D mesons in a relativistic quark model

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(Received 7 May 1993)

A relativistic quark model which was previously developed and applied to heavy-quark systems is applied to the light-heavy-quark *D* mesons. Kernels are generated for the bound-state amplitudes of unequal quark-mass systems in this model, which is essentially the Coulomb-gauge QCD Hamiltonian, augmented by a scalar linearly confining term. The model generates all the correct perturbative physics up to order α_s^4. Results are compared to experimental data.

PACS number(s): 12.40.Qq, 14.40.Jz

I. INTRODUCTION

In earlier work [1], hereafter referred to as I, we developed a relativistic quark model based on the variational method applied to quantum field theory. Although in principle one could apply the variational method to a direct solution of QCD [2] a much more modest approach was attempted. A variational ansatz was proposed that was not explicitly sensitive to the non-Abelian structure of QCD. Instead, we assumed that all the low-energy gluon modes were integrated out, leaving an effective, linearly confining potential between heavy quarks. This of course is precisely the program of Monte Carlo simulations which find the strength, form, and Lorentz structure of the long-range potential [3]. The ansatz did, however, contain transverse gluons, and as a result, all the physics of one-gluon exchange accurate to order α_s^4 was obtained.

In I we applied our model to the heavy-quark equal-mass $c\bar{c}$ and $b\bar{b}$ systems with some success. We would like to stress that a model such as this is arbitrarily accurate for arbitrarily heavy-quark masses, as the system becomes insensitive to the confining potential and sits deep in the Coulomb well, modified by small relativistic corrections. What is surprising to us is that the model also seems to give satisfactory results for the lightest of mesons [4].

In the present paper, we would like to extend this work to the light-heavy-quark systems. Previous work on heavy-quark [4–9] systems has relied on semirelativistic reductions of the Bethe-Salpeter equation or a reduction of a scattering amplitude which leads to Schrödinger-like equations. In our approach, there are no relativistic ambiguities and the variational method is of course inherently nonperturbative.

Unlike the equal-quark-mass systems, the Hamiltonian for an unequal-quark-mass system contains two distinct quark fields. In this paper, we present the integral equations in momentum space for these systems and apply them to *D* mesons. We present our Hamiltonian and ansatz in Sec. II. Also, in Sec. II, we present the integral equations and (approximately) decoupled equation for a general system. The kernels for specific quantum numbers are given in Sec. III. In Sec. IV we show our numerical results and give our conclusions.

II. HAMILTONIAN AND EQUATIONS

Our model Hamiltonian is the Hamiltonian of QCD in the Coulomb gauge augmented by a term which produces linear scalar confinement. Of course, in a true solution of the theory, one would not need this additional term as all the nonperturbative confining physics is generated by the QCD Hamiltonian alone.

As our ansatz is not explicitly sensitive to the non-Abelian terms of the Hamiltonian, our effective Hamiltonian is given by

$$H=H_q+H_g+H_c+H_{qg}+H_s \ , \tag{1}$$

where

$$H_q=\sum_{i=1}^{2}\int d^3x\,\bar{q}_i(\mathbf{x})(-i\nabla\cdot\gamma+m_i)q_i(\mathbf{x}) \ ,$$

$$H_g=\tfrac{1}{2}\int d^3x\{\,\dot{\mathbf{A}}_a^2(\mathbf{x})+[\nabla\times\mathbf{A}_a(\mathbf{x})]^2\} \ ,$$

$$H_c=\frac{\alpha_s}{2}\sum_{i=1}^{2}\int d^3x\,d^3y\,q_i^\dagger(\mathbf{x})\frac{\lambda^a}{2}q_i(\mathbf{x})\times\frac{1}{|\mathbf{x}-\mathbf{y}|}q_i^\dagger(\mathbf{y})\frac{\lambda^a}{2}q_i(\mathbf{y}) \ ,$$

$$H_{gq}=g_s\sum_{i=1}^{2}\int d^3x\,\bar{q}_i(\mathbf{x})\gamma\cdot\frac{\lambda_a}{2}\mathbf{A}_a(\mathbf{x})q_i(\mathbf{x}) \ ,$$

$$H_s=\frac{3b}{8}\sum_{i=1}^{2}\int d^3x\,d^3y\,\bar{q}_i(\mathbf{x})\frac{\lambda^a}{2}q_i(\mathbf{x})\times|\mathbf{x}-\mathbf{y}|\bar{q}_i(\mathbf{y})\frac{\lambda^a}{2}q_i(\mathbf{y}) \ ,$$

where $q_1=q$ and $q_2=Q$ are two distinct quark fields and $m_1=m$ and $m_2=M$ correspond to the masses of the two quarks, respectively. In Eq. (1), Dirac and color indices on the quark field operators are suppressed.

Our variational ansatz for unequal-mass quark systems consists of two components in Fock space:

$$|\text{meson}\rangle=|q\bar{Q}\rangle+|q\bar{Q}g\rangle \ , \tag{2}$$

where

$$|q\bar{Q}\rangle = \sum_{\substack{\sigma\delta \\ i}} \int d^3p\, F(\mathbf{p},\sigma,\delta) b_i^\dagger(\mathbf{p},\sigma) D_i^\dagger(-\mathbf{p},\delta)|0\rangle ,$$

$$|q\bar{Q}g\rangle = \sum_{\substack{s,s',\lambda \\ a,i,j}} \int d^3p\, d^3q\, G(\mathbf{p},\mathbf{q},s,s',\lambda) b_i^\dagger(\mathbf{p},s) \times D_j^\dagger(\mathbf{q},s') \frac{\lambda^a}{2} ij a_a^\dagger(-\mathbf{p}-\mathbf{q},\lambda)|0\rangle .$$

The operators $b^\dagger$, $D^\dagger$, and $a^\dagger$ are creation operators for two quarks and gluon with the momentum, color, and polarization indicated. The functions F and G are variational coefficients. The Lorentz structure of F is given by

$$F(\mathbf{p},\sigma,\delta) = f(p)\bar{u}(\mathbf{p},\sigma)\Gamma_Y V(-\mathbf{p},\delta) , \tag{3}$$

where $\bar{u}$ and V are the spinors for the two quarks. Γ_Y is a linear combination of Dirac matrices multiplied by spherical harmonic functions, which determines the desired quantum numbers. The explicit form of G is not given, as in our approximate decoupling of the generated integral equations it will be directly related to F.

