PHYSICAL REVIEW D VOLUME 48, NUMBER 2 15 JULY 1993

SO(10) cosmic strings and baryon-number violation

Chung-Pei Ma
Center for Theoretical Physics, Laboratory for Nuclear Science, and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 5 October 1992)

SO(10) cosmic strings formed during the phase transition Spin(10) $\rightarrow$ SU(5)$\times Z_2$ are studied. Two types of strings, one effectively Abelian and one non-Abelian, are constructed and the string solutions are calculated numerically. The non-Abelian string can catalyze baryon-number violation via the "twisting" of the scalar field which causes mixing of leptons and quarks in the fermion multiplet. The non-Abelian string is also found to have the lower energy, possibly for the entire range of the parameters in the theory. Scattering of fermions in the fields of the strings is analyzed, and the baryon-number-violation cross section is calculated. The role of the self-adjoint parameters is discussed and the values are computed.

PACS number(s): 98.80.Cq; 11.17.+y; 12.10.Gq

I. INTRODUCTION

Motivated by the Callan-Rubakov effect in the context of magnetic monopoles [1], studies have been carried out recently on the possibility that cosmic strings can also catalyze baryon-number violation with strongly enhanced cross sections. It has been shown that the wave function of a fermion scattering off a cosmic string can acquire a large amplification factor near the core of the string, leading to enhancement of the processes that violate the baryon number inside the string [2, 3]. The catalysis processes that have been studied include those mediated by scalar fields and by the grand-unified X and Y gauge bosons in the string core. Although strings, in contrast to monopoles, have no magnetic fields outside, fermions can interact quantum mechanically with the long-range gauge fields via the Aharonov-Bohm effect. Depending on the flux of the string and the core model used, the enhanced catalysis cross sections (per length) can be of the scale of strong interactions in comparison to the much smaller geometrical cross section $\sim \Lambda_{\rm GUT}^{-1}$, where $\Lambda_{\rm GUT} \sim 10^{16}$ GeV. In the early Universe when the density of cosmic strings is high, such processes can play important roles, washing out any primordially generated baryon asymmetry [4], or conceivably even generating the baryon to entropy ratio observed today.

Cosmic strings can be produced during certain phase transitions when a gauge group G is broken down to a subgroup H by the vacuum expectation value of some scalar field ϕ. The topological criterion for the existence of a string is a nontrivial fundamental homotopy group of the vacuum manifold G/H, denoted by $\pi_1(G/H)$. For a connected and simply connected G, the general construction of the scalar field at large distances from the string is given by

$$\phi(\theta) = g(\theta)\phi_0\,, \quad g(\theta) = e^{i\tau\theta}\,. \tag{1}$$

Here τ is some generator of G, θ is the azimuthal angle measured around the string, and $g(0)$ and $g(2\pi)$ belong to two disconnected pieces of H. In the papers referenced in the previous paragraph, the scalar field responsible for the formation of the string is taken to have the simple form $\phi(\theta) = e^{i\tau\theta}\phi_0 = e^{i\theta}\phi_0$. As a result, a non-Abelian string can be modeled by a U(1) vortex, and the scattering of fermions in the background fields of the string is governed by the Abelian Dirac equation. In general however, for a given ϕ_0, the generator τ can be chosen such that $e^{i\tau\theta}\phi_0$ "twists" around the string in more complicated fashion than a phase $e^{i\theta}$ times ϕ_0. This gives rise to dynamically different strings which are intrinsically non-Abelian [5]. One expects the complexity and rich structure of such strings to lead to interesting effects on fermions traveling around them. In particular, we will demonstrate in this paper that for certain τ's, the twisting of $\phi(\theta)$ can result in mixing of lepton and quark fields, providing a mechanism for baryon number violations distinct from the processes in Abelian strings studied previously.

Since no strings are formed in the minimal SU(5) model, we choose the gauge group SO(10) [6] in this paper as an example of grand-unified theories in investigating the B-violating process. We will construct string configurations, solve numerically for the undetermined functions, and study the baryon catalysis in the SO(10) theory, although we expect such processes to occur in other non-Abelian theories as well. In SO(10), stable strings can be formed when Spin(10), the simply connected covering group of SO(10), is broken down to SU(5)$\times Z_2$ by the vacuum expectation value of a Higgs field ϕ in the **126** representation [7]. It must be pointed out, however, that the subsequent symmetry-breaking SU(5)$\times Z_2$ $\rightarrow$SU(3)$\times$SU(2)$\times$U(1)$\times Z_2$ produces magnetic monopoles which would have to be eliminated to provide a consistent cosmological scenario. If the monopoles were eliminated by inflation, then the cosmic strings would also disappear. While other symmetry-breaking patterns for SO(10) are possible, we use Spin(10)$\rightarrow$SU(5)$\times Z_2$ because it produces strings in the simplest way. The emphasis here will be on studying the particle physics.

The generators of SO(10) transform as the adjoint **45**, which transforms as $\mathbf{24}+\mathbf{1}+\mathbf{10}+\overline{\mathbf{10}}$ under SU(5). The **24** and **1** generate the subgroup SU(5)×U(1), where the U(1) includes simultaneous rotations in the 1-2, 3-4, 5-6, 7-8, and 9-10 planes. We are interested in the generators outside SU(5) because, to have noncontractible loops at all, $g(\theta)$ in Eq. (1) has to be outside the unbroken H for some θ. We will refer to the U(1) generator as $\tau_{\rm all}$ and to any of the other 20 basis generators outside SU(5) as τ_1; we name the associated strings as string $\tau_{\rm all}$ and string τ_1, respectively. As we shall see, the scalar field of string τ_1 causes mixing of leptons and quarks while string $\tau_{\rm all}$ is effectively Abelian and no such mixing occurs. Properties of string $\tau_{\rm all}$ such as the string mass per unit length [8] and its superconducting capability in terms of fermion zero modes [9] have been studied. We will compare it with string τ_1, which will be the main subject of study of this paper.

