Parametrization relating the fermionic mass spectra

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When parametrizing the fermionic mass spectra in terms of the unit matrix and a recursive matrix $\mathcal{R}_0$, which corresponds to an underlying scaling pattern in the mass spectra, each fermionic sector is characterized by three parameters: k, α, and R. Using the set of relations displayed by the parameters of the different sectors, it is possible to formulate a "family Lagrangian" which for each sector encompasses all the families. Relations between quark masses are furthermore deduced from these "family Lagrangians." Using the relations between the parameters of the different charge sectors, it is also possible to "derive" the quark mass spectra from the (charged) leptonic mass spectrum.

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I. INTRODUCTION

The fermion masses do not seem to bear any simple mathematical relation to each other. Yet, in the standard model (SM) the same mechanism is responsible for generating all the fermion masses, so one would expect to find traces of some common underlying structure. But not even the only obvious pattern displayed by the fermion masses, namely, the family structure, is predicted by the SM, where the families are all treated on the same footing. This is because in the SM the gauge invariance does not supply any constraints on the Yukawa couplings, whereby the quark masses and mixing angles are left as free parameters.

There have been many attempts to constrain the Yukawa couplings, either by introducing extra symmetries or simply by postulating some "plausible" quark mass matrices. A central motivation in this search for fermion mass matrices is the hope to find an ansatz that would give a hint of some underlying mechanism—a common structure hidden in the fermion mass spectrum.

Instead of looking for a certain ansatz for the *mass matrices* (in the weak basis), I have studied the diagonalized mass matrices in their corresponding mass bases, i.e., the *mass spectra.* My hope was to find some parametrization of the fermion mass spectrum that would possibly reveal some underlying relations both within and between the sectors. Speculating that there is an underlying scaling pattern in the mass spectra, I express the fermion mass spectra $D_j=\mathrm{diag}(m_{j1},m_{j2},m_{j3})$, where j runs over all the fermionic charge sectors, as

$$D_j=\alpha_j\begin{bmatrix}1 & 0 & 0\\ 0 & R_j & 0\\ 0 & 0 & R_j^2\end{bmatrix}+k_j\mathbb{1}=\mathcal{R}_0+k_j\mathbb{1}\ , \qquad (1)$$

where $\mathbb{1}$ is the unit matrix and k_j, α_j, and R_j are functions of the mass eigenvalues. With some assumptions about the relations between the parameters $k_{u,d}$ and the mass eigenvalues $m_{u,d}$, this allows a set of relations to be established between the parameters of the different sectors, such as $\alpha_l/\alpha_d\approx\alpha_u/\alpha_l$, $R_u/R_d\approx R_d/R_l$, and $k_l\approx k_u k_d/\sqrt{k_u^2-k_d^2}$. I begin by introducing the parametrization, accounting for how it makes it possible to express the fermion mass spectra in terms of k_u and k_d. I then make use of the parametrization (1) in formulating a "family Lagrangian density,"

$$\mathcal{L}_n=i\bar{\psi}_n\gamma_\mu\partial^\mu\psi_n-(\beta+Rm_{n-1})\bar{\psi}_n\psi_n,\quad n=1,2,3\ ,$$

which for each sector includes all the families. Additional relations between the parameters of the different sectors are found, expressed in mass eigenvalues the relations between the parameters of the quark sectors can be formulated as $m_t\approx m_c^2/(m_d+m_u)$ and $m_b\approx m_s^2/(m_d-m_u)$. I furthermore express the quark mass spectra in terms of the (charged) leptonic parameters: with the leptonic masses as input, quark mass values can be "derived" that agree well with "experimental" (running) mass values [1,2] for d, s, b, u, and c; and a top mass value of the order of 120–180 GeV.

Parametrizing the neutrino sector in the same way as the charged fermion sectors leads to the relation $m_{\nu_\tau}\geq 2m_{\nu_\mu}$, and when inserting the upper bound limit value [3] of m_{ν_μ}, a limit on the m_{ν_τ} is obtained, which agrees with the τ-neutrino limit deduced from primordial helium considerations [4].

I also consider the form of the quark mass matrices when the mass spectra are parametrized as in (1), these mass matrices can be expressed in terms of the leptonic parameters and projection matrices in flavor space.

The approach proposed in this paper is quite different from the mass matrix procedure usually found in the literature. One of the most popular mass matrices is that of Fritzsch [5]. It advocates a scheme where only the mass of the third family is initially nonzero, and the lighter families are endowed with masses through the so-called radiative feed-down scenario.

Another popular approach is the *democratic family mixing* [6] scenario. In this scheme, as in the Fritzsch ansatz, it is the third family that is assumed to be primarily massive. The two lighter families are assumed to become massive only as the initially "democratic family mixing matrix" of the so-called Nambu form,

$$m_0 \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} , \tag{2}$$

is modified, in order to obtain mass matrices with eigenvalues that correspond to the physical fermion masses. In this scenario, the final mass matrices will have a form where the primordial "democratic" origin remains apparent, such that

$$m = a \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \Lambda , \tag{3}$$

where the magnitude of the matrix elements of Λ is assumed to be small compared to a. In my approach (1), there are also two terms, one much smaller than the other, in the sense that the k_j are assumed to be small compared to the R_j, but the family structure does not emerge like in the scenarios mentioned above, where the family structure so to speak is built step by step from the initial situation with one heavy and two massless states. In my approach the entire family structure of a sector instead appears at once, in one step.