Sandwiching the Hamiltonian in Eq. (1) between the ansatz Eq. (2) and using the variational principle leads to coupled integral equations for the bound systems:

$$\begin{aligned} EF(\mathbf{p}\sigma\delta) = {} & (\omega_p + \Omega_p) F(\mathbf{p}\sigma\delta) + \frac{4\alpha_s}{3}\frac{mM}{2\pi^2}\sum_{\sigma'\delta'}\int \frac{d^3q\, F(\mathbf{q}\sigma'\delta')}{\sqrt{\omega_p\omega_q\Omega_p\Omega_q}\,|\mathbf{p}-\mathbf{q}|^2}\bar{u}(\mathbf{p},\sigma)u(-\mathbf{q}\sigma')\bar{V}(-\mathbf{q}\delta')V(\mathbf{p}\delta) \\ & + \left[\frac{4\alpha_s}{3}\right]^{1/2}\frac{M}{2\pi}\sum_{\sigma'\lambda}\int \frac{d^3q}{(\Omega_p\Omega_q|\mathbf{p}-\mathbf{q}|)^{1/2}} G(\mathbf{p},-\mathbf{q}\sigma\sigma'\lambda)\bar{V}(-\mathbf{q}\sigma')\gamma\cdot\epsilon(\mathbf{q}-\mathbf{p},\lambda)V(-\mathbf{p}\delta) \\ & - \left[\frac{4\alpha_s}{3}\right]^{1/2}\frac{m}{2\pi}\sum_{\sigma'\lambda}\int \frac{d^3q}{(\omega_p\omega_q|\mathbf{p}-\mathbf{q}|)^{1/2}} G(\mathbf{q},-\mathbf{p}\sigma'\delta\lambda)\bar{u}(\mathbf{p}\sigma)\gamma\cdot\epsilon(\mathbf{p}-\mathbf{q},\lambda)u(\mathbf{q}\sigma') \\ & + \frac{bmM}{\pi^2}\sum_{\sigma'\delta'}\int \frac{d^3q F(\mathbf{q}\sigma'\delta')}{\sqrt{\omega_p\omega_q\Omega_p\Omega_q}\,|\mathbf{p}-\mathbf{q}|^4}\bar{u}(\mathbf{p}\sigma)u(\mathbf{q}\sigma')\bar{V}(-\mathbf{q}\delta')V(-\mathbf{p}\delta) \end{aligned} \tag{4}$$

and

$$\begin{aligned} EG(\mathbf{p},-\mathbf{q}ss'\lambda) = {} & [\omega_p + \Omega_p + |\mathbf{p}-\mathbf{q}|]G(\mathbf{p},-\mathbf{q}ss'\lambda) \\ & + \frac{4\alpha_s}{3}\frac{mM}{2\pi^2}\sum_{\sigma\sigma'}\int \frac{d^3k\, d^3k'}{(\omega_p\Omega_q\omega_k\Omega_{k'})^{1/2}} G(\mathbf{k},-\mathbf{k}'\sigma\sigma'\lambda)\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k}+\mathbf{k}')\frac{\bar{u}(\mathbf{p}s)u(-\mathbf{k}\sigma)\bar{V}(-\mathbf{k}'\sigma')V(\mathbf{q}s')}{|\mathbf{p}-\mathbf{k}|^2} \\ & + \left[\frac{4\alpha_s}{3}\right]^{1/2}\frac{M}{2\pi}\sum_{\sigma}\frac{1}{(\Omega_p\Omega_q|\mathbf{p}-\mathbf{q}|)^{1/2}}F(\mathbf{p}s\sigma)\bar{V}(-\mathbf{p}\sigma)\gamma\cdot\epsilon(\mathbf{q}-\mathbf{p},\lambda)V(-\mathbf{q}s') \\ & - \left[\frac{4\alpha_s}{3}\right]^{1/2}\frac{m}{2\pi}\sum_{\sigma}\frac{1}{(\omega_p\omega_q|\mathbf{p}-\mathbf{q}|)^{1/2}}F(\mathbf{q}\sigma s')\bar{u}(\mathbf{p}s)\gamma\cdot\epsilon(\mathbf{q}-\mathbf{p},\lambda)u(\mathbf{q}\sigma) , \end{aligned} \tag{5}$$

where $\omega_p = \sqrt{m^2+p^2}$ and $\Omega_p = \sqrt{M^2+p^2}$ are kinetic energies of the two quarks. In order to simplify the problem, we drop the second term in the second equation [10], which in perturbation theory would represent corrections of order α_s^5. Thus the second equation becomes

$$\begin{aligned} G(\mathbf{p},-\mathbf{q}\sigma\delta\lambda) = {} & \left[\frac{4\alpha_s}{3}\right]^{1/2}\frac{1}{2\pi[E-\omega_p-\Omega_p-|\mathbf{p}-\mathbf{q}|]} \\ & \times\left[\sum_{\sigma'}\frac{M}{(\Omega_p\Omega_q|\mathbf{p}-\mathbf{q}|)^{1/2}}F(\mathbf{p}\sigma\sigma')\bar{V}(-\mathbf{p}\sigma')\gamma\cdot\epsilon(\mathbf{q}-\mathbf{p},\lambda)V(-\mathbf{q}\delta)\right. \\ & \left. -\sum_{\sigma'}\frac{m}{(\omega_p\omega_q|\mathbf{p}-\mathbf{q}|)^{1/2}}F(\mathbf{q}\sigma'\delta)\bar{u}(\mathbf{p}\sigma)\gamma\cdot\epsilon(\mathbf{q}-\mathbf{p},\lambda)u(\mathbf{q}\sigma')\right] . \end{aligned} \tag{6}$$

Substituting it into the F equation, we obtain the three-dimensional integral equation

$$EF(\mathbf{p}\sigma\delta)=(\omega_p+\Omega_p)F(\mathbf{p}\sigma\delta)-\frac{4\alpha_s}{3}\frac{mM}{2\pi^2}\sum_{\sigma'\delta'}\int\frac{d^3q}{\sqrt{\omega_p\omega_q\Omega_p\Omega_q}\,|\mathbf{p}-\mathbf{q}|^2}\Bigg\{F(\mathbf{q}\sigma'\delta')u^\dagger(\mathbf{p}\sigma)u(\mathbf{q}\sigma')V^\dagger(\mathbf{q}\delta')V(\mathbf{p}\delta)-\frac{|\mathbf{p}-\mathbf{q}|}{[M-\omega_p-\Omega_q-|\mathbf{p}-\mathbf{q}|]}\times\sum_\lambda F(\mathbf{q}\sigma'\delta')\bar{u}(\mathbf{p}\sigma)\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}(\mathbf{q}-\mathbf{p},\lambda)\times u(\mathbf{q}\sigma')\bar{V}(\mathbf{q}\delta')\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}(\mathbf{q}-\mathbf{p},\lambda)V(\mathbf{p}\delta)\Bigg\}+\frac{bmM}{\pi^2}\sum_{\sigma'\delta'}\int\frac{d^3q}{\sqrt{\omega_p\omega_q\Omega_p\Omega_q}\,|\mathbf{p}-\mathbf{q}|^4}F(\mathbf{q}\sigma'\delta')\bar{u}(\mathbf{p}\sigma)u(\mathbf{q}\sigma')\bar{V}(-\mathbf{q}\delta')V(-\mathbf{p}\delta)\ , \tag{7}$$

where we can clearly identify the physical origin of the terms as fermion kinetic energy, Coulomb and transverse gluon exchange, and linear confinement, respectively. Of course, in the limit of equal quark masses we retrieve the analogous equation in I.

