\rm in}}\frac{\omega_i^2(\omega_j^2-\omega_k^2)}{D} , \qquad i,j,k=\{+,-,0\} , \quad (17)$$

where[1] $j=\text{suc}(i)$ and $k=\text{suc}(j)$.

We introduce now the energy coupling factors β_{L_i} of the modes, with $i=\{+,-,0\}$, which are defined as the ratios of the energy at the input of the d.c. SQUID amplifier for the ith mode E_{L_i} to the total energy E_s:

$$\beta_{L_i}=E_{L_i}/E_s . \quad (18)$$

The energy released to the bar by a gravitational radiation pulse is

$$E_s=f_0^2/2m_b ,$$

where f_0 is the pulse magnitude. The energy E_{L_i} stored in the input inductance $L_{\rm in}$ of the d.c. SQUID at the ith mode, is given by

$$E_{L_i}=\tfrac{1}{2}L_{\rm in}I_{s_i}^2 . \quad (19)$$

Substituting the expression (17) of I_{s_i}, we get

$$\beta_{L_i}=\frac{E_0^2}{m_b L_0}\frac{L_i L_{\rm in}k^2}{[L_{\rm in}+L_i(1-k^2)]^2}\frac{\omega_i^4(\omega_j^2-\omega_k^2)^2}{D^2} , \quad (20)$$

where we have used the relation $M=k\sqrt{L_0 L_i}$ and where k is the transformer coupling factor.

In Fig. 5 we plot the β_{L_i} factors versus the ratio ν_e/ν_b in the "tuning" case (see Table I for the parameter values used). The figure shows that for values of the electrical resonance frequency greater than the mechanical frequency of the bar ν_b, the plus mode is heavily attenuated (about all the energy is available at the minus and zero modes), whereas for values of ν_e smaller than ν_b is the minus mode to be attenuated. We notice that the central mode is always privileged with respect to the other ones, independently by the ratio value ν_e/ν_b. In Fig. 6 we report the β_{L_i} factors versus the electrical field E_0 in the tuning case. The plot shows that, for the values of the parameters we have used (see Table I), the maximum energy transfer for all the modes is obtained with $E_0\simeq 10^6$ (V/m).

[1]We define the function $i=\text{suc}(j)$ as follows: let i and j be two symbolic indices with n elements $\{a_1,\ldots,a_n\}$, if $j=a_k$ (the kth symbol) then

$$i=\begin{cases} a_{k+1} & \text{for } k<n, \\ a_1 & \text{for } k=n . \end{cases}$$

For example, if $i=j=\{+,-,0\}$, then suc(+)=−, suc(0)=+.

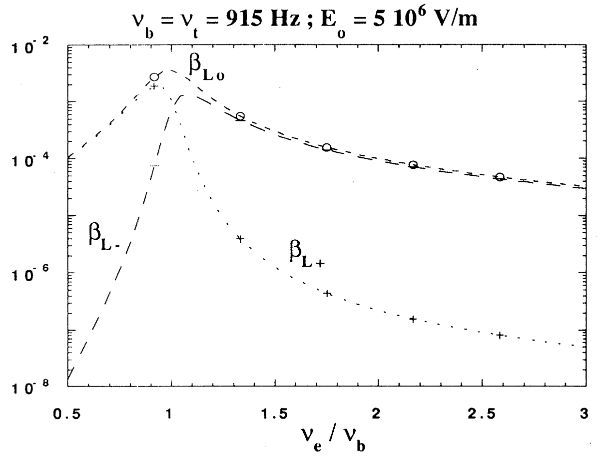

FIG. 5. Energy coupling factors of modes versus ν_e/ν_b.

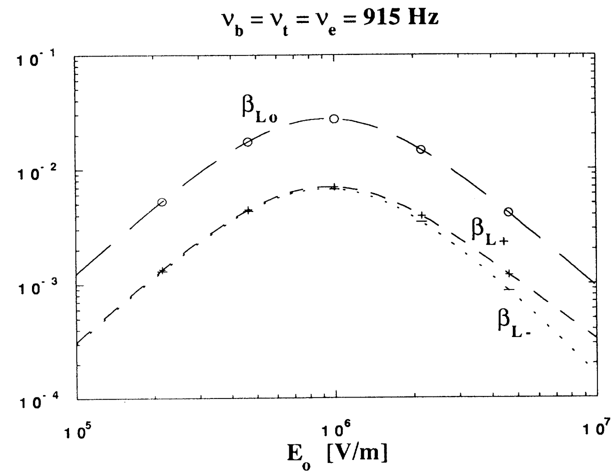

FIG. 6. Energy coupling factors of modes versus E_0.

IV. QUALITY FACTORS OF MODES

In this section we shall derive the merit factor Q_i expressions of the modes using their usual definition, $Q_i=\omega_i/\Delta\omega_i$, where ω_i is the angular resonance frequency of the mode and $\Delta\omega_i$ is the -3 dB bandwidth. The analytical expression of Q_i will be derived by expanding in series the transfer function $W(s)$, that was introduced in the previous paragraph, around the angular resonance frequency of ith mode [19].