II. SCALE TRANSFORMATION

The family pattern is similar for all the fermion sectors. There have been many attempts to interpret this pattern; the most successful is possibly the radiative feed-down scenario mentioned above. There have also been attempts to understand the family structure as a set of excited states, while still others believe that the families can be understood in terms of some underlying horizontal symmetry pattern.

My parametrization of the fermion mass spectra can be related to some speculations made a long time ago, about the possibility of interpreting the family pattern as resulting from some kind of "mass scaling" [7]. The Lagrangian density for a Dirac field is given by

$$\mathcal{L} = i\bar{\psi}\gamma_\mu \partial^\mu \psi - m\bar{\psi}\psi , \tag{4}$$

and the Dirac field has dimension $l = -\frac{3}{2}$, so under a scale transformation

$$x_\mu \to x'_\mu = R x_\mu \tag{5}$$

ψ transforms as

$$\psi \to \psi'(x') = R^{-3/2} \psi(Rx) \tag{6}$$

and the Lagrangian density transforms as

$$\mathcal{L} \to \mathcal{L}' = R^{-4}(i\bar{\psi}'\gamma_\mu \partial'^\mu \psi' - Rm\bar{\psi}'\psi') . \tag{7}$$

Now assume that the scale transformation is discrete, i.e., the value of R in (5) is fixed, and consider the Lagrangian

$$\mathcal{L}_0 = i\bar{\psi}_0 \gamma_\mu \partial^\mu \psi_0 - \tilde{m}_0 \bar{\psi}_0 \gamma_0 \tag{8}$$

corresponding to a Dirac field with mass $\tilde{m}_0$.

Under the discrete scale transformation, $\mathcal{L}_0$ transforms as

$$\mathcal{L}_0 \to R^{-4}(i\bar{\psi}'\gamma^\mu \partial'_\mu \psi' - R\tilde{m}_0 \bar{\psi}'\psi') = \mathcal{L}_1 , \tag{9}$$

$\mathcal{L}_1$ is thus of the same form as $\mathcal{L}_0$, with $\tilde{m}_0$ replaced by $R\tilde{m}_0$. It can be interpreted as indicating a simple scaling pattern, so that the existence of a Dirac field whose quanta have mass $\tilde{m}_0$ implies the existence of other Dirac fields whose quanta have the masses $\tilde{m}_0 R$, $\tilde{m}_1 R, \ldots$. That is, "family Lagrangian density" $\mathcal{L}_n$ of a given charge sector can be expressed as

$$\mathcal{L}_n = i\bar{\psi}_n \gamma^\mu \partial_\mu \psi_n - R\tilde{m}_{n-1} \bar{\psi}_n \psi_n , \tag{10}$$

where $n = 1,2,3$ is the family index.

It is clear that the fermion masses do not exhibit a structure such as in (10). But is it conceivable that this pattern could be traced in the mass spectrum of a fermion sector, however, "blurred" by some (smaller) additional contribution. That would mean that each fermionic mass spectrum $D_j = \text{diag}(m_{j1}, m_{j2}, m_{j3})$, hiding an "unblurred" mass spectrum,

$$\begin{aligned} \hat{m}_1 &= \alpha , \\ \hat{m}_2 &= \hat{m}_1 R , \\ \hat{m}_3 &= \hat{m}_2 R , \end{aligned} \tag{11}$$

should be on the form

$$D_j = \alpha \begin{bmatrix} 1 & 0 & 0 \\ 0 & R_j & 0 \\ 0 & 0 & R_j^2 \end{bmatrix} + k_j \mathbb{1} = \mathcal{R}_{0j} + k_j \mathbb{1} , \tag{12}$$

where the "additional contribution" $k_j \mathbb{1}$ is supposed to be smaller than the first term, in the sense that $\text{tr}(\mathcal{R}_{0j}) \approx \text{tr}(D_j)$ and

$$\{\text{tr}(\mathcal{R}_{0j}^2) - [\text{tr}(\mathcal{R}_{0j})]^2\}/2 \approx \{\text{tr}(D_j^2) - [\text{tr}(D_j)]^2\}/2 .$$

The parametrization (12) fixes the parameters k_j, α_j, and R_j, such that

$$\begin{aligned} k_j &= \frac{m_{j1} m_{j3} - m_{j2}^2}{(m_{j1} - m_{j2}) - (m_{j2} - m_{j3})} , \\ \alpha_j &= \frac{(m_{j1} - m_{j2})^2}{(m_{j1} - m_{j2}) - (m_{j2} - m_{j3})} , \\ R_j &= \frac{(m_{j2} - m_{j3})}{(m_{j1} - m_{j2})} , \end{aligned} \tag{13}$$

$$j = d, u, l, \nu .$$

It is interesting to relate this parametrization to the flavor symmetries of the weak mixing matrix. The matrix elements of the weak mixing matrix can be expressed by means of projection operators in flavor space [8], as