III. KERNELS

The eigenvalue equation (7) can be written in a spin-independent form in which kernels are determined by specified quantum numbers. Substituting F into Eq. (7), multiplying both sides by Γ_Y, summing over all the spins, and after some trace algebra, one arrives at the integral equation

$$Ef(p)=(\omega_p+\Omega_p)f(p)-\frac{1}{2\pi^2}\int d^3q\, f(q)\left[\frac{4\alpha_s}{3}K_g(\mathbf{p},\mathbf{q})-2bK_S(\mathbf{p},\mathbf{q})\right]\ , \tag{8}$$

where $K_g(\mathbf{p},\mathbf{q})$ and $K_S(\mathbf{p},\mathbf{q})$ are the gluon-exchange and confining kernels, respectively. They have the general forms

$$K_g(\mathbf{p},\mathbf{q})=\frac{mM}{N^{1/2}}\mathrm{tr}\left[\frac{\tilde{\not P}-M}{2M}\tilde{\Gamma}_Y\frac{\not p+m}{2m}\gamma^\mu\frac{\not q+m}{2m}\Gamma_Y\frac{\tilde{\not Q}-M}{2M}\gamma^\nu\right]D_{\mu\nu}(|\mathbf{p}-\mathbf{q}|) \tag{9}$$

and

$$K_S(\mathbf{p},\mathbf{q})=\frac{mM}{N^{1/2}}\mathrm{tr}\left[\frac{\tilde{\not P}-M}{2M}\tilde{\Gamma}_Y\frac{\not p+m}{2m}\frac{\not q+m}{2m}\Gamma_Y\frac{\tilde{\not Q}-M}{2M}\right]\frac{1}{|\mathbf{p}-\mathbf{q}|^4}\ , \tag{10}$$

where N is a normalization factor,

$$N=\frac{1}{\omega_p\omega_q\Omega_p\Omega_q}\mathrm{tr}\left[\frac{\tilde{\not P}-M}{2M}\tilde{\Gamma}_Y\frac{\not p+m}{2m}\Gamma_Y\right]\mathrm{tr}\left[\frac{\tilde{\not Q}-M}{2m}\tilde{\Gamma}_Y\frac{\not q+m}{2m}\Gamma_Y\right]\ , \tag{11}$$

and $D_{\mu\nu}(|\mathbf{p}-\mathbf{q}|)$ is

$$D_{00}(|\mathbf{p}-\mathbf{q}|)=\frac{1}{|\mathbf{p}-\mathbf{q}|^2} \tag{12a}$$

and

$$D_{ij}(|\mathbf{p}-\mathbf{q}|)=\frac{1}{[E-\omega_p-\Omega_q-|\mathbf{p}-\mathbf{q}|]|\mathbf{p}-\mathbf{q}|}\left[\delta_{ij}-\frac{(p-q)_i(p-q)_j}{|\mathbf{p}-\mathbf{q}|^2}\right]\ . \tag{12b}$$

Taking $E=\omega_p+\Omega_q$ in the denominator, we obtain the transverse gluon propagator

$$D_{ij}(|\mathbf{p}-\mathbf{q}|)=-\frac{1}{|\mathbf{p}-\mathbf{q}|^2}\left[\delta_{ij}-\frac{(p-q)_i(p-q)_j}{|\mathbf{p}-\mathbf{q}|^2}\right]\ . \tag{13}$$

Since one of the main motivations of this work was to establish the unequal-mass kernels, we now explicitly list the kernels for $J^P=0^\mp$ and $J^P=1^\pm$.

For $J^P=0^\mp$ singlet S states and triplet P states, the kernels are

$$K_g(\mathbf{p},\mathbf{q})=\frac{1}{4[\omega_p\omega_q\Omega_p\Omega_q(\omega_p\Omega_p\pm mM+p^2)(\omega_q\Omega_q\pm mM+q^2)]^{1/2}}\frac{1}{|\mathbf{p}-\mathbf{q}|^2}$$
$$\times\left\{3(\omega_p\Omega_p\pm mM+p^2)(\omega_q\Omega_q\pm mM+q^2)-(m\Omega_p\pm M\omega_p)(m\Omega_q\pm M\omega_q)\right.$$
$$\left.+[3(m\mp M)^2-(\omega_p+\Omega_p)(\omega_q+\Omega_q)]\mathbf{p}\cdot\mathbf{q}-2[(m\mp M)^2-(\omega_p+\Omega_p)(\omega_q+\Omega_q)]\frac{p^2q^2-(\mathbf{p}\cdot\mathbf{q})^2}{|\mathbf{p}-\mathbf{q}|^2}\right\} \tag{14a}$$

and

$$K_S(\mathbf{p},\mathbf{q})=-\frac{1}{4[\omega_p\omega_q\Omega_p\Omega_q(\omega_p\Omega_q\pm mM+p^2)(\omega_q\Omega_q\pm mM+q^2)]^{1/2}}\frac{1}{|\mathbf{p}-\mathbf{q}|^4}$$
$$\times\left\{(\omega_p\omega_q+m^2)(\Omega_p\Omega_w+M^2)\pm mM(\omega_p+\omega_q)(\Omega_p+\Omega_q)\right.$$
$$\left.+p^2q^2+p^2\omega_q\Omega_q+q^2\omega_p\Omega_p\pm mM(p^2+q^2)-[(m\mp M)^2+(\omega_p+\Omega_p)(\omega_q+\Omega_q)]\mathbf{p}\cdot\mathbf{q}\right\}. \tag{14b}$$

For $J^P=1^-$ triplet S states, the kernels are

$$K_g(\mathbf{p},\mathbf{q})=\frac{1}{12[\omega_p\omega_q\Omega_p\Omega_q(\omega_p+m)(\Omega_p+M)(\omega_q+m)(\Omega_q+M)]^{1/2}}\frac{1}{|\mathbf{p}-\mathbf{q}|^2}$$
$$\times\left\{3(\omega_p+m)(\Omega_p+M)(\omega_q+m)(\Omega_q+M)-p^2q^2-2(\omega_q+m)(\Omega_q+M)p^2-2(\omega_p+m)(\Omega_p+M)q^2\right.$$
$$+[2(\omega_p+m)(\Omega_q+M)+2(\Omega_p+M)(\omega_q+m)$$
$$+3(\omega_p+m)(\omega_q+m)+3(\Omega_p+M)(\Omega_q+M)]\mathbf{p}\cdot\mathbf{q}+4(\mathbf{p}\cdot\mathbf{q})^2$$
$$+2[2(\omega_p+m)(\Omega_p+M)+2(\omega_q+m)(\Omega_q+M)+\omega_p\Omega_q+\omega_q\Omega_p$$
$$\left.+m(\Omega_p+\Omega_q)+M(\omega_p+\omega_q)+2mM]\frac{p^2p^2-(\mathbf{p}\cdot\mathbf{q})^2}{|\mathbf{p}-\mathbf{q}|^2}\right\} \tag{15a}$$