Introducing the losses of the oscillators and using Eqs. (4), the expression of the system matrix results:

$$|M|=\left[2\beta_b m_b+sm_b\left[1+\mu+\frac{\omega_b^2}{s^2}\right]\right]\left\{\left[2\beta_t m_t+sm_t\left[1+\frac{\omega_t^2}{s^2}\right]\right]\left[R_d+\frac{1}{sC_t}+s\hat{L}_0\right]-\frac{E_0^2}{s^2}\right\}-s^2m_t^2\left[R_d+\frac{1}{sC_t}+s\hat{L}_0\right],$$

where we have neglected the resistor R_0, since in the cases of interest $R_0 \gg \omega_i L_0$. From this expression we get

$$|M|=\frac{m_b m_t \hat{L}_0 P(s)}{s^3}, \tag{21}$$

where $P(s)$ is the polynomial

$$P(s)=\sum_{i=0}^{6} a_i s^i=a_0+a_1 s+\cdots+a_6 s^6, \tag{22}$$

the coefficients a_i are

$$\begin{aligned}
a_0&=\omega_e^2\omega_t^2\omega_b^2-\frac{\omega_b^2E_0^2}{m_t\hat{L}_0},\\
a_1&=2\beta_t\omega_e^2\omega_b^2+2\beta_b\omega_e^2\omega_t^2-2\beta_b\frac{E_0^2}{m_t\hat{L}_0}+2\beta_e\omega_t^2\omega_b^2,\\
a_2&=\omega_b^2\omega_t^2+4\beta_b\beta_t\omega_e^2+\omega_e^2\omega_b^2+\omega_e^2\omega_t^2(1+\mu)-\frac{(1+\mu)E_0^2}{m_t\hat{L}_0}\\
&\quad+4\beta_e\beta_t\omega_b^2+4\beta_e\beta_b\omega_t^2,\\
a_3&=2\beta_t\omega_b^2+2\beta_b\omega_t^2+2\beta_t\omega_e^2(1+\mu)+2\beta_b\omega_e^2+8\beta_e\beta_t\beta_b\\
&\quad+2\beta_e\omega_b^2+2\beta_e\omega_t^2(1+\mu),\\
a_4&=4\beta_b\beta_t+\omega_b^2+\omega_t^2(1+\mu)+\omega_e^2\\
&\quad+4\beta_e\beta_t(1+\mu)+4\beta_e\beta_b,\\
a_5&=2\beta_t(1+\mu)+2\beta_b+2\beta_e,\\
a_6&=1,
\end{aligned}\tag{23}$$

and where the electrical dissipation coefficient $\beta_e=\omega_e/2Q_e=R_d/2\hat{L}_0$ has been introduced.

Setting $s=j\omega$, $P(s)$ can be written as

$$P(j\omega)=A(\omega)+j\omega B(\omega), \tag{24}$$

where $A(\omega)$ and $B(\omega)$ are given by

$$\begin{aligned}
A(\omega)&=-(\omega^2-\omega_-^2)(\omega^2-\omega_+^2)(\omega^2-\omega_0^2),\\
B(\omega)&=a_5(\omega^2-\omega_1^2)(\omega^2-\omega_2^2)
\end{aligned}\tag{25}$$

with

$$\omega_{1,2}^2=\frac{a_3\pm\sqrt{a_3^2-4a_1a_5}}{2a_5}. \tag{26}$$

The angular resonance frequencies of the modes ω_i in presence of dissipations are now given by the roots of $A(\omega)=0$. By comparing a_0 with C, a_2 with B, and a_4 with A, i.e., with the lossless case, we deduce that these coefficients are identical except for the β^2 terms. Thus we neglect the difference under the condition that $4\beta_{\max}^2\omega_{\max}^2 \ll \omega_{\min}^4$, where $\beta_{\max}$ is the largest among β_b, β_e, β_t, and $\omega_{\max}$ ($\omega_{\min}$) the largest (smallest) among ω_b, ω_t, and ω_e.[2] Usually the above condition is satisfied, since $Q_e > 10^4$, thus we can evaluate the angular resonance frequencies of the modes using the expressions (13) found in absence of dissipations.

Now let us calculate the expressions of the merit factors Q_i: from Eq. (14),

$$W(s)=\frac{E_0}{m_b\hat{L}_0}\frac{M}{L_i+L_{\rm in}}\frac{s^2}{P(s)}\ ;$$

by substituting $s=j\omega$, we get

$$|W(j\omega)|^2=\left[\frac{E_0}{m_b\hat{L}_0}\frac{M}{L_i+L_{\rm in}}\right]^2\frac{\omega^6}{A^2(\omega)+\omega^2B^2(\omega)}\ . \tag{27}$$

At the angular resonance frequency of ith mode ω_i [where $A(\omega_i)=0$], expression (27) becomes

$$|W(j\omega)|^2_{\omega=\omega_i}=\left[\frac{E_0}{m_b\hat{L}_0}\frac{M}{L_i+L_{\rm in}}\right]^2\frac{\omega_i^4}{B^2(\omega_i)}\ . \tag{28}$$