$$|V_{\alpha j}|^2 = \text{tr}[P_\alpha(m) P'_j(m')] , \tag{14}$$

where $P_\alpha(m)$ and $P'_j(m')$ are the projection operators corresponding to the mass matrices m and m' of the charge $\frac{2}{3}$- and $-\frac{1}{3}$-quark sectors, respectively. A projection operator projects out the appropriate flavor in any frame, i.e., $mP_\alpha(m) = m_\alpha P_\alpha(m)$, where m is the mass matrix and m_α is the αth mass eigenvalue. That the projection operators are unchanged as the mass spectra change by

$$D_j \rightarrow D_j + q\mathbb{1} \tag{15}$$

and

$$D_j \rightarrow qD_j \ , \tag{16}$$

where q is a coefficient, implies that the mixing matrix is not a function of the absolute mass eigenvalues of the quarks, but of the mass differences $m_j - m_k$ and $m'_j - m'_k$ and the ratios m_j/m_k and m'_j/m'_k. It is clear that R_j is invariant under (16), while α_j and R_j are invariant, and $k_j \rightarrow k_j + q$, under (15).

Note that we are not interested in the (singular-looking) case where $m_1 - m_2 = m_2 - m_3$, because then the $R_j = 1$ and the matrix $\mathcal{R}_{0j}$ is proportional to the unit matrix. The mass spectrum (12) should come about in such a way that the family structure emerges at once in the final step, with α_j nonzero and R_j different from zero and from ± 1, so that no pair of equal masses occur. There is thus no trace of any stepwise procedure as far as family nondegenration is concerned. Likewise, the limit where $m_1/m_2 = m_2/m_3$ is excluded, as it corresponds to $k_j = 0$, that is, to the "unblurred situation."

III. THE PARAMETRIZATION

In the SM, before the spontaneous symmetry breaking, the fermions are massless and degenerate, whereby they do not mix. Now suppose that in the first step of symmetry breaking the fermions became massive but remained degenerate within a given sector (up-type, down-type, etc.). Thus at this first stage, there is still a "family degeneracy" in each sector, whereas the $SU(3)_L \times SU(3)_R$ chiral symmetry is broken down to $SU(3)_{L+R}$ with the introduction of these "primary" nonzero masses.

The degenerate primary mass matrix in each fermionic sector is then of the form

$$D_{0j} = k_j \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \tag{17}$$

where the degeneracy of the D_0's is to be subsequently broken, in order to obtain the physical mass spectra and a weak mixing matrix that differs from the unit matrix. We know from experiment that the final mixing matrix remains close to the unit matrix. Therefore, it can be written as $V = \mathbb{1} + \Sigma$, where the magnitude of the matrix elements of Σ is much smaller than 1. This "close-to-unity" structure of the mixing matrix is a further motivation for the actual approach.

The parametrization (13) indeed shows that α_j and R_j are pure "family symmetry-breaking" parameters. Only the k_j can be though to be related to the first step of mass endowment, with merely "sectorial flavor symmetry breaking," which in the quark sector can be interpreted as a predecessor of the breaking of the strong isospin (if we let the masses in each sector approach the same degenerate "family" value, the k_j remain finite and nonzero, whereas the α_j and R_j vanish).

The different sectors are characterized by their mass coefficient k_j as well as by their charges. If we naively relate these "characterizing parameters" by setting

$$\frac{k_u}{k_d} \approx \frac{Q_u}{Q_d} = -2 \ , \tag{18}$$

we get that $k_u \approx -2k_d$. This is an accordance with the expectation that $|k_u| > |k_d|$, in agreement with the overall feature of the d-sector mass values of being smaller than the u-sector mass values. In order to reflect that $m_u < m_d$, we may then choose that $k_u < 0$ and $k_d > 0$. The strength of the strong isospin breaking measured by $(m_u - m_d)/(m_u + m_d)$ would at this stage of the procedure then correspond to

$$\frac{k_d + k_u}{k_d - k_u} \approx -\frac{1}{3} \ , \tag{19}$$

which agrees with the numerical value

$$(m_u - m_d)/(m_u + m_d) \approx -0.28 \ .$$

This can be considered as yet another argument for the assumption (18).

If we accordingly suppose that

$$\frac{k_d + k_u}{k_d - k_u} \approx \frac{m_u - m_d}{m_u + m_d} \tag{20}$$

we get that

$$\frac{k_u}{k_d} \approx -\frac{m_d}{m_u} \ . \tag{21}$$

The simplest interpretation of (21), namely, that $k_u \approx -m_d$ and $k_d \approx m_u$, corresponds to the relations

$$\begin{aligned} k_d + k_u &\approx m_u - m_d \ , \\ k_d - k_u &\approx m_u + m_d \ , \end{aligned} \tag{22}$$

which implies

$$\alpha_u \approx k_d - k_u \quad \text{and} \quad \alpha_d \approx -k_d - k_u \ . \tag{23}$$

We can then express $m_u + m_d$ and its heavier colleagues in terms of the family breaking parameters

$$\begin{aligned} m_u + m_d &\approx \alpha_u \ , \\ m_c + m_s &\approx \alpha_u R_u + a_d(R_d - 1) \ , \\ m_b + m_t &\approx \alpha_u R_u^2 + \alpha_d(R_d^2 - 1) \ . \end{aligned} \tag{24}$$