and

$$K_S(\mathbf{p},\mathbf{q})=\frac{1}{12[\omega_p\omega_q\Omega_p\Omega_q(\omega_p+m)(\Omega_p+M)(\omega_q+m)(\Omega_q+M)]^{1/2}}\frac{1}{|\mathbf{p}-\mathbf{q}|^4}$$
$$\times\{-3(\omega_p+m)(\Omega_p+M)(\omega_q+m)(\Omega_q+M)+p^2q^2$$
$$+3[(\omega_p+m)(\omega_q+m)+(\Omega_p+M)(\Omega_q+M)]\mathbf{p}\cdot\mathbf{q}-4(\mathbf{p}\cdot\mathbf{q})^2\}. \tag{15b}$$

The $J^P=1^+$ states are more complicated. Unlike the equal-mass case where $\Gamma_Y=\sigma^{ij}$ and $\Gamma_Y=\gamma^5\gamma^I$ generate the kernels for singlet and triplet P states, here they lead to two different mixtures of the singlet and triplet states. However, linear combinations of those forms generate the kernels for pure singlet and triplet states. For the triplet P states, we have $\Gamma_Y=\gamma^5(a_p\gamma^0-1)\gamma^i$, where

$$a_p=(\Omega_p+M-\omega_p-m)/(\Omega_p+M+\omega_p+m).$$

The gluon-exchange kernel is

$$K_g(\mathbf{p},\mathbf{q})=a_pa_qK_g^{11}(\mathbf{p},\mathbf{q})+a_pK_g^{12}(\mathbf{p},\mathbf{q})+a_qK_g^{21}(\mathbf{p},\mathbf{q})+K_g^{22}(\mathbf{p},\mathbf{q}), \tag{16a}$$

where

$$K_g^{22}(\mathbf{p},\mathbf{q})=\frac{1}{4[\omega_p\omega_q\Omega_p\Omega_q N_p N_q]^{1/2}}\frac{1}{|\mathbf{p}-\mathbf{q}|^2}$$
$$\times\Big\{5(\omega_p\Omega_p-mM)(\omega_q\Omega_q-mM)+m^2\Omega_p\Omega_q+M^2\omega_p\omega_q$$
$$-mM(\omega_p\Omega_q+\omega_q\Omega_p)-p^2\omega_q\Omega_q-q^2\omega_p\Omega_p+mM(p^2+q^2)-3p^2q^2$$
$$+[5m^2+5M^2-2mM+3\omega_p\Omega_q+3\omega_q\Omega_p+\omega_p\omega_q+\Omega_p\Omega_q]\mathbf{p}\cdot\mathbf{q}+8(\mathbf{p}\cdot\mathbf{q})^2$$
$$+2[-(m+M)^2-4mM+(\omega_p+\Omega_p)(\omega_q+\Omega_q)+2(\omega_p\Omega_p+\omega_q\Omega_q)]\frac{p^2q^2-(\mathbf{p}\cdot\mathbf{q})^2}{|\mathbf{p}-\mathbf{q}|^2}\Big\}, \tag{16b}$$

where

$$N_p=3(\omega_p\Omega_p-mM)(1+a_p^2)+(1-a_p^2)p^2+6a_p(m\Omega_p-M\omega_p),$$
$$N_q=3(\omega_q\Omega_q-mM)(1+a_q^2)+(1-a_q^2)q^2+6a_q(m\Omega_q-M\omega_q),$$

$$K_g^{11}(\mathbf{p},\mathbf{q})=\frac{1}{4[\omega_p\omega_q\Omega_p\Omega_q N_p N_q]^{1/2}}\frac{1}{|\mathbf{p}-\mathbf{q}|^2}$$
$$\times\Big\{(\omega_p\Omega_p-mM)(\omega_q\Omega_q-mM)+5(m\Omega_p-M\omega_p)(m\Omega_q-M\omega_q)$$
$$-3p^2\omega_q\Omega_q-3q^2\omega_p\Omega_p+3mM(p^2+q^2)+p^2q^2$$
$$+[(m-M)^2-4mM+\omega_p\Omega_q+\omega_q\Omega_p+5\omega_p\omega_q+5\Omega_p\Omega_q]\mathbf{p}\cdot\mathbf{q}$$
$$+2[(m-M)^2-4mM-(\omega_p-\Omega_p)(\omega_q-\Omega_q)+2(\omega_p\Omega_p+\omega_q\Omega_q)]\frac{p^2q^2-(\mathbf{p}\cdot\mathbf{q})^2}{|\mathbf{p}-\mathbf{q}|^2}\Big\}, \tag{16c}$$

$$K_g^{12}(\mathbf{p},\mathbf{q})=\frac{1}{4[\omega_p\omega_q\Omega_p\Omega_q N_p N_q]^{1/2}}\frac{1}{|\mathbf{p}-\mathbf{q}|^2}$$
$$\times\Big\{5(\omega_p\Omega_p-mM)(m\Omega_q-M\omega_q)+(\omega_q\Omega_q-mM)(m\Omega_p-M\omega_p)+(m\Omega_q-M\omega_q)p^2+3(m\Omega_p-M\omega_p)^2$$
$$+[5(m\omega_q-M\Omega_q)-3(m\Omega_p-M\omega_p)-(m\Omega_q-M\omega_q)+(m\omega_p-M\Omega_p)]\mathbf{p}\cdot\mathbf{q}$$
$$-2[3(m\Omega_q-M\omega_q)+3(m\Omega_p-M\omega_p)-m(\omega_q-\omega_q)+M(\Omega_p-\Omega_q)]\frac{p^2q^2-(\mathbf{p}\cdot\mathbf{q})^2}{|\mathbf{p}-\mathbf{q}|^2}\Big\}, \tag{16d}$$

and

$$K_g^{21}(\mathbf{p},\mathbf{q})=\frac{1}{4[\omega_p\omega_q\Omega_p\Omega_q N_p N_q]^{1/2}}\frac{1}{|\mathbf{p}-\mathbf{q}|^2}$$
$$\times\Big\{(\omega_p\Omega_p-mM)(m\Omega_q-M\omega_q)+5(\omega_q\Omega_q-mM)(m\Omega_p-M\omega_p)+3(m\Omega_q-M\omega_q)p^2+(m\Omega_p-M\omega_p)q^2$$
$$+[-3(m\Omega_q-M\omega_q)+5(m\omega_p-M\Omega_p)+(m\omega_q-M\Omega_q)-(m\Omega_p-M\omega_p)]\mathbf{p}\cdot\mathbf{q}$$
$$-2[3(m\Omega_q-M\omega_q)+3(m\Omega_p-M\omega_p)+m(\omega_q-\omega_q)-M(\Omega_p-\Omega_q)]\frac{p^2q^2-(\mathbf{p}\cdot\mathbf{q})^2}{|\mathbf{p}-\mathbf{q}|^2}\Big\}. \tag{16e}$$