In Sec. II, we give more detailed discussion of the Higgs **126** and the breaking of Spin(10) to SU(5)×Z_2, and elaborate on the B-violating mechanism due to the nontrivial winding of the Higgs field. In Sec. III, we write down an ansatz for the field configuration of each string and derive the corresponding equations of motion. The numerical solutions and the energy of the strings are presented in Sec. IV, where we find that τ_1 strings have lower energy than $\tau_{\rm all}$ strings, probably for the entire range of the parameters in the theory. Having shown that such strings are energetically favorable, we turn to the scattering problem in Sec. V, where the Dirac equation in the background fields of the strings is solved, and the differential cross section for the B-violating processes in string τ_1 is calculated. We also comment on the role of the self-adjoint parameters and compute their values using our string solutions. To establish a common notation and to facilitate reading of this paper, we include in the Appendix a discussion about the relevant aspects of the spinor representation **16** of SO(10), which accommodates a single generation of left-handed fermions.

II. SO(10) STRINGS

There is considerable freedom in the breakings of SO(10) down to the low-energy gauge group SU(3)×U(1). Two commonly studied examples include the breaking via an intermediate SU(5), SO(10)→SU(5), and the one via an intermediate Pati-Salam SU(4)×SU(2)$_L$×SU(2)$_R$ [10]. Details of the symmetry-breaking patterns and the Higgs fields inducing the breakings can be found in Ref. [6] and the papers by Slansky and Rajpoot [11]. Kibble, Lazarides, and Shafi argued that the strings formed during the phase transition SO(10) →SU(4)×SU(2)$_L$×SU(2)$_R$ become boundaries of domain walls [7]. Thus in this paper we choose the SU(5) breaking pattern instead for its simplicity. More precisely, we study strings formed when Spin(10)→SU(5)×Z_2 by the vacuum expectation value of a Higgs **126** ϕ. The nontrivial element of Z_2 corresponds to rotation by 2π in SO(10). The homotopy group $\pi_1[\mathrm{Spin}(10)/\mathrm{SU}(5)\times Z_2]$ is Z_2; therefore a Z_2 string is formed during this phase transition. The subsequent symmetry breakings can be implemented by the adjoint **45** of SO(10) and the fundamental **10** in the usual fashion:

$$\begin{aligned}\mathrm{Spin}(10) &\xrightarrow{\mathbf{126}} \mathrm{SU}(5)\times Z_2\\ &\xrightarrow{\mathbf{45}} \mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)\times Z_2\\ &\xrightarrow{\mathbf{10}} \mathrm{SU}(3)\times\mathrm{U}(1)_{\rm em}\times Z_2\,. \end{aligned}\tag{2}$$

This Z_2 string survives all the symmetry breakings since Z_2 is preserved at low energies.

The **126** representation consists of fifth-rank antisymmetric tensors satisfying the self-duality condition

$$\phi_{i_1\cdots i_5}=\frac{i}{5!}\epsilon_{i_1\cdots i_{10}}\phi_{i_6\cdots i_{10}}.\tag{3}$$

The component which acquires an expectation value $\langle\phi\rangle$ transforms as an SU(5) singlet, and to write it down explicitly, we first specify how the SU(5) subgroup is embedded in SO(10). The fundamental representation of SO(10) consists of 10×10 matrices, which can be labeled by indices $i,i=1,\ldots,10$. The generators of SO(10) in this representation can be written as antisymmetric, purely imaginary matrices. The generators of SU(5) in the fundamental representation are Hermitian, traceless 5×5 matrices which can be written as

$$\tau_{\alpha\beta}=S_{\alpha\beta}+iA_{\alpha\beta}\,,\tag{4}$$

where $\alpha,\beta=1,\ldots,5$ label the matrix elements, and S,A are real 5×5 matrices, representing the real and imaginary parts of τ. Hermiticity and tracelessness of τ require $S_{\alpha\beta}=S_{\beta\alpha}$, $A_{\alpha\beta}=-A_{\beta\alpha}$, and $\mathrm{Tr}S=0$. A natural way to embed SU(5) in SO(10) is to treat five-dimensional complex vectors as ten-dimensional real vectors, i.e., replace the paired indices (α,a), where $\alpha=1,\ldots,5$ label a five-dimensional vector and $a=1,2$ label its real and imaginary parts, by the index i, $i=1,\ldots,10$. Then, the generators of the subgroup SU(5) of SO(10) can be expressed as

$$\tau_{\alpha a,\,\beta b}=i(A_{\alpha\beta}I_{ab}+S_{\alpha\beta}M_{ab})\,,\tag{5}$$

where I is the 2×2 identity matrix and $M=i\sigma_2$, σ_2 being the second 2×2 Pauli matrix. One can convince oneself that in this (α,a) notation, the rank-five antisymmetric Levi-Civita tensor $\epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}$ which transforms as an SU(5) singlet in the SU(5) notation becomes

$$i^{f(a_1\cdots a_5)}\epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}\,,\tag{6}$$

where $f(a_1\cdots a_5)$ is defined to equal the number of a_i that takes the value 2. It is also straightforward to check that this expression satisfies the self-duality condition [Eq. (3)]. Thus $\langle\phi\rangle$ is written as

$$\langle\phi_{\alpha_1a_1\cdots\alpha_5a_5}\rangle=\mu\, i^{f(a_1\ldots a_5)}\epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}\,,\tag{7}$$

where μ is a parameter.

Some words about our notation. The tensor indices $i_1,\ldots,i_5$ of $\phi_{i_1\cdots i_5}$ will be suppressed for convenience and legibility whenever no ambiguity should arise. In the expressions like $\tau\phi$ and $e^{i\tau\theta}\phi$ where τ operates on ϕ, τ is understood to be in the same representation of ϕ; i.e., τ is the shorthand for

$$\tau_{i_1\cdots i_5 j_1\cdots j_5} = \tau_{i_1 j_1}\delta_{i_2 j_2}\cdots\delta_{i_5 j_5} + \delta_{i_1 j_1}\tau_{i_2 j_2}\cdots\delta_{i_5 j_5} + \cdots . \tag{8}$$

With the symmetry-breaking Spin(10)→SU(5)×Z_2, strings are formed. At spatial infinity, the general form of ϕ is given by Eq. (1). For the energy to be finite, the covariant derivative of ϕ, $D_\mu\phi \equiv \partial_\mu\phi + eA_\mu\phi$, has to vanish at spatial infinity; therefore the gauge field A_μ takes the form $A^\theta = i\frac{1}{er}\tau$, $A^r = 0$, as $r \to \infty$. In the core of the string, there is a magnetic flux $\oint \mathbf{A}\cdot dl = \frac{2\pi}{e}\tau$ pointing in the direction of τ in group space. Strings carrying flux pointing in different directions in group space are topologically equivalent since the only nontrivial winding number here is one, but dynamically they can differ. Because the scalar field $\phi(\theta)$ varies with θ, the embedding of the unbroken subgroup SU(5) in SO(10) outside the string also varies with θ. More precisely, the generators $\tau_\theta^a, a = 1,\ldots,24$ of the unbroken SU(5) at θ are related to the generators τ_0^a of the unbroken SU(5) at $\theta = 0$ by the similarity transformation

$$\tau_\theta^a = g(\theta)\tau_0^a g^{-1}(\theta)\,, \quad g(\theta) = e^{i\tau\theta}\,. \tag{9}$$

Consequently, the fermion fields which transform as **1**, $\bar{\mathbf{5}}$, and **10** under SU(5) are also rotated as one goes around the string. How the fields mix depends on which direction in group space $\phi(\theta)$ winds.