From the above, we obtain the squared modulus of the normalized transfer function as

$$W_{N_i}^2(\omega)=\left[\frac{|W(j\omega)|}{|W(j\omega)|_{\omega=\omega_i}}\right]^2=\frac{\omega^6}{\omega_i^4}\frac{B^2(\omega_i)}{A^2(\omega)+\omega^2B^2(\omega)}$$
$$=\left[\frac{\omega}{\omega_i}\right]^4\left[\frac{B(\omega_i)}{B(\omega)}\right]^2\frac{1}{1+G^2(\omega)}\ , \tag{29}$$

where $G(\omega)$ is given by

$$G(\omega)=\frac{A(\omega)}{\omega B(\omega)}=-\frac{(\omega^2-\omega_-^2)(\omega^2-\omega_+^2)(\omega^2-\omega_0^2)}{\omega a_5(\omega^2-\omega_1^2)(\omega^2-\omega_2^2)} \tag{30}$$

and

$$\frac{B(\omega_i)}{B(\omega)}=\frac{(\omega_i^2-\omega_1^2)(\omega_i^2-\omega_2^2)}{(\omega^2-\omega_1^2)(\omega^2-\omega_2^2)}\ .$$

By assuming $\omega_1^2=\omega_i^2(1+\delta_1)$, $\omega_2^2=\omega_i^2(1+\delta_2)$, and expanding in series at the first order around ω_i, we obtain

$$\left[\frac{\omega}{\omega_i}\right]^4\simeq 1+4\left[\frac{\omega-\omega_i}{\omega_i}\right]\ ,$$
$$\left[\frac{B(\omega_i)}{B(\omega)}\right]^2\simeq 1+4\frac{\delta_1+\delta_2}{\delta_1\delta_2}\left[\frac{\omega-\omega_i}{\omega_i}\right]\ . \tag{31}$$

In the bandwidth $\Delta\omega_i$ by ω_i, Eqs. (31) will be considered with good approximation equal to unity.

[2]If, for example, $\omega_e=\omega_{\max}=2\pi$ 1800 rad/s, $\omega_{\min}=\omega_b=2\pi$ 900 rad/s, $\beta_{\max}=\beta_e=\omega_e/2Q_e$, then we must have $Q_e \gg (\omega_e/\omega_b)^2\sim 4$.

In such a case the functions (29) become

$$W_{N_i}^2(\omega)\simeq\frac{1}{1+G^2(\omega)}\ . \tag{32}$$

It can be shown [6] that on the condition that Q_i be much greater than the square root of the smallest among the uncoupled merit factors ($Q_i \gg \sqrt{Q_{\min}}$), with good approximation, it is possible to expand $G(\omega)$ around ω_i, so that (32) becomes

$$W_{N_i}^2(\omega)\approx\frac{1}{1+k_{1_i}^2(\omega-\omega_i)^2}\ , \tag{33}$$

with

$$k_{1_i}=\left.\frac{dG(\omega)}{d\omega}\right|_{\omega=\omega_i}=\frac{2Q_i}{\omega_i}\ . \tag{34}$$

From Eqs. (30) and (34) we obtain

$$\frac{2Q_+}{\omega_+}=\left|\frac{-2}{a_5}\frac{(\omega_+^2-\omega_-^2)(\omega_+^2-\omega_0^2)}{(\omega_+^2-\omega_1^2)(\omega_+^2-\omega_2^2)}\right|\ ,$$
$$\frac{2Q_-}{\omega_-}=\left|\frac{-2}{a_5}\frac{(\omega_-^2-\omega_+^2)(\omega_-^2-\omega_0^2)}{(\omega_-^2-\omega_1^2)(\omega_-^2-\omega_2^2)}\right|\ , \tag{35}$$
$$\frac{2Q_0}{\omega_0}=\left|\frac{-2}{a_5}\frac{(\omega_0^2-\omega_+^2)(\omega_0^2-\omega_-^2)}{(\omega_0^2-\omega_1^2)(\omega_0^2-\omega_2^2)}\right|\ ,$$

or, in a much closer form, we rewrite [19]

$$Q_i=\frac{\omega_i}{a_5}=\left|\frac{(\omega_i^2-\omega_j^2)(\omega_i^2-\omega_k^2)}{(\omega_i^2-\omega_1^2)(\omega_i^2-\omega_2^2)}\right|\ ,\qquad i,j,k=\{+,-,0\} \tag{36}$$

in which i is the symbolic index of the mode, $j=\mathrm{suc}(i)$, and $k=\mathrm{suc}(j)$.

We recall that such a procedure for calculating the merit factors is valid only if

$$Q_i \gg \sqrt{Q_{\min}}\ , \tag{37}$$

where $Q_{\min}$ is the smallest among the uncoupled merit factors. We notice also that Eq. (36) depends on the frequencies of the modes ω_i, on ω_1 and ω_2 and, finally, on the coefficient a_5.