The confining kernel is

$$K_S(\mathbf{p},\mathbf{q})=a_pa_q(K_S^{11}(\mathbf{p},\mathbf{q})+a_pK_S^{12}(\mathbf{p},\mathbf{q})+a_qK_S^{21}(\mathbf{p},\mathbf{q})+K_S^{22}(\mathbf{p},\mathbf{q}), \tag{17a}$$

where

$$K_S^{22}(\mathbf{p},\mathbf{q})=\frac{1}{4[\omega_p\omega_q\Omega_p\Omega_q N_p N_q]^{1/2}}\frac{1}{|\mathbf{p}-\mathbf{q}|^4}\times\{3(\omega_p\omega_q+m^2)(\Omega_p\Omega_q+M^2)-3mM(\omega_p+\omega_q)(\Omega_p+\Omega_q)+p^2\omega_q\Omega_q+q^2\omega_p\Omega_p-mM(p^2+q^2)-p^2q^2+4(\mathbf{p}\cdot\mathbf{q})^2-[3m^2+3M^2+2mM+\omega_p\Omega_q+\omega_q\Omega_p+3\omega_p\omega_q+3\Omega_p\Omega_q]\mathbf{p}\cdot\mathbf{q}\}\ , \tag{17b}$$

$$K_S^{11}(\mathbf{p},\mathbf{q})=\frac{1}{4[\omega_p\omega_q\Omega_p\Omega_q N_p N_q]^{1/2}}\frac{1}{|\mathbf{p}-\mathbf{q}|^4}\times\{3(\omega_p\omega_q+m^2)(\Omega_p\Omega_q+M^2)-3mM(\omega_p\omega_q+\Omega_p\Omega_q)-p^2q^2-p^2\omega_q\Omega_q-q^2\omega_p\Omega_p+mM(p^2+q^2)+4(\mathbf{p}\cdot\mathbf{q})^2+[-3m^2-3M^2+2mM+\omega_p\Omega_q+\omega_q\Omega_p-3\omega_p\omega_q-3\Omega_p\Omega_q]\mathbf{p}\cdot\mathbf{q}\}\ , \tag{17c}$$

$$K_S^{12}(\mathbf{p},\mathbf{q})=\frac{1}{4[\omega_p\omega_q\Omega_p\Omega_q N_p N_q]^{1/2}}\frac{1}{|\mathbf{p}-\mathbf{q}|^4}\times\{(3\omega_p\Omega_p-3mM-p^2-\mathbf{p}\cdot\mathbf{q})(m\Omega_q-M\omega_q)+(3\omega_q\Omega_q-3mM+q^2+\mathbf{p}\cdot\mathbf{q})(m\Omega_p-M\omega_p)+3[M(\Omega_p+\Omega_q)-m(\omega_p+\omega_q)]\mathbf{p}\cdot\mathbf{q}\}\ , \tag{17d}$$

and

$$K_S^{21}(\mathbf{p},\mathbf{q})=\frac{1}{4[\omega_p\omega_q\Omega_p\Omega_q N_p N_q]^{1/2}}\frac{1}{|\mathbf{p}-\mathbf{q}|^4}\times\{(3\omega_p\Omega_p-3mM+p^2+\mathbf{p}\cdot\mathbf{q})(m\Omega_q-M\omega_q)+(3\omega_q\Omega_q-3mM-q^2-\mathbf{p}\cdot\mathbf{q})(m\Omega_p-M\omega_p)+3[M(\Omega_p+\Omega_q)-m(\omega_p+\omega_q)]\mathbf{p}\cdot\mathbf{q}\}\ . \tag{17e}$$

For the singlet P states, we take $\Gamma_Y=\gamma^5(\gamma^0-a)\gamma^i$, which generates the gluon-exchange kernel

$$K_g(\mathbf{p},\mathbf{q})=a_pa_qK_g^{22}(\mathbf{p},\mathbf{q})+a_pK_g^{21}(\mathbf{p},\mathbf{q})+a_qK_g^{12}(\mathbf{p},\mathbf{q})+K_g^{11}(\mathbf{p},\mathbf{q}) \tag{18a}$$

and the confining kernel

$$K_S(\mathbf{p},\mathbf{q})=a_pa_qK_S^{22}(\mathbf{p},\mathbf{q})+a_pK_S^{21}(\mathbf{p},\mathbf{q})+a_qK_S^{12}(\mathbf{p},\mathbf{q})+K_S^{11}(\mathbf{p},\mathbf{q})\ , \tag{18b}$$

where K_g^{ij} and K_S^{ij} with $i,j=1,2$ are the same as those for the triplet P states with $J^P=1^+$. An eigenstate of 1^+ is a mixture of triplet and singlet states. Its eigenenergy is obtained by solving the matrix eigenvalue equation

$$\begin{bmatrix}\langle{}^3P_1|{}^3P_1\rangle & \langle{}^3P_1|{}^1P_1\rangle\\ \langle{}^1P_1|{}^3P_1\rangle & \langle{}^1P_1|{}^1P_1\rangle\end{bmatrix}E=\begin{bmatrix}\langle{}^3P_1|H|{}^3P_1\rangle & \langle{}^3P_1|H|{}^1P_1\rangle\\ \langle{}^1P_1|H|{}^3P_1\rangle & \langle{}^1P_1|H|{}^1P_1\rangle\end{bmatrix}. \tag{19}$$

The matrix elements on the left-hand side (LHS) are

$$\langle{}^3P_1|{}^3P_1\rangle=\frac{1}{mM}\int d^3p\, f^2(p)[3(\omega_p\Omega_p-mM)(1+a_p^2)+(1-a_p^2)p^2+6a_p(m\Omega_p-M\omega_p)]\ , \tag{20}$$

$$\langle{}^1P_1|{}^1P_1\rangle=\frac{1}{mM}\int d^3p\, f^2(p)[3(\omega_p\Omega_p-mM)(1+a_p^2)-(1-a_p^2)p^2+6a_p(m\Omega_p-M\omega_p)]\ , \tag{21}$$

and

$$\langle{}^3P_1|{}^1P_1\rangle=\langle{}^1P_1|{}^3P_1\rangle=\frac{3}{mM}\int d^3p\, f^2(p)[(a_p^2+1)(m\Omega_p-M\omega_p)+2a_p(\omega_p\Omega_p-mM)]\ . \tag{22}$$