The SO(10) generators can be written as 10×10 matrices of the form $(\tau^{ab})_{ij} = -i(\delta_i^a\delta_j^b - \delta_i^b\delta_j^a)$, where a, b label the group indices, i, j label the matrix elements, and a, b, i, j all run from 1 to 10. In this notation τ_{all} is given by

$$\tau_{\text{all}} \equiv \frac{1}{5}(\tau^{12} + \tau^{34} + \cdots + \tau^{9\,10})\,, \tag{10}$$

where the factor of 1/5 is included for $\phi(\theta)$ to have a 2π rotational period. It takes a little more effort to write down the τ_1's. Let us first write the SU(5) generators specified by Eq. (5) in terms of τ^{ab} given above. The four diagonal generators are trivial. For the other 20 generators, one can group the 10×10 space into 2×2 blocks, and write the 45 τ^{ab}'s as $\tau^{2\alpha-1,\,2\beta-1}$, $\tau^{2\alpha-1,\,2\beta}$, $\tau^{2\alpha,\,2\beta-1}$, and $\tau^{2\alpha,2\beta}$, where α, β both run from 1 to 5. Then it is not hard to see that the 20 linear combinations

$$\begin{aligned}&\frac{1}{2}(\tau^{2\alpha-1,\,2\beta} - \tau^{2\alpha,\,2\beta-1})\,,\\ &\frac{1}{2}(\tau^{2\alpha-1,\,2\beta-1} + \tau^{2\alpha,\,2\beta})\,, \quad \alpha < \beta,\end{aligned} \tag{11}$$

are all of the form of Eq. (5), and therefore can be chosen to be the 20 off-diagonal generators of SU(5). Note that the superscripts α, β above label the group indices while the subscripts α, β in Eq. (5) label the matrix elements. The 20 τ_1's outside SU(5) then can be expressed by the other 20 linear combinations as

$$\begin{aligned}\tau_1 \equiv\ &\frac{1}{2}(\tau^{2\alpha-1,\,2\beta} + \tau^{2\alpha,\,2\beta-1})\,,\\ &\frac{1}{2}(\tau^{2\alpha-1,\,2\beta-1} - \tau^{2\alpha,\,2\beta})\,, \quad \alpha < \beta\,.\end{aligned} \tag{12}$$

Other than the SU(5) group properties, the linear combinations above can also be classified under the group SO(4), which is locally isomorphic to SU(2)×SU(2). For a given α and β where $\alpha < \beta$, the two generators of Eq. (11) plus the diagonal

$$\frac{1}{2}(\tau^{2\alpha-1,\,2\alpha} - \tau^{2\beta-1,\,2\beta}) \tag{13}$$

can be easily shown to obey the SU(2) algebra. Similarly, the two generators of Eq. (12) plus

$$\frac{1}{2}(\tau^{2\alpha-1,\,2\alpha} + \tau^{2\beta-1,\,2\beta}) \tag{14}$$

generate another SU(2). Thus, for a given α and β ($\alpha < \beta$), the six generators of Eqs. (11)–(14) generate rotations in the four-dimensional space spanned by vectors in the $2\alpha-1, 2\alpha, 2\beta-1, 2\beta$ directions.

III. FIELD CONFIGURATIONS

The relevant part of the Lagrangian for the SO(10) theory is given by

$$\mathcal{L} = \frac{1}{4}\text{tr}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - V(\phi), \tag{15}$$

where $F_{\mu\nu} = -iF_{\mu\nu}^a\tau_a$, $A_\mu = -iA_\mu^a\tau_a$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + e[A_\mu, A_\nu]$, $D_\mu = \partial_\mu + eA_\mu$; $A_\mu^a, a = 1,\ldots,45$, are the SO(10) gauge fields and ϕ is the Higgs **126**. The most general gauge-invariant and renormalizable potential $V(\phi)$ contains all the distinct contractions of two and four ϕ's:

$$\begin{aligned}V(\phi) = {}& v_1\phi_{i_1\cdots i_5}\phi^*_{i_1\cdots i_5} + v_2(\phi_{i_1\cdots i_5}\phi^*_{i_1\cdots i_5})^2 + v_3\phi_{i_1 n_2 n_3 n_4 n_5}\phi^*_{j_1 n_2 n_3 n_4 n_5}\phi_{i_1 \ell_2 \ell_3 \ell_4 \ell_5}\phi^*_{j_1 \ell_2 \ell_3 \ell_4 \ell_5}\\ &+ v_4\phi_{i_1 i_2 n_3 n_4 n_5}\phi^*_{j_1 j_2 n_3 n_4 n_5}\phi_{i_1 i_2 \ell_3 \ell_4 \ell_5}\phi^*_{j_1 j_2 \ell_3 \ell_4 \ell_5} + v_5\phi_{i_1 j_2 n_3 n_4 n_5}\phi^*_{j_1 i_2 n_3 n_4 n_5}\phi_{i_1 i_2 \ell_3 \ell_4 \ell_5}\phi^*_{j_1 j_2 \ell_3 \ell_4 \ell_5}\\ &+ v_6\phi_{i_1 i_2 j_3 n_4 n_5}\phi^*_{j_1 j_2 i_3 n_4 n_5}\phi_{i_1 i_2 i_3 \ell_4 \ell_5}\phi^*_{j_1 j_2 j_3 \ell_4 \ell_5}\,.\end{aligned} \tag{16}$$

In writing down the v_3 through v_6 terms above, one has to consider two things: (1) the possible ways to contract the indices, and (2) which ϕ's are to be complex conjugated. One can deal with (1) without the complication of (2) by adopting an equivalent real **252** representation for ϕ because a complex, self-dual 126-dimensional tensor can be thought of as a real, 252-dimensional tensor by dropping the self-duality condition and taking the real parts of the resulting complex, 252-dimensional tensor. One can see there are only four distinct terms and they are terms v_3 through v_6 in Eq. (16) above. Then when ϕ is taken to be complex, two out of the four ϕ's have to be complex conjugated to make the potential real. There are three possibilities: $\phi\phi^*\phi\phi^*$, $\phi^*\phi\phi\phi^*$, $\phi\phi\phi^*\phi^*$, for each of the four contractions $\phi\phi\phi\phi$ when ϕ is real. But after the self-duality condition is applied, one can show that only one of the three terms is actually independent.