The merit factors of the three modes are reported in Fig. 7 versus the ratio ν_e/ν_b, using the values of the parameters given in Table II. For ν_e greater than ν_b, Q_+ becomes equal to Q_e, whereas Q_- and Q_0 tend to the same limit value. We notice that for $\nu_e > 2\nu_b$ or $\nu_e < \nu_b/2$ (see also Figs. 3 and 5) the electrical oscillator

TABLE II. Typical parameter values of a resonant gravitational wave detector.

$m_b=1160$ kg	$Q_b=10^7$
$\nu_b=915$ Hz	$m_t=0.35$ kg
$\nu_t=915$ Hz	$Q_t=10^7$
$C_t=4$ nF	$L_{\rm in}=1.5\ \mu$H
$L_i=1.5\ \mu$H	$k=0.77$
$Q_e=10^4$	$E_0=5\times10^6$ (V/m)

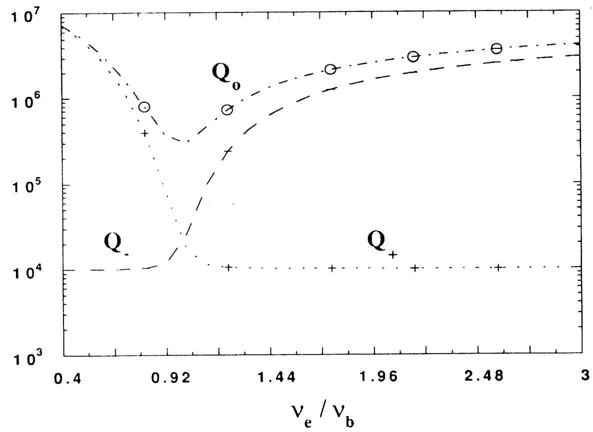

FIG. 7. Merit factors of modes versus ν_e/ν_b.

is practically uncoupled and the system behavior is the same as the simpler case of two coupled mechanical oscillators. The merit factors of modes as a function of the electrical field E_0, are shown in Fig. 8, using the same parameters of Fig. 7.

By using the above results we derive now a relation between the factors β_{L_i} and Q_i that we shall use in the next paragraph for computing the equivalent temperature of the modes. To this aim we observe that in the expression (20) of β_{L_i} there is the quantity

$$\frac{(\omega_j^2-\omega_k^2)^2}{D^2} . \tag{38}$$

This can be properly transformed if we take account that [Eq. (16)]

$$D=\omega_0^4(\omega_+^2-\omega_-^2)+\omega_+^4(\omega_-^2-\omega_0^2)+\omega_-^4(\omega_0^2-\omega_+^2)$$

or

$$D=(\omega_-^2-\omega_0^2)(\omega_+^2-\omega_-^2)(\omega_+^2-\omega_0^2) .$$

In such a case (38) becomes

$$\frac{(\omega_j^2-\omega_k^2)^2}{D^2}=\frac{1}{[(\omega_i^2-\omega_j^2)(\omega_i^2-\omega_k^2)]^2} \tag{39}$$

and substituting Eq. (39) in the expression of β_{L_i}, we have

$$\beta_{L_i}=\frac{E_0^2}{m_b L_0}\frac{L_i L_{\text{in}} k^2}{[L_{\text{in}}+L_i(1-k^2)]^2}\frac{\omega_i^4}{[(\omega_i^2-\omega_j^2)(\omega_i^2-\omega_k^2)]^2} . \tag{40}$$

From Eq. (36) and from the expression of $B(\omega)$ in (26), we get

$$|(\omega_i^2-\omega_j^2)(\omega_i^2-\omega_k^2)|=\frac{a_5|(\omega_i^2-\omega_1^2)(\omega_i^2-\omega_2^2)|Q_i}{\omega_i}=\frac{B_i Q_i}{\omega_i} , \tag{41}$$

where we have indicated $B_i=B(\omega_i)$.

Finally, substituting Eq. (41) in Eq. (40), we obtain the desired relation

$$\beta_{L_i}=\frac{E_0^2}{m_b L_0}\frac{L_i L_{\text{in}} k^2}{[L_{\text{in}}+L_i(1-k^2)]^2}\frac{\omega_i^6}{B_i^2 Q_i^2} . \tag{42}$$

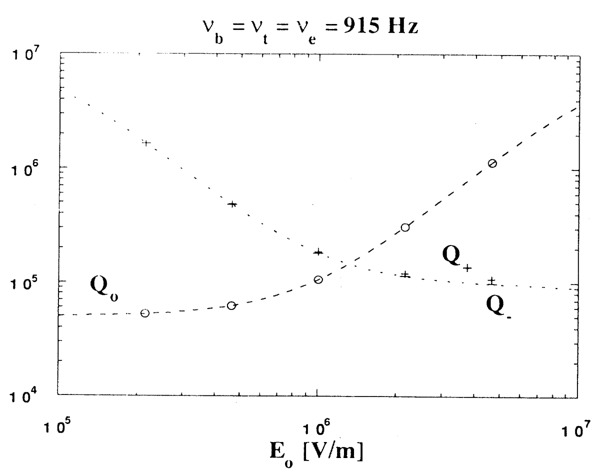

FIG. 8. Merit factors of modes versus E_0.