The parameters of the d sector can be calculated from (12), using the (running) quark mass values [1,2] (at $\mu = 1$ GeV)

$$m_d = 8.9 \pm 2.6 \text{ MeV}, \quad m_s = 157 \pm 36 \text{ MeV} \ ,$$

$$m_b = 5.7 \pm 0.07 \text{ GeV}$$

and (25)

$$m_u = 5.1 \pm 1.5 \text{ MeV}, \quad m_c = 1.36 \pm 0.02 \text{ GeV} \ ,$$

the numerical value of R_u can be evaluated, and a top mass value of the order of 120–180 GeV is deduced.

Furthermore, with the whole body of quark mass values inserted in (13) and (19), the measure of the strength of the isospin breaking, $(k_d+k_u)/(k_d-k_u)$, is found to be about -0.27, which is of the same order of magnitude as the $-\frac{1}{3}$ of the relation (18).

IV. THE MASS SPECTRA IN TERMS OF THE "PRIMARY" QUARK MASSES

So far, (12) is just a parametrization of the fermionic mass spectra. Its relevance lies in the possibility of relating the parameters of the different sectors. It turns out that, for the charged lepton (l) mass values

$$m_e=0.511\ \text{MeV},\quad m_\mu=105.66\ \text{MeV}\ ,$$

and (26)

$$m_\tau=1782.4\ \text{MeV}\ ,$$

the parameters of the l, d, and u sectors can be related as follows:

$$\frac{\alpha_u}{\alpha_l}\approx\frac{\alpha_l}{\alpha_d},\quad \alpha_l k_l\approx k_u k_d,\quad \text{and}\quad \frac{R_u}{R_d}\approx\frac{R_d}{R_l}$$

or

$$\alpha_l\approx\sqrt{\alpha_u\alpha_d}\approx\sqrt{k_u^2-k_d^2},$$
$$k_l\approx\frac{k_u k_d}{\sqrt{\alpha_u\alpha_d}}\approx\frac{k_u k_d}{\sqrt{k_u^2-k_d^2}}\ , \tag{27}$$

and

$$R_l\approx R_d^2/R_u\ .$$

Again making use of the idea that the fermionic sectors are primarily characterized by their k_j values and their charges, it is interesting to note that we can alternatively express the leptonic parameters in terms of $k_{u,d}$ and the charges, in the sense that

$$Q_l\alpha_l\approx-2Q_u k_d\ ,$$
$$Q_l k_l\approx Q_u k_d+Q_d k_u \tag{28}$$

so that $Q_l m_e\approx Q_d k_u-Q_u k_d$.

In (23), (27), and (28), k_u and k_d appear as "generating" the α's and k_l. Is it conceivable that the family structure could simply be due to this initial "breaking of the sectorial symmetry"?

In this perspective, k_u and k_d are "primary," they should thus be explained by means of some fundamental structure. One conjecture is that the origin of k_u and k_d would be the "quark-antiquark condensates" $\langle\bar{u}u\rangle$ and $\langle\bar{d}d\rangle$, the "quark-antiquark condensate" $\langle\bar{q}q\rangle$ is invariant under flavor SU(3) transformations, but not under $SU(3)_L\times SU(3)_R$ (and in the limit $m_u=m_d=0$, both k_u and k_d are negative).

If we express the u, d, and l spectra in terms of k_u and k_d, that is, if we consider the spectra normalized in k_d and parametrized in terms of the ratio $\omega=k_u/k_d$, with

$$S=1-\omega$$

and (29)

$$\eta=\frac{\omega}{1+\omega}$$

[which according to (27) means that $S/\eta\approx-\alpha_l k_l$], and with $\hat{D}_u=D_u/k_d, \hat{D}_d=D_d/k_d$, and $\hat{D}_l=D_l/k_d$,

$$\hat{D}_u\approx\omega\mathbb{1}+(1-\omega)\begin{bmatrix}1&0&0\\0&\eta^4S&0\\0&0&\eta^8S^2\end{bmatrix}, \tag{30}$$

$$\hat{D}_d\approx\mathbb{1}-(1+\omega)\begin{bmatrix}1&0&0\\0&\eta^3S&0\\0&0&\eta^6S^2\end{bmatrix}, \tag{31}$$

$$\hat{D}_l\approx\frac{1}{\sqrt{\omega^2-1}}\left[\omega\mathbb{1}-(\omega^2-1)\begin{bmatrix}1&0&0\\0&\eta^2S&0\\0&0&\eta^4S^2\end{bmatrix}\right], \tag{32}$$

it is clear that, in the limit where $\omega\to1$, the family structure disappears, since $S\to0$.