The Euler-Lagrange equations of motion for ϕ and A_μ are given by

$$D_\mu D^\mu \phi = -\frac{\partial V}{\partial \phi^*}\,, \tag{17}$$

$$\mathrm{Tr}(\tau^{a\,2})(\partial_\mu F^{a\,\mu\nu} + ef^{abc}A^b_\mu F^{c\,\mu\nu}) = ie\{(D^\nu\phi)^*(\tau^a\phi) - (\tau^a\phi)^*(D^\nu\phi)\}\,, \tag{18}$$

where a is not summed over, and where a basis has been chosen so that $\mathrm{Tr}(\tau^a\tau^b) = \delta^{ab}\mathrm{Tr}(\tau^{a\,2})$.

We construct for string $\tau_{\rm all}$ a solution of the following form.

Ansatz I:

$$\begin{aligned} \phi &= f(r)e^{i\tau_{\rm all}\theta}\phi_0 = f(r)e^{i\theta}\phi_0\,,\\ A^\theta &= i\frac{g(r)}{er}\tau_{\rm all}\,,\\ A^r &= 0\,, \end{aligned} \tag{19}$$

where $\phi_0 \equiv \langle\phi\rangle$ as defined in Eq. (7). The boundary conditions on the functions are

$$\begin{aligned} f(0) &= 0\,, \qquad f(r) \overset{r\to\infty}{\longrightarrow} \mu\,,\\ g(0) &= 0\,, \qquad g(r) \overset{r\to\infty}{\longrightarrow} 1\,, \end{aligned} \tag{20}$$

$V(\phi)$ is minimized at $f = \mu$. Inserting this ansatz into the equations of motion and using the relations $\tau_{\rm all}\tau_{\rm all}\phi_0 = \phi_0$ and $(\tau_{\rm all}\phi_0)^*(\tau_{\rm all}\phi_0) = \phi_0^*\phi_0 = 3840 \equiv N$, we obtain two coupled differential equations for $f(r)$ and $g(r)$:

$$\begin{aligned} f'' + \frac{1}{r}f' - \frac{(1-g)^2}{r^2}f &= f(v_1 + 2Nv_2f^2)\,,\\ \mathrm{Tr}(\tau_{\rm all}^2)\left(g'' - \frac{1}{r}g'\right) &= -2Ne^2(1-g)f^2\,, \end{aligned} \tag{21}$$

where the prime denotes differentiation with respect to r, and $\mathrm{Tr}(\tau_{\rm all}^2) = \frac{2}{5}$ from Eq. (10). An expansion of $f(r)$ and $g(r)$ in powers of r around the origin reveals that $f(r)$ is odd in r with a linear leading term, whereas $g(r)$ is even in r with a quadratic leading term.

Inserting Ansatz I for string $\tau_{\rm all}$ into the Lagrangian gives

$$\begin{aligned} -\mathcal{L}^{\rm all} = {} & \frac{\mathrm{Tr}(\tau_{\rm all}^2)}{2e^2r^2}g'^2 + Nf'^2 + N\frac{(1-g)^2}{r^2}f^2\\ & + N(v_1f^2 + Nv_2f^4)\,. \end{aligned} \tag{22}$$

As a consistency check, note that the equations of motion obtained by varying $\mathcal{L}^{\rm all}$ with respect to the functions g and f are identical to those in Eq. (21).

Note that the parameters v_3 through v_6 in the potential V are absent from Eq. (21) and $\mathcal{L}^{\rm all}$ above. This is because whenever one index of a given ϕ is contracted with one index of another ϕ, this index is summed over from 1 through 10, or in the (α, a) notation discussed earlier, from $\alpha = 1$ through 5 and $a = 1, 2$. For a given α, the term with $a = 2$ by definition has an extra factor of $i^2 = -1$ compared to the term with $a = 1$. These two terms cancel each other when they are added. Because this is true for every α, the third through the sixth terms in V vanish identically for the string-$\tau_{\rm all}$ ansatz.

To construct an ansatz for string τ_1, we need to consider separately the two sets of generators in Eq. (12), which will be referred to as

$$\begin{aligned} \tau_{1+} &= \frac{1}{2}(\tau^{2\alpha-1,\,2\beta} + \tau^{2\alpha,\,2\beta-1})\,,\\ \tau_{1-} &= \frac{1}{2}(\tau^{2\alpha-1,\,2\beta-1} - \tau^{2\alpha,\,2\beta})\,, \quad \alpha < \beta\,. \end{aligned} \tag{23}$$

As we shall see, it is sufficient to derive the equations of motion for an ansatz based on a generator of the form τ_{1+}. By a simple redefinition, it will then be possible to construct an ansatz based on a generator of the form τ_{1-}. For now, we consider the case when τ_1 has the form τ_{1+}. The simple extension of ansatz I with $\tau_{\rm all}$ replaced by τ_1 does not work for string τ_1. The problem arises from the term $\tau_1\tau_1\phi$ on the left-hand side of Eq. (17) in which a new tensor ϕ_0^A,

$$\tau_1\tau_1\phi_0 = \phi_0^A\,, \tag{24}$$

is generated, where

$$\phi^A_{0\,i_1\cdots i_5} \equiv \begin{cases} \phi_{0\,i_1\cdots i_5} & \text{if two indices take the values } (2\alpha-1, 2\beta) \text{ or } (2\alpha, 2\beta)\,,\\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

As a result, the differential equations for $g(r)$ and $f(r)$ are satisfied only if $g(r) = 1$ or $f(r) = 0$ everywhere, which is not consistent with the boundary conditions given by Eq. (20). (Note that the solution $g = 1$ and $f = \mu$ is the vacuum field configuration expressed in a singular gauge.)