V. EQUIVALENT TEMPERATURE, BACK ACTION AND MODE BANDWIDTHS

The noise sources of the d.c. SQUID e_n and i_n excite the bar through the transducer as shown in Fig. 2. This effect of *back action* is taken into account in the definition of the equivalent temperature of the modes: we imagine replacing the "true" system, the bar at temperature T, and the noise sources of the d.c. SQUID, with a "virtual" system made up by the bar alone at the equivalent temperature T_{eq} without the noise sources.

If we suppose the detector be at thermodynamical equilibrium, i.e., at constant temperature in all the experimental space, then the thermal noise of both the resistor R_d and of the transducer do not give any back action contribution [20,21], because they cannot exchange energy with the bar, as established by the thermodynamics' laws.

We can prove that the mean square displacement velocity at the ith mode of the resonant bar face at temperature T, when it is loaded with a resonant capacitive transducer and with an electrical circuit, can be written as

$$\sigma_{\dot{x}_T}^2=\frac{k}{m_b}\gamma_i T , \tag{43}$$

where the factor γ_i, $i=\{+,-,0\}$, takes into account the features at the ith mode both of the resonant transducer and of the electrical circuit (see later). In the case of the bar alone, we have only its longitudinal vibration mode at the frequency ν_0, and the factor γ_i is, naturally, $\gamma_0=1$.

The equivalent temperature T_{eq} is defined so that the equivalent resultant mean-square displacement velocity ($\langle \dot{x}_{\text{eq}}^2 \rangle$) of the bar is equal to that obtained adding the thermodynamical and back action effects.

According to Eq. (43) we write

$$\sigma^2_{\dot{x}_{eq}}=\frac{k}{m_b}\gamma_i T_{eq} . \quad (44)$$

For this reason T_{eq} is always greater or equal to T, and is equal only in the ideal case. The back action spectrum of the d.c. SQUID amplifier is due only to e_n and is given by [22]

$$S_{\dot{x}}=\frac{E_0^2}{m_b^2 L_0}\frac{L_i k^2}{[L_{in}+L_i(1-k^2)]^2}\frac{\omega^6}{A^2(\omega)+\omega^2 B^2(\omega)}e_n^2 , \quad (45)$$

where all spectrums are considered single sided.

The mean-square value of $\dot{x}$ due to back action [22] is

$$\sigma^2_{\dot{x}}=\frac{1}{2\pi}\int_0^{+\infty}S_{\dot{x}}(\omega)d\omega . \quad (46)$$

Introducing a summation on the modes in the expression of $S_{\dot{x}}$, we get

$$S_{\dot{x}}(\omega)\simeq\sum_i S_{\dot{x}}(\omega_i)W_i(\omega) , \quad (47)$$

where the functions $W_i(\omega)$, with $W_i(\omega_i)=1$, are such to approximate $S_{\dot{x}}$ around the resonance of the ith mode and to vanish far from the mode. From Eq. (46), substituting Eq. (47), we obtain

$$\sigma^2_{\dot{x}}\simeq\frac{1}{2\pi}\int_0^{+\infty}\sum_i S_{\dot{x}_i}W_i(\omega)d\omega$$
$$=\sum_i S_{\dot{x}_i}B_{N_i}=\sum_i\sigma^2_{\dot{x}_i} , \quad (48)$$

in which $S_{\dot{x}_i}=S_{\dot{x}}(\omega_i)$, with

$$B_{N_i}=\frac{1}{2\pi}\int_0^{+\infty}W_i(\omega)d\omega \quad (49)$$

and

$$\sigma^2_{\dot{x}_i}=S_{\dot{x}_i}B_{N_i} . \quad (50)$$

To understand the effective significance of B_{N_i}, we consider the case of one mode alone.

Equation (47) becomes $S_{\dot{x}}(\omega)=S_{\dot{x}}(\omega_0)W_0(\omega)$; thus

$$B_{N_0}=\frac{1}{2\pi}\int_0^{+\infty}W_0(\omega)d\omega=\frac{1}{2\pi}\int_0^{+\infty}\frac{S_{\dot{x}}(\omega)}{S_{\dot{x}}(\omega_0)}d\omega . \quad (51)$$

From the last expression we deduce that B_{N_0} is the equivalent noise bandwidth. It is natural to consider B_{N_i} as the equivalent noise bandwidth of the ith mode.