The matrix elements of energy expectation on the RHS are

$$\langle{}^3P_1|H|{}^3P_1\rangle=-\frac{1}{2\pi^2}\int\int d^3p\, d^3q\, f(q)f(q)(N_pN_q)^{1/2}\left[\frac{4\alpha_s}{3}K_g(\mathbf{p},\mathbf{q})-2bK_S(\mathbf{p},\mathbf{q})\right]\ , \tag{23}$$

where $K_g(\mathbf{p},\mathbf{q})$ and $K_S(\mathbf{p},\mathbf{q})$ are given by Eqs. (16) and (17), and

$$\langle{}^1P_1|H|{}^1P_1\rangle=-\frac{1}{2\pi^2}\int\int d^3p\, d^3q\, f(p)f(q)(N_pN_q)^{1/2}\left[\frac{4\alpha_3}{3}K_g(\mathbf{p},\mathbf{q})-2bK_S(\mathbf{p},\mathbf{q})\right]\ , \tag{24}$$

where $K_g(\mathbf{p},\mathbf{q})$ and $K_S(\mathbf{p},\mathbf{q})$ are given by Eqs. (18), and

$$\langle {}^3P_1|H|{}^1P_1\rangle = -\frac{1}{2\pi^2}\int\int d^3p\, d^3q\, f(p)f(q)(N_pN_q)^{1/2}\left[\frac{4\alpha_s}{3}K_g(\mathbf{p},\mathbf{q})-2bK_S(\mathbf{p},\mathbf{q})\right], \tag{25a}$$

where the gluon-exchange kernel is

$$K_g(\mathbf{p},\mathbf{q})=a_pK_g^{11}(\mathbf{p},\mathbf{q})+a_pa_qK_g^{12}(\mathbf{p},\mathbf{q})+K_g^{21}(\mathbf{p},\mathbf{q})+a_qK_g^{22}(\mathbf{p},\mathbf{q}) \tag{25b}$$

and the confining kernel is

$$K_S(\mathbf{p},\mathbf{q})=a_pK_S^{11}(\mathbf{p},\mathbf{q})+a_pa_qK_S^{12}(\mathbf{p},\mathbf{q})+K_S^{21}(\mathbf{p},\mathbf{q})+a_qK_S^{22}(\mathbf{p},\mathbf{q}), \tag{25c}$$

where K_g^{ij} and K_S^{ij} with $i,j=1,2$ are given by Eqs. (16) and (17). Finally,

$$\langle {}^1P_1|H|{}^3P_1\rangle = -\frac{1}{2\pi^2}\int\int d^3p\, d^3q\, f(p)f(q)(N_pN_q)^{1/2}\left[\frac{4\alpha_s}{3}K_g(\mathbf{p},\mathbf{q})-2bK_S(\mathbf{p},\mathbf{q})\right], \tag{26a}$$

where the gluon-exchange kernel is

$$K_g(\mathbf{p},\mathbf{q})=a_qK_g^{11}(\mathbf{p},\mathbf{q})+K_g^{12}(\mathbf{p},\mathbf{q})+a_pa_qK_g^{21}(\mathbf{p},\mathbf{q})+a_pK_g^{22}(\mathbf{p},\mathbf{q}) \tag{26b}$$

and the confining kernel is

$$K_S(\mathbf{p},\mathbf{q})=a_qK_S^{11}(\mathbf{p},\mathbf{q})+K_S^{12}(\mathbf{p},\mathbf{q})+a_pa_qK_S^{21}(\mathbf{p},\mathbf{q})+a_pK_S^{22}(\mathbf{p},\mathbf{q}), \tag{26c}$$

where K_g^{ij} and K_S^{ij} with $i,j=1,2$ are given by Eqs. (16) and (17).

Performing the angular integration over Eq. (8), one obtains the radial kernel equation

$$Ef(p)=[\omega_p+\Omega_p]f(p)-\frac{1}{4\pi}\int_0^\infty \frac{q}{p}dq\, f(q)\left[\frac{4}{3}\alpha_sK_g(p,q)-bK_S(p,q)\right], \tag{27}$$

where $K_g(p,q)$ and $K_S(p,q)$ describe one-gluon-exchange interactions and scalar linear confinement, respectively. Their forms depend on the quantum number J^P selected. For $J^P=0^{\mp}$, the radial kernels are

$$\begin{aligned}K_g(p,q)=&\frac{1}{[\omega_p\omega_q\Omega_p\Omega_q(\omega_p\Omega_p\pm mM+p^2)(\omega_q\Omega_q\pm mM+q^2)]^{1/2}}\\ &\times\Bigg\{-(m\mp M)^2pq-(\omega_p+\Omega_p)(\omega_q+\Omega_q)pq+\tfrac{1}{2}(p^2+q^2)[(m\mp M)^2+(\omega_p+\Omega_p)(\omega_q+\Omega_q)]\ln\left|\frac{p+q}{p-q}\right|\\ &\quad+[3(\omega_p\Omega_p\pm mM+p^2)(\omega_q\Omega_q\pm mM+q^2)-(m\Omega_p\pm M\omega_p)(m\Omega_q\pm M\omega_q)]\ln\left|\frac{p+q}{p-q}\right|\Bigg\}\end{aligned} \tag{28a}$$

and

$$\begin{aligned}K_S(p,q)=&\frac{\pm1}{[\omega_p\omega_q\Omega_p\Omega_q(\omega_p\Omega_p\pm mM+p^2)(\omega_q\Omega_q\pm mM+g^2)]^{1/2}}\\ &\times\Bigg\{\frac{4pq}{(p^2-q^2)^2}[-(\omega_p\omega_q+m^2)(\Omega_p\Omega_q+M^2)-\omega_p\Omega_pq^2-\omega_q\Omega_qp^2-p^2q^2\\ &\quad\mp mM(p^2+q^2)\mp mM(\omega_p+\omega_q)(\Omega_p+\Omega_q)+\tfrac{1}{2}(p^2+q^2)(\omega_p+\Omega_p)(\omega_q+\Omega_q)\\ &\quad+\tfrac{1}{2}(p^2+q^2)(m\mp M)^2]-[(m\mp M)^2+(\omega_p+\Omega_p)(\omega_q+\Omega_q)]\ln\left|\frac{p+q}{p-q}\right|\Bigg\}.\end{aligned} \tag{28b}$$