We construct a nontrivial solution for string τ_1 by replacing $f(r)\phi_0$ and $\tau_{\rm all}$ in ansatz I with $[f_1(r)\phi_0 + f_2(r)\phi_0^A]$ and τ_1, respectively. Note that ϕ_0 is not orthogonal to ϕ_0^A because $\phi^A_{0\,i_1\cdots i_5}\phi^*_{0\,i_1\cdots i_5} \neq 0$. Therefore instead of expanding ϕ in ϕ_0 and ϕ_0^A, we will use the more convenient basis ϕ_0^A and ϕ_0^B where

$$\phi_0^B \equiv \phi_0 - \phi_0^A \tag{26}$$

and ϕ_0^B is orthogonal to ϕ_0^A:

$$\phi^A_{0\,i_1\cdots i_5}\phi^{B\,*}_{0\,i_1\cdots i_5} = 0\,. \tag{27}$$

From the definition of ϕ_0^A [Eq. (25)] and the properties of ϕ_0, one can see that

$$\phi^B_{0\,i_1\cdots i_5} = \begin{cases} \phi_{0\,i_1\cdots i_5} & \text{if two indices take the values } (2\alpha-1,2\beta) \text{ or } (2\alpha,2\beta-1)\,, \\ 0 & \text{otherwise} \end{cases} \tag{28}$$

and ϕ_0^B is annihilated by τ_1:

$$\tau_1\phi_0^B = 0\,. \tag{29}$$

The solution constructed for string τ_1 is of the following form. Ansatz II:

$$\begin{aligned} \phi &= e^{i\tau_1\theta}\left\{f_o(r)\phi_0^A + f_e(r)\phi_0^B\right\}\,, \\ A^\theta &= i\frac{g(r)}{er}\tau_1\,, \\ A^r &= 0\,, \end{aligned} \tag{30}$$

where, as will become clear in the next two paragraphs, the functions $f_o(r)$ and $f_e(r)$ are named after their odd and even parities in r.

At the origin, we require the fields to be regular. Since ϕ_0^B is left invariant by $e^{i\tau_1\theta}$ [Eq. (29)] but ϕ_0^A is not, at the origin $f_e(0)$ can be any constant but $f_o(0)$ has to vanish. At large r, the scalar field ϕ has to take the form

$$\phi \overset{r\to\infty}{\longrightarrow} \mu\, e^{i\tau_1\theta}\phi_0 = \mu\, e^{i\tau_1\theta}(\phi_0^A + \phi_0^B) \tag{31}$$

for the unbroken gauge group to be SU(5), so both $f_o(r)$ and $f_e(r)$ approach μ at large r. The boundary conditions on the functions are

$$\begin{aligned} f_o(0) &= 0\,, & f_o(r) &\overset{r\to\infty}{\longrightarrow} \mu\,, \\ f_e(0) &= a_0\,, & f_e(r) &\overset{r\to\infty}{\longrightarrow} \mu\,, \\ g(0) &= 0\,, & g(r) &\overset{r\to\infty}{\longrightarrow} 1\,, \end{aligned} \tag{32}$$

where a_0 is a constant.

The equations of motion for ϕ and A_μ are closed when the fields take the form in ansatz II. We obtain three coupled differential equations for $f_o(r), f_e(r)$, and $g(r)$. The algebra involved in extracting these three equations, however, is considerably more tedious than in the $\tau_{\rm all}$ case mainly because the forms of ϕ_0^A, ϕ_0^B, and τ_1 are less symmetric. We will not present the algebra involved and simply quote the results:

$$\begin{aligned} & f_e'' + \frac{1}{r}f_e' \\ &\quad = f_e\left\{v_1 + Nv_2(f_o^2+f_e^2) - \frac{N}{25}e^2\lambda_3(f_o^2-f_e^2)\right\}, \\ & f_o'' + \frac{1}{r}f_o' - \frac{(1-g)^2}{r^2}f_o \\ &\quad = f_o\left\{v_1 + Nv_2(f_o^2+f_e^2) + \frac{N}{25}e^2\lambda_3(f_o^2-f_e^2)\right\}, \\ & \mathrm{Tr}(\tau_1^2)\left(g'' - \frac{1}{r}g'\right) = -Ne^2(1-g)f_o^2\,, \end{aligned} \tag{33}$$

where $e^2\lambda_3 \equiv v_3 + \frac{v_4}{4} + \frac{v_5}{4} + \frac{v_6}{12}$, and $\mathrm{Tr}(\tau_1^2) = 1$ from Eq. (12). An expansion of g, f_o, and f_e in powers of r around the origin gives

$$\begin{aligned} f_o(r) &= a_1 r + a_3 r^3 + \cdots\,, \\ f_e(r) &= a_0 + a_2 r^2 + \cdots\,, \\ g(r) &= b_2 r^2 + b_4 r^4 + \cdots\,, \end{aligned} \tag{34}$$

where the coefficients of all the higher terms are related to a_0, a_1, and b_2, recursively. The function f_o is indeed odd and f_e even in r as claimed earlier.

Inserting ansatz II for string τ_1 into the Lagrangian gives

$$\begin{aligned} -\mathcal{L}^1 = {}& \frac{\mathrm{Tr}(\tau_1^2)}{2e^2r^2}g'^2 + \frac{N}{2}\left(f_e'^2 + f_o'^2\right) \\ & + \frac{N}{2}\frac{(1-g)^2}{r^2}f_o^2 + V_{\rm ans}, \end{aligned} \tag{35}$$

where

$$\begin{aligned} V_{\rm ans} = \frac{N}{2}\Big\{ & v_1(f_o^2+f_e^2) + \frac{N}{2}v_2(f_o^2+f_e^2)^2 \\ & + \frac{N}{50}e^2\lambda_3(f_o^2-f_e^2)^2\Big\}. \end{aligned} \tag{36}$$

Here again, note that the equations of motion obtained by varying $\mathcal{L}^1$ with respect to the functions g, f_o, and f_e are identical to those in Eq. (33).