V. THE FAMILY LAGRANGIAN DENSITIES

The parametrization (12) of the fermionic mass spectra can be reformulated as

$$m_1=k+\alpha=\beta+Rm_0\ ,$$
$$m_2=k+\alpha R=\beta+Rm_1\ , \tag{33}$$
$$m_3=k+\alpha R^2=\beta+Rm_2\ ,$$

where $m_0=k+\alpha/R$ and the k_j, R_j, and α_j are given by (13), and $\beta_j=k_j(1-R_j)$. This means that the masses can be considered as functions of R:

$$m_1(R)=\beta+Rm_0=k(1-R)+Rm_0\ ,$$
$$m_2(R)=\beta+Rm_1=k(1-R)+Rm_1\ , \tag{34}$$
$$m_3(R)=\beta+Rm_2=k(1-R)+Rm_2\ ,$$

which permits us, while m_0 is considered as a constant, to interpret the $k_j\mathbb{1}$ term in (12) as the extremum matrix (the j indices suppressed):

$$D_{\text{extremum}}=\begin{vmatrix}m_1(R)&0&0\\0&m_2(R)&0\\0&0&m_3(R)\end{vmatrix}_{|\partial m_n/\partial R=0}$$
$$=k\mathbb{1}\ ,\qquad n=1,2,3\ . \tag{35}$$

Furthermore, in the parametrization (33) the masses can be regarded as generated by a "mass boost" $\mathcal{B}_m$, such that

$$m_1=\mathcal{B}_m(m_0)=\beta+Rm_0\ ,$$
$$m_2=\mathcal{B}_m(m_1)=\beta+Rm_1=\beta+R\beta+R^2m_0\ , \tag{36}$$
$$m_3=\mathcal{B}_m(m_2)=\beta+Rm_2=\beta+R\beta+R^2\beta+R^3m_0\ ,$$

where the β's, R's, and m_0's are as given above.

This permits us to introduce, in analogy with (10), starting from the "initial" Lagrangian density

$$\mathcal{L}_i = i\bar{\psi}\gamma_\mu\partial\mu\psi - m_0\bar{\psi}\psi ,$$

the "family Lagrangian density"

$$\mathcal{L}_n = i\bar{\psi}_n\gamma_\mu\partial^\mu\psi_n - (\beta + Rm_{n-1})\bar{\psi}_n\psi_n , \tag{37}$$

where m_n, $n=1,2,3$ are the fermion mass eigenvalues within the given charge sector. According to (27), $R_u \approx R_d^2/R_l$, and with the mass values (25) and (26) we also have that

$$\beta_u \approx -\beta_d^3/\beta_l^2 . \tag{38}$$

In the limit where the first family is massless, the m_0 satisfies $m_0 = -\beta/R$. Still with a massive first family, these relations are approximately satisfied, and $m_0 \approx -\beta/R$. So the form of the "family Lagrangian density" in the chiral limit of the light fermions is also possibly relevant for the massive case.

Vanishing masses in the first family corresponds to

$$\begin{aligned} \mathcal{L}_1 &= i\bar{\psi}\gamma_\mu\partial^\mu\psi , \\ \mathcal{L}_2 &= i\bar{\psi}\gamma_\mu\partial^\mu\psi - \beta\bar{\psi}\psi , \\ \mathcal{L}_3 &= i\bar{\psi}\gamma_\mu\partial^\mu\psi - \beta(1+R)\bar{\psi}\psi . \end{aligned} \tag{39}$$

However, when $\mathcal{L}_2$ and $\mathcal{L}_3$ are simply considered as part of a *series* of Lagrangian densities, with mass terms

$$(\ldots, \mathcal{L}_{m_2}, \mathcal{L}_{m_3}) = (\ldots, -\beta\bar{\psi}\psi, -\beta(1+R)\bar{\psi}\psi, \ldots) , \tag{40}$$

the predecessor of $\mathcal{L}_2$ in this "series" (40) should have the mass term

$$\hat{\mathcal{L}}_{m_1} = -\frac{\beta}{(1+R)}\bar{\psi}\psi . \tag{41}$$

It turns out that the coefficient $\beta/(1+R)$ is "relevant," in the sense that from the parameters β_u^0, β_d^0 and R_u^0, R_d^0, corresponding to the chiral limit of the light quarks, the nonzero masses m_u, and m_d can be "deduced," as

$$\begin{aligned} m_u &\approx \frac{1}{2}\left[\left[\frac{\beta}{1+R}\right]_u^0 - \left[\frac{\beta}{1+R}\right]_d^0\right] , \\ m_d &\approx \frac{1}{2}\left[\left[\frac{\beta}{1+R}\right]_u^0 + \left[\frac{\beta}{1+R}\right]_d^0\right] . \end{aligned} \tag{42}$$

With a massless first family and the following quark mass values for the second and third family, $m_s = 150$ MeV, $m_b = 5.7$ GeV, $m_c = 1.35$ GeV, and $m_t = 130$ GeV, we get, according to (42), $m_d \approx 8.9$ MeV and $m_u \approx 5$ MeV.