The equivalent mean-square value of $\dot{x}$ at the mode is obtained by summing two contributions: the back action noise at the mode, $\sigma^2_{\dot{x}_i}$, and the thermodynamical noise $\sigma^2_{\dot{x}_T}$, i.e.,

$$\sigma^2_{\dot{x}_{eq}}=\sigma^2_{\dot{x}_T}+\sigma^2_{\dot{x}_i} . \quad (52)$$

From Eqs. (43) and (44) we get

$$T_{eq_i}=T+\frac{m_b\sigma^2_{\dot{x}_i}}{k\gamma_i}=T+\frac{m_b S_{\dot{x}_i}B_{N_i}}{k\gamma_i} , \quad (53)$$

where T is the thermodynamical temperature of the bar and

$$\gamma_i=\frac{2\beta_b\omega_i}{B_i^2 Q_i}\{[(\omega_i^2-b_1^2)(\omega_i^2-b_2^2)]^2+[\omega_i b_3(\omega_i^2-b_4^2)]^2\} ,$$

with

$$b_{1,2}^2=\frac{\omega_e^2+\omega_t^2\pm\left[(\omega_e^2+\omega_t^2)^2-4\left[\omega_e^2\omega_t^2-\frac{E_0^2}{m_t\hat{L}_0}\right]\right]^{1/2}}{2} ,$$

$$b_3=4\beta_e\beta_t,\quad b_4=\left[\frac{\beta_t\omega_e^2+\beta_e\omega_t^2}{2\beta_e\beta_t}\right]^{1/2} .$$

In order to relate T_{eq_i} with the parameters β_{L_i} and Q_i, we calculate the equivalent noise bandwidth of the mode:

$$B_{N_i}=\frac{1}{2\pi}\int_0^{+\infty}W_i(\omega)d\omega . \quad (54)$$

The analytical relation between the displacement velocity back action noise and its value at the ith mode is

$$S_{\dot{x}}(\omega)|_{\omega\approx\omega_i}=W^2_{N_i}(\omega)S_{\dot{x}}(\omega_i) ,$$

where the functions $W^2_{N_i}(\omega)$ were introduced in the previous paragraph. Thus we deduce

$$W_i(\omega)|_{\omega\approx\omega_i}=W^2_{N_i}(\omega)|_{\omega\approx\omega_i} .$$

The detector response, for frequencies around the ith mode, is that of a resonant second-order system with angular resonance frequency ω_i and merit factor Q_i; thus,

$$B_{N_i}=\frac{\omega_i}{4Q_i} . \quad (55)$$

By using Eq. (45) the displacement velocity back action spectrum calculated at the ith mode is given by

$$S_{\dot{x}_i}=\frac{E_0^2}{m_b^2 L_0}\frac{L_i k^2}{[L_{in}+L_i(1-k^2)]^2}\frac{\omega_i^4}{B_i^2}2kT_n R_n , \quad (56)$$

where $R_n=e_n/i_n$ is the noise resistance of the d.c. SQUID amplifier and $T_n=\sqrt{e_n^2 i_n^2}/2k$ its equivalent noise temperature.

From Eqs. (55) and (56), we have

$$\frac{T_{eq_i}}{T}=1+\frac{E_0^2}{m_b^2 L_0}\frac{L_i k^2}{[L_{in}+L_i(1-k^2)]^2}\frac{T_n}{T}\frac{m_b R_n}{2}\frac{\omega_i^5}{Q_i B_i^2\gamma_i} \quad (57)$$

and, using relation (42),

$$\frac{T_{eq_i}}{T}=1+\frac{\beta_{L_i}Q_i}{\gamma_i}\frac{T_n}{T}\frac{1}{2\lambda_0} , \quad (58)$$

where

$$\lambda_0=\omega_i L_{\text{in}}/R_n \tag{59}$$

is the ratio of the input impedance of d.c. SQUID at the ith mode to its noise resistance. We notice that λ_0 should be indicated as λ_{0_i} since it depends on the considered mode, but that will be implicit.

By means of Eq. (58) we rewrite an expression that relates T_{eq_i} with T_n:

$$\frac{T_{\text{eq}_i}}{T_n}=\frac{\beta_{L_i}Q_i}{2\lambda_0}\left[\frac{1}{\gamma_i}+\frac{2\lambda_0}{\beta_{L_i}Q_i}\frac{T}{T_n}\right] . \tag{60}$$

VI. EFFECTIVE TEMPERATURE AND MATCHING CONDITIONS

Several algorithms to detect a gravitational wave pulse signal in the presence of noise have been developed. An algorithm widely used is based on the theory of Wiener and Kolmogorov and in this work we shall use it.

In the effective temperature definition the parameter Γ defined as [23,24]

$$\Gamma=S_{nn}/S_{uu} \tag{61}$$

plays a fundamental role. S_{nn} represents the observed wide band noise spectrum, while S_{uu} is the narrow band noise spectrum; S_{nn} and S_{uu} are both output noise spectrums; therefore, Γ is dimensionless.

Indicating Z_0 as the impedance observed from the secondary of the transformer, the d.c. SQUID electrical network can be represented as in Fig. 9. In such a case the input current noise spectrum of the amplifier results:

$$S_{i_s}=i_n^2+\frac{e_n^2}{|Z_0(\omega)+Z_6(\omega)|^2}+8km_b\beta_b T|W(j\omega)|^2 , \tag{62}$$

where the square module of $W(j\omega)$ is given by Eq. (27).