Now let us consider the other case when τ_1 has the form of τ_{1-}. One can show that Eq. (24) now is $\tau_1\tau_1\phi_0 = \phi_0^B$, and instead of $\tau_1\phi_0^B = 0$, one has $\tau_1\phi_0^A = 0$. Therefore by switching the definitions of ϕ_0^A and ϕ_0^B in Eqs. (25) and (28), all the equations between (24) and (32) are preserved, and one can show that the equations of motion are unchanged. We conclude that ansatz II applies to all 20 τ_1's, where, for τ_{1+}, ϕ_0^A and ϕ_0^B are defined by Eqs. (25) and (28), respectively, but for τ_{1-}, the definitions of the two are reversed. The equations of motion are given by Eq. (33) for all cases.

IV. NUMERICAL CALCULATIONS

In this section we present the numerical solutions to the two sets of differential equations (21) and (33) with the appropriate boundary conditions at the origin and some large value of r. We implemented two methods: the "shooting" and the relaxation methods to handle this two-point boundary value problem. In the "shooting" method [12], an initial guess for the free parameters at $r = 0$ was made and then the equations were integrated out to large r where the boundary conditions were specified. As the name of the method suggests, the true solutions were found by adjusting the parameters at $r = 0$ in the beginning of each iteration to reduce the discrepancies from the desired boundary conditions at large r computed in the previous iteration. For string τ_1, the small-r expansion of the functions in Eq. (34) gives $g(0) = g'(0) = 0\,, f_o(0) = f_o''(0) = f_e'(0) = 0\,,$ and $f_e''(0) = 2a_2\,,$ where a_2 is related to a_0, a_1, and b_2, but the values of

$$\begin{aligned} f_e(0) &= a_0\,, \\ f_o'(0) &= a_1\,, \\ g''(0) &= 2b_2\,, \end{aligned} \tag{37}$$

were adjusted to match the boundary conditions at large r. For string-$\tau_{\rm all}$, we have shown that $f(r)$ is odd and $g(r)$ is even in r, with $f(r) = ar + \cdots$ and $g(r) = br^2 + \cdots$. Thus only the two values $f'(0), g''(0)$ were free parameters. At large r, discrepancies from the boundary condition were corrected by the multidimensional Newton-Raphson method which computed the corrections to the initial parameters. With an initial guess for the parameters at $r = 0$, this "shooting" process was iterated until the "targets" were met. The fourth-order Runge-Kutta method was used to integrate the equations.

We have also implemented a relaxation scheme for comparison. In this method the first step is to express the string energy as a function of the values of the functions f and g (or f_e, f_o, and g) defined on an evenly spaced mesh of points. While a Simpson's rule approximation worked well for the middle range of parameters, a more sophisticated approximation was used to extend the range of parameters that could be treated. For each interval of two lattice spacings, smooth functions $\tilde{f}$ and $\tilde{g}$ were defined by second-order polynomial interpolation from the three mesh points (midpoint and two end points); with the help of a symbolic integration program, the integral defining the energy was carried out exactly for the interpolated functions. (By this method the energy obtained is a rigorous upper limit on the true ground state string energy.) To avoid divergences caused by the explicit factors of $1/r^2$ in the energy density, the first interval had to be treated more carefully— instead of fitting the functions with a second-order polynomial, we fitted the coefficients of the analytically determined power series, such as Eq. (34). Trial functions f and g were chosen, and then the energy was minimized by varying each mesh point one at a time, successively going through the lattice many times. We found it efficient to begin with a coarse mesh which was made successively finer by factors of 2, interpolating the solution at each stage to obtain the first trial solution for the next stage. For the final run in each case we used 2048 points.

We found the results by the two methods to agree to approximately one part in a million or better. In general we were able to explore a wider parameter range with the relaxation method than with the "shooting" method, but the qualitative features given by the "shooting" method remained the same. (The author wishes to thank Alan Guth for implementing the relaxation part of the calculations.)

The dependence of the equations on the parameters in the theory can be simplified if r, f, f_o, and f_e are rescaled as $(v_1 < 0)$

$$\begin{aligned} r &\to \sqrt{-v_1}\,r\,, \\ \{f\,, f_o\,, f_e\} &\to \sqrt{\frac{2Nv_2}{-v_1}}\{f\,, f_o\,, f_e\}\,. \end{aligned} \tag{38}$$

Then only the following combinations of parameters appear in the differential equations:

$$\begin{aligned} \lambda_2 &\equiv \frac{v_2}{e^2}\,, \\ \lambda_3 &\equiv \frac{1}{e^2}\left(v_3 + \frac{v_4}{4} + \frac{v_5}{4} + \frac{v_6}{12}\right)\,. \end{aligned} \tag{39}$$

The Hamiltonian densities $\mathcal{H}^{\rm all}$ and $\mathcal{H}^1$ for the two strings are simply $-\mathcal{L}^{\rm all}$ and $-\mathcal{L}^1$ given by Eqs. (22) and (35) because all fields are assumed to be time independent. With the same rescaling, one obtains

$$\frac{v_2}{(-v_1)^2}\mathcal{H}^{\rm all} = \frac{1}{2}\left\{\frac{2\lambda_2}{5r^2}g'^2 + f'^2 + \frac{(1-g)^2}{r^2}f^2 + \frac{1}{2}(1-f^2)^2\right\} \tag{40}$$

and

$$\frac{v_2}{(-v_1)^2}\mathcal{H}^1 = \frac{1}{2}\left\{\frac{\lambda_2}{r^2}g'^2 + \frac{f_o'^2 + f_e'^2}{2} + \frac{(1-g)^2}{2r^2}f_o^2 + \frac{1}{2}\left(1 - \frac{f_o^2+f_e^2}{2}\right)^2 + \frac{\lambda_3}{200\lambda_2}(f_o^2 - f_e^2)^2\right\}, \tag{41}$$

where the $\tau_{\rm all}$ equation depends on λ_2 only but the τ_1 equation depends on both λ_2 and λ_3.