For $J^P=1^-$ triplet S states, the kernels become

$$K_g(p,q)=\frac{pq}{3[\omega_p\omega_q\Omega_p\Omega_q(\omega_p+m)(\Omega_p+M)(\omega_q+m)(\Omega_q+M)]^{1/2}}$$
$$\times\Big\{-4(\omega_p+m)(\Omega_p+M)-4(\omega_q+m)(\Omega_q+M)-4(\omega_p+m)(\Omega_q+M)-4(\omega_q+m)(\Omega_p+M)$$
$$-3(\omega_p+m)(\omega_q+m)-3(\Omega_p+M)(\Omega_q+M)-2(p^2+q^2)$$
$$+\{6(\omega_p+m)(\Omega_p+M)(\omega_q+m)(\Omega_q+M)-2p^2q^2-4(\omega_q+m)(\Omega_q+M)p^2-4(\omega_p+m)(\Omega_p+M)q^2$$
$$+(p^2+q^2)[4(\omega_p+m)(\Omega_p+M)+4(\omega_q+m)(\Omega_q+M)+4(\omega_p+m)(\Omega_q+M)$$
$$+4(\omega_q+m)(\Omega_p+M)+3(\omega_p+m)(\omega_q+m)$$
$$+3(\Omega_p+M)(\Omega_q+M)+2(p^2+q^2)]\}\frac{1}{2pq}\ln\left|\frac{p+q}{p-q}\right|\Big\} \tag{29a}$$

and

$$K_S(p,q)=\frac{2pq}{3[\omega_p\omega_q\Omega_p\Omega_q(\omega_p+m)(\Omega_p+M)(\omega_q+m)(\Omega_q+M)]^{1/2}}$$
$$\times\Big\{-4+\frac{3}{(p^2-q^2)^2}[-2(\omega_p+m)(\Omega_p+M)(\omega_q+m)(\Omega_q+M)-2p^2q^2$$
$$+(p^2-q^2)(\omega_p+m)(\omega_q+m)+(p^2+q^2)(\Omega_p+M)(\Omega_q+M)]$$
$$+[-3(\omega_p+m)(\omega_q+m)-3(\Omega_p-M)(\Omega_q+M)+4(p^2+q^2)]\frac{1}{2pq}\ln\left|\frac{p+q}{p-q}\right|\Big\}. \tag{29b}$$

For $J^P=1^+$ triplet P states, the corresponding gluon-exchange kernel is

$$K_g(p,q)=a_pa_qK_g^{11}(p,q)+a_pK_g^{12}(p,q)+a_qK_g^{21}(p,q)+K_g^{22}(p,q)\ , \tag{30a}$$

where

$$K_g^{22}(p,q)=\frac{pq}{[\omega_p\omega_q\Omega_p\Omega_qN_pN_q]^{1/2}}$$
$$\times\Big\{-3(m-M)^2+8mM-4(p^2+q^2)-3(\omega_p+\Omega_p)(\omega_q+\Omega_q)-2(\omega_p\Omega_q+\omega_q\Omega_p)-4(\omega_p\Omega_p+\omega_q\Omega_q)$$
$$+\{10(\omega_p\Omega_p-mM)(\omega_q\Omega_q-mM)+2(m\Omega_p-M\omega_p)(m\Omega_q-M\omega_q)$$
$$-2p^2\omega_q\Omega_q-2q^2\omega_p+2mM(p^2+q^2)-6p^2q^2$$
$$+[3(m-M)^2-8mM+4(p^2+q^2)+3(\omega_p+\Omega_p)(\omega_q+\Omega_q)$$
$$+2(\omega_p\Omega_q+\omega_q\Omega_p)+4(\omega_p\Omega_p+\omega_q\Omega_q)](p^2+q^2)\}\frac{1}{2pq}\ln\left|\frac{p+q}{p-q}\right|\Big\}, \tag{30b}$$

$$K_g^{11}(p,q)=\frac{pq}{[\omega_p\omega_q\Omega_p\Omega_qN_pN_q]^{1/2}}$$
$$\times\Big\{[(\omega_p\Omega_p-mM)(\omega_q\Omega_q-mM)+5(m\Omega_p-M\omega_p)(m\Omega_q-M\omega_q)$$
$$+p^2q^2-3p^2\omega_q\Omega_q-3q^2\omega_p\Omega_p+3mM(p^2+q^2)]\frac{1}{pq}\ln\left|\frac{p+q}{p-q}\right|$$
$$+[3(m-M)^2-12mM+3(\omega_p+\Omega_p)(\omega_q+\Omega_q)$$
$$+4\omega_p\Omega_p+4\omega_q\Omega_q]\left[\frac{p^2+q^2}{2pq}\ln\left|\frac{p+q}{p-q}\right|-1\right]\Big\}, \tag{30c}$$

$$K_g^{12}(p,q)=\frac{pq}{[\omega_p\omega_q\Omega_p\Omega_q N_p N_q]^{1/2}}\times\Bigg\{[5(\omega_p\Omega_p-mM)(m\Omega_q-M\omega_q)+(\omega_q\Omega_q-mM)(m\Omega_p-M\omega_p)+(m\Omega_q-M\omega_q)p^2+3(m\Omega_p-M\omega_p)q^2]\frac{1}{pq}\ln\left|\frac{p+q}{p-q}\right|+[3(m\omega_q-M\Omega_q)+3(m\omega_p-M\Omega_p)-9(m\Omega_p-M\omega_p)-7(m\Omega_q-M\omega_q)]\left[\frac{p^2+q^2}{2pq}\ln\left|\frac{p+q}{p-q}\right|-1\right]\Bigg\}, \tag{30d}$$

and

$$K_g^{21}(p,q)=\frac{pq}{[\omega_p\omega_q\Omega_p\Omega_q N_p N_q]^{1/2}}\times\Bigg\{[(\omega_p\Omega_p-mM)(m\Omega_q-M\omega_q)+5(\omega_q\Omega_q-mM)(m\Omega_p-M\omega_p)+3(m\Omega_q-M\omega_q)p^2+(m\Omega_p-M\omega_p)q^2]\frac{1}{pq}\ln\left|\frac{p+q}{p-q}\right|+[3(m\omega_q-M\Omega_q)+3(m\omega_p-M\Omega_p)-7(m\Omega_p-M\omega_p)-9(m\Omega_q-M\omega_q)]\left[\frac{p^2+q^2}{2pq}\ln\left|\frac{p+q}{p-q}\right|-1\right]\Bigg\}. \tag{30e}$$