In terms of mass eigenvalues, this "prediction" corresponds to the relations

$$\begin{aligned} m_t &\approx \frac{m_c^2 m_b}{2m_d m_b - m_s^2} , \\ m_d &\approx m_u + \frac{m_s^2}{m_b} , \end{aligned} \tag{43}$$

which leads to the relations

$$\begin{aligned} m_b &\approx \frac{m_s^2}{m_d - m_u} \approx \frac{-m_s^2}{k_d + k_u} , \\ m_t &\approx \frac{m_c^2}{m_d + m_u} \approx \frac{m_c^2}{k_d - k_u} . \end{aligned} \tag{44}$$

In the same spirit as in (28), the corresponding leptonic expression would be

$$m_\tau \approx \frac{-Q_l m_\mu^2}{Q_u k_d + Q_d k_u} . \tag{45}$$

VI. QUARK MASS VALUES "DERIVED" FROM THE LEPTONIC MASS VALUES

Since the lepton masses so to speak exist at an other ontological level than the semitheoretical, semiempirical objects that are the quark masses, one way of testing the obtained relations between the parameters of the different sectors, is to attempt to express the *l*, *d*, and *u* mass spectra in terms of the leptonic masses.

Using the relations (27), I express all the spectra in terms of the parameters k_l and $F = -\rho - \sqrt{\rho^2+4}$, where $\rho = \alpha_l/k_l$. Rewriting $R_l/R_d \approx R_d/R_u$ as

$$\begin{aligned} R_d &= \eta R_l , \\ R_u &= \eta^2 R_l , \end{aligned} \tag{46}$$

where

$$\eta = \frac{2}{2+F} , \tag{47}$$

the spectra expressed in terms of the lepton masses are

$$D_l = G\left[\frac{2F}{\sqrt{4-F^2}}\mathbb{1} + \sqrt{4-F^2}\begin{bmatrix}1 & 0 & 0\\ 0 & R_l & 0\\ 0 & 0 & R_l^2\end{bmatrix}\right] , \tag{48}$$

$$D_d \approx G\left[-F\mathbb{1} + (2+F)\begin{bmatrix}1 & 0 & 0\\ 0 & \eta R_l & 0\\ 0 & 0 & \eta^2 R_l^2\end{bmatrix}\right] , \tag{49}$$

and

$$D_u \approx G\left[-2\mathbb{1} + (2-F)\begin{bmatrix}1 & 0 & 0\\ 0 & \eta^2 R_l & 0\\ 0 & 0 & \eta^4 R_l^2\end{bmatrix}\right] , \tag{50}$$

where

$$G = \frac{k_l}{2F}\sqrt{4-F^2} . \tag{51}$$

Using the charged lepton mass values (26), this parametrization corresponds to the quark mass values

$$m_d \approx 8.8 \text{ MeV}, \quad m_s \approx 145 \text{ MeV}, \quad m_b \approx 5542 \text{ MeV} ,$$

and (52)

$$m_u \approx 5.2 \text{ MeV}, \quad m_c \approx 1365 \text{ MeV}, \quad m_t \approx 135 \text{ GeV} ,$$

corresponding to running (1 GeV) masses [1,2].

VII. DEPENDENCE ON ONLY TWO LEPTONIC PARAMETERS

If R_l is expressed as a function of the other leptonic parameters, i.e.,

$$R_l \approx \eta^2 S , \tag{53}$$

the mass spectra can be expressed as functions of only m_e and m_μ (and m_τ is "predicted" from the formalism). The mass spectra then take the form

$$D_l = \tilde{G} \left[\frac{2\tilde{F}}{\sqrt{4-\tilde{F}^2}} \mathbb{1} + \sqrt{4-\tilde{F}^2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \tilde{\eta}^2 S & 0 \\ 0 & 0 & \tilde{\eta}^4 S^2 \end{bmatrix} \right] , \tag{54}$$

$$D_d \approx \tilde{G} \left[-\tilde{F}\mathbb{1} + (2+\tilde{F}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & \tilde{\eta}^3 S & 0 \\ 0 & 0 & \tilde{\eta}^6 S^2 \end{bmatrix} \right] , \tag{55}$$

and

$$D_u \approx \tilde{G} \left[-2\mathbb{1} + (2-\tilde{F}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & \tilde{\eta}^4 S & 0 \\ 0 & 0 & \tilde{\eta}^8 S^2 \end{bmatrix} \right] , \tag{56}$$

where

$$\tilde{G} = \frac{\tilde{k}_l}{2\tilde{F}} \sqrt{4-\tilde{F}^2} \tag{57}$$

and

$$S = \frac{\tilde{F}-2}{\tilde{F}} , \tag{58}$$

$$\tilde{\eta} = \frac{2}{2+\tilde{F}} , \tag{59}$$

and $\tilde{F} = -\tilde{\rho} - \sqrt{\tilde{\rho}^2+4}$, where $\tilde{\rho} = \tilde{\alpha}_l / \tilde{k}_l$.

Like α_l and k_l, the parameters $\tilde{\alpha}_l$ and $\tilde{k}_l$ satisfy the relations $\tilde{\alpha}_l + \tilde{k}_l = m_e$, etc., but these parameters can no longer simply be determined by (13). If we suppose that m_e and m_μ are given, and we equate $\tilde{\eta}^2 S$ with

$$R_l = (m_\mu - \tilde{k}) / (m_e - \tilde{k}) ,$$

$\tilde{k}$ can be found, and thereby $\tilde{\alpha}$ as well. With m_e and m_μ fixed, we can thus "derive" a m_τ value, $m_\tau \approx 1832.3$ MeV. Furthermore, with these $\tilde{\alpha}_l$ and $\tilde{k}_l$ values we get another set of quark mass values: viz.,

$$m_d \approx 8.5 \text{ MeV}, \quad m_s \approx 145 \text{ MeV}, \quad m_b \approx 5677 \text{ MeV} ,$$

and (60)

$$m_u \approx 5.1 \text{ MeV}, \quad m_c \approx 1357 \text{ MeV}, \quad m_t \approx 138 \text{ GeV} .$$

These values of m_d, m_s, m_b, m_u, and m_c, as well as the corresponding ones in (52), obviously agree well with (25).