Typical solutions for the two strings calculated from the "shooting" method are shown in Figs. 1 and 2, where $\lambda_2 = 0.132$ and $\lambda_3 = 10.25$. For the same λ_2 and λ_3, the solutions given by the relaxation method appear indistinguishable visually from those in Figs. 1 and 2. For string $\tau_{\rm all}$, we were able to find solutions in the approximate range $10^{-2} < \lambda_2 < 10$ using the "shooting" method and $10^{-4} < \lambda_2 < 10^3$ using the relaxation method. For string τ_1, we explored the range $5\times10^{-2} < \lambda_2 < 1$ and $0.5 < \lambda_3 < 10^2$. In general, the functions converged more slowly near the two ends of each range above, and we did not attempt to find solutions beyond these limits. We numerically integrated $\mathcal{H}^{\rm all}$ and $\mathcal{H}^1$ for the solutions we computed, and found string τ_1 to have the lower energy for all the parameters we explored. In Fig. 3, the energy density $2\pi r\mathcal{H}$ of the two solutions shown in Figs. 1 and 2 is plotted, and the energy of string τ_1 is clearly lower. For comparison, we point out that the energy per unit length of string $\tau_{\rm all}$ in the range $0.9 < \lambda_2 < 4.0$ has been calculated by Aryal and Everett [8]. Our values in this range of parameters agree with theirs to within 1%.

One of the most important properties of the two strings

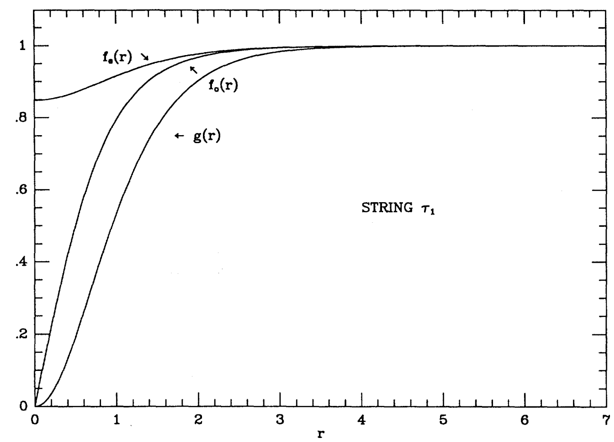

FIG. 1. The solution of string τ_1, $g(r), f_o(r), f_e(r)$, as a function of dimensionless r for the case $\lambda_2 = 0.132, \lambda_3 = 10.25$. The function $g(r)$ represents the spatial dependence of the gauge field, and $f_o(r), f_e(r)$ represent that of the Higgs field.

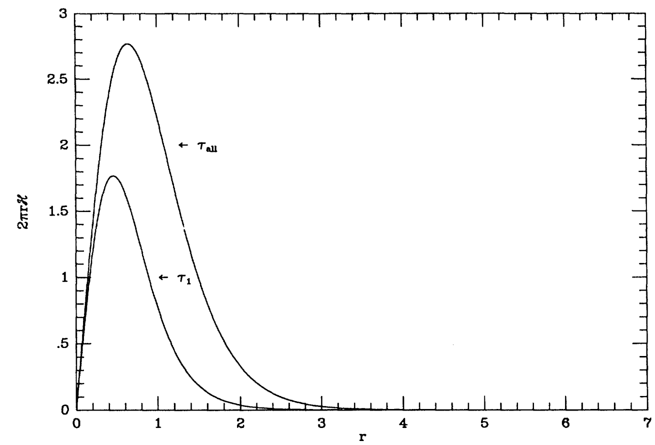

FIG. 3. The radial energy density $2\pi r\mathcal{H}(r)$ (in units of v_1^2/v_2) of string τ_1 and τ_{all}, computed from the solutions in Figs. 1 and 2.

we investigate in this paper is whether string τ_1 has lower energy than string τ_{all}. We just showed that this is true for some range of the parameters. To systematically explore a wider parameter range, however, it is very laborious and time consuming to calculate the τ_1 solutions for different λ_2 and λ_3 first and then compute the corresponding energy. Instead, we employ an upper-bound argument to reduce the two-dimensional parameter space (λ_2, λ_3) to one. We set $f_o = f_e \equiv f_1$ in the Lagrangian and take $g(r), f_1(r)$ as trial functions for string τ_1. The advantage in using $f_o = f_e$ is that the last term in Eq. (41) vanishes, and the equations no longer depend on λ_3. Moreover, Eqs. (40) and (41) then have the same functional form, differing only in the coefficients of the first and the third terms, and one can solve the equations for string τ_1 the same way as for string τ_{all} using different values of λ_2. The corresponding energy, denoted by $E_1(f_o = f_e)$, gives an upper bound on the true energy of string τ_1 by the variational principle. If $E_1(f_o = f_e) < E_{all}$ for a given λ_2, then one can conclude that string τ_1 has the lower energy for that value of λ_2 and all values of λ_3. (Note that in the limit of $\lambda_3 \to \infty$, the trial functions approach the true string solution because for the energy to be finite, the last term in Eq. (41) requires $f_o \to f_e$.) Our result is presented in Fig. 4, where the ratio $E_1(f_o = f_e)/E_{all}$ is plotted as a function of $\log_{10}\lambda_2$ for $10^{-4} < \lambda_2 < 2.5 \times 10^3$. Note that $E_1(f_o = f_e)/E_{all} < 1$ for all seven decades of λ_2, and is approaching an asymptote of 1 (or possibly less than 1) as $\lambda_2 \to 0$. For large λ_2, we find the individual curves of E_{all} vs $\log_{10}\lambda_2$ and E_1 vs $\log_{10}\lambda_2$ approach straight

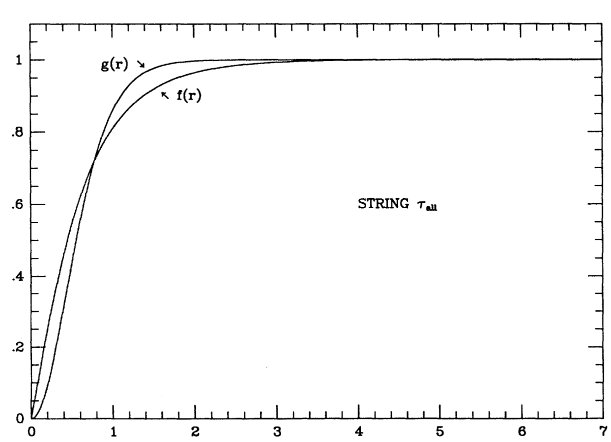

FIG. 2. The solution of string τ_{all}, $g(r), f(r)$, as a function of dimensionless r for the same case as in Fig. 1. Here $g(r)$ represents the spatial dependence of the gauge field and $f(r)$ that of the Higgs field.