The spectrum S_{i_s} includes the contributions both of the narrow band and the wideband noise of the amplifier, and of the resonant Brownian noise.

S_{uu}, i.e., the total narrow band noise spectrum, can be calculated using the following argument: the mean-square value of the displacement velocity at the ith mode due to the thermodynamical noise and back action effect is given by Eq. (52). The same noise can be thought to be due to a wideband force noise generator (N^2/Hz) with the value

$$S_{ff_i}=8kT_{\text{eq}_i}m_b\beta_b . \tag{63}$$

In such a case, the narrow band current spectrum at the input of the d.c. SQUID amplifier results:

$$S_{uu_i}=S_{ff_i}|W(j\omega)|^2_{\omega=\omega_i} . \tag{64}$$

From Eq. (28), using expression (42) we have

$$S_{uu_i}=\frac{8kT_{\text{eq}_i}\beta_{L_i}Q_i^2\beta_b}{\omega_i^2 L_{\text{in}}} . \tag{65}$$

To evaluate the wideband current noise S_{nn}, we start from the expression of S_{i_s} (62) neglecting the resonant Brownian noise term, since the thermodynamical noise of the bar gives its contribution only as resonant noise.

Handling the expression (62) and evaluating the output impedance Z_0 far enough from ω_i, the wideband current noise at the ith mode results:

$$S_{nn_i}=2k\frac{T_n}{R_n}\left[1+\left[\frac{\lambda_1}{\lambda_0}\right]^2\right] , \tag{66}$$

where λ_0 was defined in (59) and

$$\lambda_1\simeq\frac{L_{\text{in}}}{L_i+L_{\text{in}}} . \tag{67}$$

If the system has more independent resonance modes [24] we can compute the effective temperature of the apparatus from the effective temperature of each mode T_{eff_i}, using the relation

$$\frac{1}{T_{\text{eff}}}=\sum_i\frac{1}{T_{\text{eff}_i}} . \tag{68}$$

The expression of the effective temperature of the ith mode is [24,25]

$$T_{\text{eff}_i}=4T_{\text{eq}_i}\frac{\beta_t}{\beta_{L_i}}\sqrt{\Gamma_i} , \tag{69}$$

where T_{eq_i} is the equivalent temperature of the ith mode, and Γ_i is defined by (61)

$$\Gamma_i=\Gamma|_{\omega=\omega_i}=\frac{S_{nn_i}}{S_{uu_i}} . \tag{70}$$

The quantity β_t/β_{L_i} takes account of a possible unequal energy distribution at the modes, with

$$\beta_t=\sum_i\beta_{L_i} . \tag{71}$$

Using Eqs. (60), (65), and (66) we obtain from Eq. (69) the general expression of the effective temperature of the ith mode:

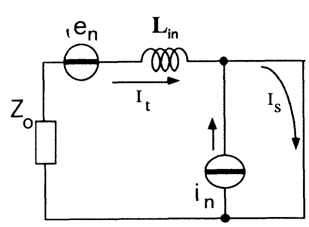

FIG. 9. Output impedance and d.c. SQUID input network.

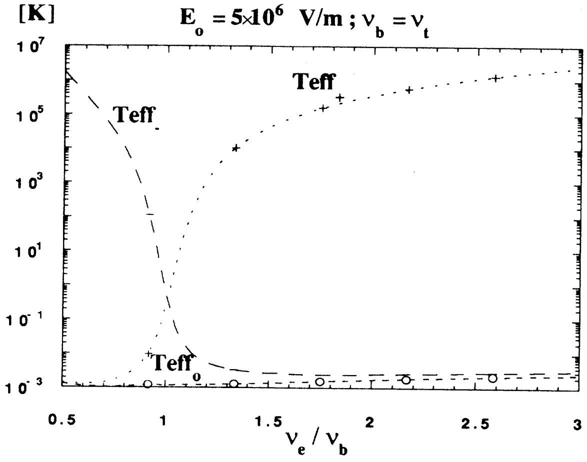

FIG. 10. Effective temperature of modes versus ν_e/ν_b.

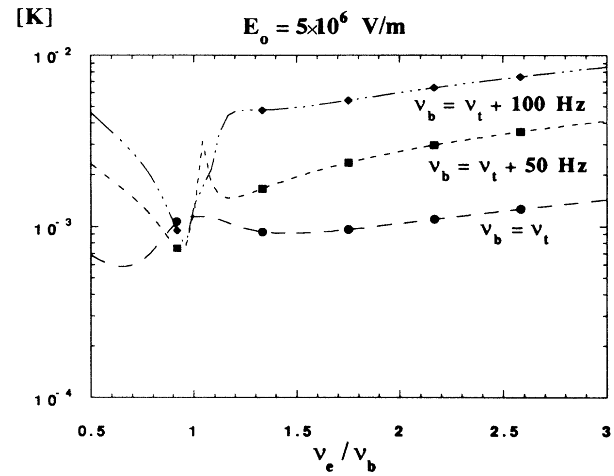

FIG. 11. Effective temperature versus ν_e/ν_b for different values of ν_t.