VIII. THE NEUTRINO MASS SPECTRUM

Even with the assumption that the neutral leptons are massless, $m_{\nu_j} = 0$, $j = e, \mu, \tau$, the neutrino mass spectrum can still be written as

$$D_\nu = k_\nu \mathbb{1} + \alpha_\nu \begin{bmatrix} 1 & 0 & 0 \\ 0 & R_\nu & 0 \\ 0 & 0 & R_\nu^2 \end{bmatrix} , \tag{61}$$

where, in the massless case, $R_\nu = 1$ and $\alpha_\mu = -k_\nu$. If we instead assume that the neutrinos have nonzero masses, the m_{ν_μ} and m_{ν_τ} are related by

$$m_{\nu_\tau} = \frac{(m_{\nu_\mu} - k_\nu)^2 + \alpha_\nu k_\nu}{\alpha_\nu} . \tag{62}$$

Even in the massive case, m_{ν_e} presumably is quite small, hence $\alpha_\nu \approx -k_\nu$. Inserted in (62), this leads to

$$m_{\nu_\tau} \approx 2 m_{\nu_\mu} - m_{\nu_\mu}^2 / k_\nu . \tag{63}$$

$\alpha_\nu \approx -k_\nu$ also implies that $m_{\nu_\mu} \approx k_\nu (1-R)$ and $m_{\nu_\tau} \approx k_\nu (1-R^2)$. In any case, $R \geq 1$, which means that in order to have positive masses k_ν must be negative, which according to (63) means that $m_{\nu_\tau} \geq 2 m_{\nu_\mu}$. A ν_μ mass value less than 0.27 MeV [3], gives $2 m_{\nu_\mu} < 0.54$ MeV. This corresponds to a limit on m_{ν_τ} in accordance with the τ-neutrino range

$$0.5 \text{ MeV} < m_{\nu_\tau} < 0.74 \text{ MeV}$$

deduced from primordial helium considerations [4].

IX. THE QUARK MASS MATRICES

With the mass spectra given by (12), the mass matrices m and m' of the charge $\frac{2}{3}$- and $-1/3$-quark sectors correspondingly, take the form $m_j = k_j \mathbb{1} + \mathcal{R}_j$, which in the mass basis of the $\frac{2}{3}$ sector corresponds to

$$m = D_u = k_u \mathbb{1} + \alpha_u \begin{bmatrix} 1 & 0 & 0 \\ 0 & R_u & 0 \\ 0 & 0 & R_u^2 \end{bmatrix} \tag{64}$$

and

$$m' = V D_d V^\dagger , \tag{65}$$

where D_d is given by (12), and V is the weak mixing matrix. Explicitly, m' can be written as

$$m' = k_d \mathbb{1} + \alpha_d F_1 + \alpha_d R_d F_2 + \alpha_d R_d^2 F_3 , \tag{66}$$

where F_j, $j = 1,2,3$, are projection matrices, given by

$$F_j = \begin{bmatrix} V_{1j} & 0 & 0 \\ 0 & V_{2j} & 0 \\ 0 & & 0 V_{3j} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} V_{1j}^* & 0 & 0 \\ 0 & V_{2j}^* & 0 \\ 0 & 0 & V_{3j}^* \end{bmatrix} . \tag{67}$$

That is, $(F_j)_{\alpha\beta}=V_{\alpha j}V^*_{\beta j}$, where V_{ij} are the matrix elements of the weak mixing matrix. Each F_j corresponds to a (mass) eigenstate, so naturally, since the mixing matrix is unitary, $F_1F_3=\mathbb{0}$, $F_1F_2=\mathbb{0}$ and $F_2F_3=\mathbb{0}$, $F_1+F_2+F_3=\mathbb{1}$, and $F_j^2=F_j, j=1,2,3$.

X. IN AN ARBITRARY BASIS

The quark mass matrices (64) and (66) clearly retain traces of their initial degenerate form in any basis, presumably also in the weak basis. The transformation from the mass basis of m to an arbitrary basis B_a is made by means of the unitary transformation matrix X_a, whereby the quark mass matrices m and m' take the form

$$m=X_a\left[k_u\mathbb{1}+\alpha_u\begin{bmatrix}1&0&0\\0&R_u&0\\0&0&R_u^2\end{bmatrix}\right]X_a^\dagger \tag{68}$$

$$=k_u\mathbb{1}+\alpha_u A_1+\alpha_u R_u A_2+\alpha_u R_u^2 A_3$$
$$=G[-2\mathbb{1}+(2-F)(A_1+R_uA_2+R_u^2A_3)] \tag{69}$$