The confining kernel becomes

$$K_S(p,q)=a_p a_q K_S^{11}(p,q)+a_p K_S^{12}(p,q)+a_q K_S^{21}(p,q)+K_S^{22}(p,q)\ , \tag{31a}$$

where

$$K_S^{22}(p,q)=\frac{2pq}{[\omega_p\omega_q\Omega_p\Omega_q N_p N_q]^{1/2}}\times\Bigg\{4+\frac{2}{(p^2-q^2)^2}\Bigg[3p^2q^2+3(\omega_p\omega_q+m^2)(\Omega_p\Omega_q+M^2)-3mM(\omega_p+\omega_q)(\Omega_p+\Omega_q)+p^2\omega_q\Omega_q+q^2\omega_p\Omega_p-mM(p^2+q^2)-(3m^2+3M^2+2mM+3\omega_p\omega_q+3\Omega_p\Omega_q+\omega_p\Omega_q+\omega_q\Omega_p)\frac{p^2+q^2}{2}\Bigg]+[3m^2+3M^2+2mM+3\omega_p\omega_q+3\Omega_p\Omega_q+\omega_p\Omega_q+\omega_q\Omega_p-4(p^2+q^2)]\frac{1}{2pq}\ln\left|\frac{p+q}{p-q}\right|\Bigg\}, \tag{31b}$$

$$K_S^{11}(p,q)=\frac{2pq}{[\omega_p\omega_q\Omega_p\Omega_q N_p N_q]^{1/2}}\times\Bigg\{\frac{2}{(p^2-q^2)^2}\Bigg[3(\omega_p\omega_q+m^2)(\Omega_p\Omega_q+M^2)-3mM(\omega_p+\omega_q)(\Omega_p+\Omega_q)+3p^2q^2-p^2\omega_q\Omega_q+mM(p^2+q^2)+\frac{p^2+q^2}{2}(\omega_p\Omega_q+\omega_q\Omega_p+2mM-3m^2-3M^2-3\omega_p\omega_p-3\Omega_p\Omega_q)-q^2\omega_p\Omega_p\Bigg]+4-\frac{1}{2pq}\ln\left|\frac{p+q}{p-q}\right|[\omega_p\Omega_q+\omega_q\Omega_p+2mM+4(p^2+q^2)-3(m^2+M^2)-3(\omega_p\omega_q+\Omega_p\Omega_q]\Bigg\}, \tag{31c}$$

$$K_S^{12}(p,q)=\frac{pq}{2[\omega_p\omega_q\Omega_p\Omega_q N_p N_q]^{1/2}}\times\left\{\frac{2}{(p^2-q^2)^2}\left[(3\omega_p\Omega_p-3mM-p^2)(m\Omega_q-M\omega_q)+(3\omega_q\Omega_q-3mM+q^2)(m\Omega_p-M\omega_p)\right.\right.$$
$$\left.+\frac{p^2+q^2}{2}(-m\Omega_q+M\omega_q+m\Omega_p-M\omega_p+3M\Omega_p+3M\Omega_q-3m\omega_p-3m\omega_q)\right]$$
$$\left.-\frac{1}{2pq}\ln\left|\frac{p+q}{p-q}\right|[-m\Omega_q+M\omega_q+m\Omega_p-M\omega_p+3M\Omega_p+3M\Omega_q-3m\omega_p-3m\omega_q]\right\}, \tag{31d}$$

and

$$K_S^{21}(p,q)=\frac{pq}{2[\omega_p\omega_q\Omega_p\Omega_q N_p N_q]^{1/2}}\times\left\{\frac{2}{(p^2-q^2)^2}\left[(3\omega_p\Omega_p-3mM+p^2)(m\Omega_q-M\omega_q)+(3\omega_q\Omega_q-3mM-q^2)(m\Omega_p-M\omega_p)\right.\right.$$
$$\left.+\frac{p^2+q^2}{2}(m\Omega_q-M\omega_q-m\Omega_p+M\omega_p+3M\Omega_p+3M\Omega_q-3m\omega_p-3m\omega_q)\right]$$
$$\left.-\frac{1}{2pq}\ln\left|\frac{p+q}{p-q}\right|[m\Omega_q-M\omega_q-m\Omega_p+M\omega_p+3M\Omega_p+3M\Omega_q-3m\omega_p-3m\omega_q]\right\}. \tag{31e}$$

For $J^P=1^+$ singlet P states, the radial kernels are

$$K_g(p,q)=a_p a_q K_g^{22}(p,q)+a_p K_g^{21}(p,q)+a_q K_g^{12}(p,q)+K_g^{11}(p,q) \tag{32a}$$

and

$$K_S(p,q)=a_p a_q K_S^{22}(p,q)+a_p K_S^{21}(p,q)+a_q K_S^{12}(p,q)+K_S^{11}(p,q)\ . \tag{32b}$$

Performing an angular integration over all the elements in the matrix equation (19) gives a one-dimensional integral matrix equation. Diagonalization of this equation leads to the true eigenvalues of 1^+ states. When $m=M$, all the above kernels reproduce the kernels for the quark and its antiquark bound states [1].

IV. NUMERICAL RESULTS AND CONCLUSIONS

We now apply our model to the D mesons ($c\bar{d}$, $c\bar{u}$, and $c\bar{s}$). The numerical procedure used to solve the integral equation was outlined in I, and the interested reader is referred there for details.

In this work we use identical quark-mass parameters [$m_c=1.49$ GeV and $m_{u(d)}(d)=0.27$ GeV] and the string tension ($b=0.18$ GeV^2) of earlier work [1,4]. We must of course introduce a new quark mass ($m_s=0.40$ GeV) and we choose $\alpha_s=\frac{3}{4}\times 0.4575$ for the strong-coupling constant. The latter value was chosen to optimize the fit to the ground state (0^-) $c\bar{d}$ sector.

Our results are displayed in Table I. We see that the agreement with experiment [11] is quite good both in the qualitative pattern of splittings as well as in the quantitative agreement of individual states.

Although no data exist for the 0^+ sector, it is interesting to note that our predictions are somewhat lower than previous calculations [7]. New data on these states will constitute an interesting test of this approach.

We have applied our model to the B mesons as well. We again use the b-quark mass of earlier work [1,4]

TABLE I. D mesons. Comparison between theory and experiment [11]. Note all experimental entries other than for 0^- states need confirmation of I, J, and P quantum numbers. All entries are given in MeV.

J^P	$n^{2S+1}L_J$	Theory ($c\bar{d}$)	Experiment ($c\bar{d}$)	Theory ($c\bar{s}$)	Experiment ($c\bar{s}$)
0^-	1^1s_0	1870	1869.3±0.5	1970	1968.8±0.7
1^-	1^3s_1	2010	2010.1±0.6	2130	2110.3±2
0^+	1^3p_0	2200		2310	
1^+	1^3p_1	2430	2424±6	2550	2536.5±0.8
1^+	1^1p_1	2450	2443±7±5	2600	2564.3±4.4

($m_b = 4.784$ GeV), and we set $\alpha_s = \frac{3}{4}\times 0.3975$ to obtain a $B(b\bar{d}, J^P = 0^-)$ mass of 5278 MeV (input) as compared to the experimental value [11] of 5278.6 ± 2.0 MeV. The model generates a B^*-B mass difference of 40 MeV compared to the experimental value [11] of 46.0 ± 0.6 MeV and a B_s^*-B_{s^*} mass difference of 55 MeV compared to the experimental value [11] of 47.0 ± 2.6 MeV. In view of the fact that α_s is essentially input, one should view this latter success with some caution.

ACKNOWLEDGMENTS

The Natural Sciences and Engineering Research Council is acknowledged for financial assistance.

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