$$T_{\mathrm{eff}_i}=2T_n\frac{\beta_t}{\beta_{L_i}}\left\{\frac{\omega_i}{2Q_i\beta_b}\left[1+\left[\frac{\lambda_1}{\lambda_0}\right]^2\right]\left[\frac{1}{\gamma_i}+\frac{2\lambda_0}{\beta_{L_i}Q_i}\frac{T}{T_n}\right]\right\}^{1/2}. \tag{72}$$

We have plotted the effective temperature of the modes (Fig. 10) versus the ratio ν_e/ν_b (see Table II) with $\nu_b=\nu_t$. The parameter values of the d.c. SQUID used are $T_n=5$ μK, $R_n=10^{-4}$ Ω, and $T=100$ mK.

We notice that for values of the electrical resonance frequency near the mechanical frequency ($\nu_b=\nu_t=915$ Hz), the mode 0 has the best sensitivity with an effective temperature of about 1 mK against 1 K of the other two modes.

In Figs. 11 and 12 we have plotted the effective temperature of the detector [Eq. (68)] versus the ratio ν_e/ν_b for different values of the uncoupled resonance frequency of the resonant capacitive transducer ν_t.

Inspecting the plots, we conclude that the best sensitivity is obtained, generally, for $\nu_b=\nu_t$, and that, if $\nu_e>1.2\nu_b\simeq 1100$ Hz, the sensitivity does not depend on the electrical resonance frequency. This is true, obviously, for a detector with the parameter values listed in Table II.

We define the "matching conditions" as those that give

$$T_{\mathrm{eff}_i}=2T_n\frac{\beta_t}{\beta_{L_i}} \tag{73}$$

or, from Eq. (68) using Eq. (70), the minimum value of the effective temperature [7,26]

$$(T_{\mathrm{eff}})_{\min}=2T_n .$$

The matching conditions are easily derived from Eqs. (72) and (73) and are given by

$$\frac{2\lambda_0\gamma_i}{\beta_{L_i}Q_i}\frac{T}{T_n}\ll 1 \quad \text{(mechanical matching)},$$
$$\left[\frac{\lambda_1}{\lambda_0}\right]^2\ll 1 \quad \text{(electrical matching)}, \tag{74}$$

with the further condition $\omega_i/2Q_i\gamma_i\beta_b=1$ (the bandwidth matching).

Most of the efforts of the experimental groups engaged in the search of gravitational waves with resonant bars are spent trying to satisfy the matching conditions for obtaining the best sensitivity of the detector. The mechanical matching condition is more difficult to reach: high values of $\beta_{L_i}Q_i$ are necessary. This requires electrical and mechanical oscillators with large merit factors especially when the thermodynamical temperature of the detector is much greater than the noise temperature of d.c. SQUID. At present the most sensitive gravitational wave detectors in the world make use of cryogenic techniques in order to reach ultra low temperatures. With

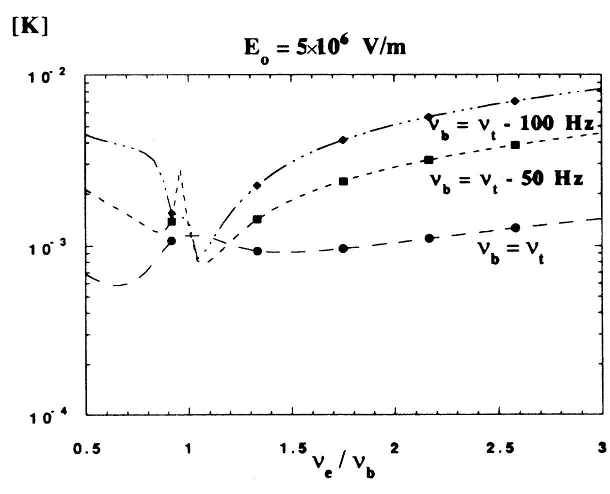

FIG. 12. Effective temperature versus ν_e/ν_b for different values of ν_t.

the third generation antennas it will be possible to have a working temperature of 50 mK [27].

The second condition in (74) is referred as the *electrical matching condition:* with the present experimental setup this condition is more easily satisfied.

VII. CONCLUSIONS

We have developed a model of a resonant gravitational wave antenna, based on the interaction of three harmonic oscillators: the bar, the resonant transducer, and the d.c. SQUID amplifier. Starting from the fundamental equations of motion of the system we have analyzed the detector response at the three normal modes when a gravitational wave burst impinges onto the bar. We have computed the resonance frequencies, the energy coupling factors, and the quality factors of the modes.

Using these results we have analyzed both the effect of the Brownian noise of the bar and of the backaction noise of the d.c. SQUID amplifier, deriving the expressions for the narrow band noise and for the wide band noise at the modes.

Finally, by using the Wiener theory, we have calculated the effective temperature of the modes, establishing new and more general matching conditions to be satisfied for reaching the optimum sensitivity.

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