and

$$m'=X_a(k_d\mathbb{1}+\alpha_dF_1+\alpha_dR_dF_2+\alpha_dR_d^2F_3)X_a^\dagger$$
$$=k_d\mathbb{1}+\alpha_dB_1+\alpha_dR_dB_2+\alpha_dR_d^2B_3$$
$$=G[-F\mathbb{1}+(2+F)(B_1+R_dB_2+R_d^2B_3)]\ , \tag{70}$$

where F and G are given in Sec. VI and A_j and B_j are projection matrices with $A_iA_j=\mathbb{0}$, $B_iB_j=\mathbb{0}$ for $i\neq j$, and $A_1+A_2+A_3=\mathbb{1}$, $B_1+B_2+B_3=\mathbb{1}$, $A_j^2=A_j$, $B_j^2=B_j$. The matrices m and m' are formally similar, with A_j and B_j, $j=1,2,3$, $\mathbb{0}$ given by

$$A_j=\begin{bmatrix}X_{1j}&0&0\\0&X_{2j}&0\\0&0&X_{3j}\end{bmatrix}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}X^*_{1j}&0&0\\0&X^*_{2j}&0\\0&0&X^*_{3j}\end{bmatrix}, \tag{71}$$

where X_{ij} are the matrix elements of X_a; and

$$B_j=\begin{bmatrix}Z_{1j}&0&0\\0&Z_{2j}&0\\0&0&Z_{3j}\end{bmatrix}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}Z^*_{1j}&0&0\\0&Z^*_{2j}&0\\0&0&Z^*_{3j}\end{bmatrix}, \tag{72}$$

where

$$Z_{1j}=V_{1j}X_{11}+V_{2j}X_{12}+V_{3j}X_{13}\ ,$$
$$Z_{2j}=V_{1j}X_{21}+V_{2j}X_{22}+V_{3j}X_{23}\ , \tag{73}$$
$$Z_{3j}=V_{1j}X_{31}+V_{2j}X_{32}+V_{3j}X_{33}\ ,$$

and X_{ij} and V_{ij} are the matrix elements of X_a and the weak mixing matrix, respectively. The "close-to-unity" structure of the mixing matrix implies that the Z_{ij} are, in a sense, not so different from the X_{ij}. That is, the structure of the B_j is close to that of the A_j, because the "close-to-unity" structure of the mixing matrix implies that the V_{jj} that multiplies the X_{ij} in each Z_{ij}, is close to one, $V_{jj}\approx 1$.

In order to illustrate this, consider m and m' in some basis which is reached from the mass basis of m by a simple unitary rotation such as, for instance,

$$X=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}. \tag{74}$$

The matrices A_j and B_j then take the form

$$A_1=\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}=\mathcal{A}_1\mathcal{Y}\mathcal{A}_1^*\ , \tag{75}$$

$$A_2=\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}=\mathcal{A}_2\mathcal{Y}\mathcal{A}_2^*\ , \tag{76}$$

$$A_3=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}=\mathcal{A}_3\mathcal{Y}\mathcal{A}_3^* \tag{77}$$

and

$$B_1=\begin{bmatrix}V_{31}&0&0\\0&V_{11}&0\\0&0&V_{21}\end{bmatrix}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\times\begin{bmatrix}V^*_{31}&0&0\\0&V^*_{11}&0\\0&0&V^*_{21}\end{bmatrix}=\mathcal{B}_1\mathcal{Y}\mathcal{B}_1^*\ , \tag{78}$$

$$B_2=\begin{bmatrix}V_{32}&0&0\\0&V_{12}&0\\0&0&V_{22}\end{bmatrix}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\times\begin{bmatrix}V^*_{32}&0&0\\0&V^*_{12}&0\\0&0&V^*_{22}\end{bmatrix}=\mathcal{B}_2\mathcal{Y}\mathcal{B}_2^*\ , \tag{79}$$

$$B_3=\begin{bmatrix}V_{33}&0&0\\0&V_{13}&0\\0&0&V_{23}\end{bmatrix}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\times\begin{bmatrix}V^*_{33}&0&0\\0&V^*_{13}&0\\0&0&V^*_{23}\end{bmatrix}=\mathcal{B}_3\mathcal{Y}\mathcal{B}_3^*\ , \tag{80}$$

where

$$\mathcal{Y}=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix} \tag{81}$$

and

$$\mathcal{A}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \tag{82}$$

$$\mathcal{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \tag{83}$$

etc. The only nonzero element in $\mathcal{A}_1$ is $\mathcal{A}_{1(22)}$, and correspondingly in $\mathcal{B}_1$, the biggest element is $\mathcal{B}_{1(22)} = V_{11} \approx 1$. Similarly the only nonzero element in $\mathcal{A}_2$ is $\mathcal{A}_{2(33)}$, and $\mathcal{B}_{2(33)} = V_{33} \approx 1$ is the biggest element in $\mathcal{B}_2$; and so on.

In this sense the structure of $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ is similar to that of $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$. It is the "close-to-unity" structure of the mixing matrix that brings about a similar pattern in the B_j and the A_j, corresponding to a formal similarity between the quark mass matrices m and m'.

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