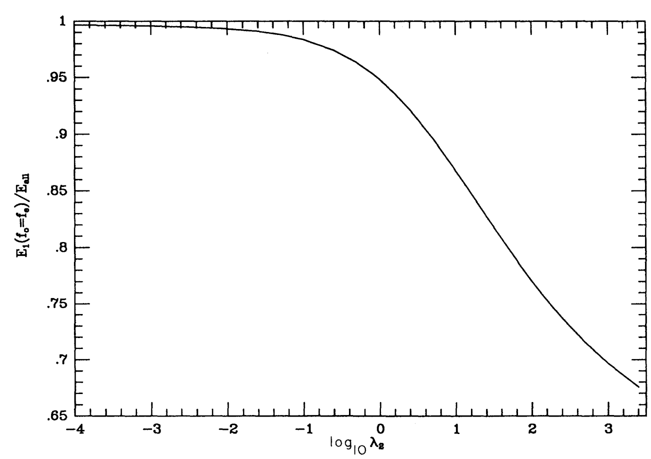

FIG. 4. The ratio of the upper bound on τ_1 energy, $E_1(f_o = f_e)$, over the τ_{all} energy, E_{all}, as a function of λ_2. $E_1(f_o = f_e)$ is calculated by setting $f_o = f_e$ in the Lagrangian.

lines, suggesting that the ratio $E_1(f_o = f_e)/E_{\rm all}$ levels off at a constant for large λ_2. We conclude that string τ_1 has lower energy than string $\tau_{\rm all}$ for $10^{-4} < \lambda_2 < 2.5 \times 10^3$ and all λ_3, and probably is the ground state for the entire range of the parameters in the theory.

V. SCATTERING SOLUTIONS

To study the scattering of fermions by an SO(10) cosmic string, one first needs to understand the 16-dimensional spinor representation of SO(10) to which the left-handed fermions are assigned. Spinor representations certainly have been discussed in the literature [13], but to establish a common notation, we discuss in the Appendix the construction of the generators, the 16 states, and the identification of states with fermions that are relevant to this paper.

Now we proceed to study the Dirac equation

$$(i \not\partial - e \not A^a \tau^a - m)\psi = 0 \tag{42}$$

in the background fields of string $\tau_{\rm all}$ and τ_1: $A^a_\mu \tau^a = A^{\rm all}_\mu \tau_{\rm all}$ and $A^1_\mu \tau_1$. As shown in the Appendix, the fermion fields can be written as a 16-dimensional column vector where each component is identified with a fermion given by Eq. (A16). The generators $\tau_{\rm all}$ and τ_1 can be written as 16×16 Hermitian matrices, where $\tau_{\rm all}$ is diagonal with one diagonal entry equal to $\frac{1}{2}$, ten entries equal to $\frac{1}{10}$ and five entries equal to $-\frac{3}{10}$. For τ_1, we choose $\tau_1 = \frac{1}{2}(\tau^{58} + \tau^{67})$ for illustration. We find that τ_1 takes the block-diagonal form

$$-\tau_1 = \frac{1}{2}\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}, \tag{43}$$

where

$$B = \begin{pmatrix} 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \end{pmatrix}, \tag{44}$$

and I is the 2×2 identity matrix. For string $\tau_{\rm all}$, since $\tau_{\rm all}$ is diagonal, Eq. (42) decouples into 16 equations, one for each component of the wave function, and there is no mixing of leptons and quarks due to twisting of the Higgs field. However, since the 16 eigenvalues of $\tau_{\rm all}$ are all fractional, all 16 fermions scatter nontrivially off the string via the Aharonov-Bohm effect. As pointed out by previous studies, the wave functions of these fermions can be strongly enhanced near the core of the string, leading to strong B-violating processes inside the string.

In the case of string τ_1, the analysis is the easiest if one works in the basis where τ_1 is diagonal. Upon diagonalizing τ_1 by a unitary matrix U and simultaneously rotating the fermion basis ψ_0 in Eq. (A16) to $\tilde\psi_0 \equiv U\psi_0$, one can write $\tilde\psi_0$ as

$$\tilde\psi_0 = \left(\frac{e^- + u_1^c}{\sqrt2}, \frac{e^- - u_1^c}{\sqrt2}, \frac{\nu^c + d_1}{\sqrt2}, \frac{\nu^c - d_1}{\sqrt2}, u_2^c, u_3^c, d_3, d_2, \frac{u_3 + d_2^c}{\sqrt2}, \frac{u_3 - d_2^c}{\sqrt2}, \frac{u_2 + d_3^c}{\sqrt2}, \frac{u_2 - d_3^c}{\sqrt2}, u_1, \nu, e^+, d_1^c\right)_L. \tag{45}$$

We note that $\tilde\psi_0$ above is the fermion field in the gauge where ϕ does not wind with θ and $A_\mu \to 0$ at large r everywhere except on a sheet of singularities at $\theta = 0$. This gauge will be referred to as the "sheet" gauge in analogy with the "string" gauge of a magnetic monopole. The particle content is probably most transparent in this gauge since both the scalar and fermion fields do not wind with θ. The Dirac equation, however, can be easily solved analytically (in the limit of zero string width) in a different gauge where the scalar field ϕ winds with θ and the gauge field falls off as r^{-1} at large r. The fields in ansatz II [see Eq. (30)] for string τ_1 were constructed in this gauge, which will be referred to as the $1/r$ gauge. In the diagonalized basis of τ_1, the fermion field $\tilde\psi$ in the $1/r$ gauge is related to the fermion field $\tilde\psi_0$ in the "sheet" gauge by the gauge transformation

$$\tilde\psi = e^{i\tau_1(\pi-\theta)}\tilde\psi_0 . \tag{46}$$

Since τ_1 is diagonal, the ith components of $\tilde\psi$ and $\tilde\psi_0$ are simply related by a phase.

In the diagonalized basis of τ_1, Eq. (42) decouples into 16 equations of the form

$$(i \not\partial + e\lambda_i \not A^1 - m)\tilde\psi_i = 0, \tag{47}$$

where each $\tilde\psi_i$ interacts with the gauge field with coupling strength $e\lambda_i$; λ_i are the eigenvalues of $-\tau_1$. The eigenvalues are $\lambda_i = \frac{1}{2}$ for $e+u_1^c$, ν^c+d_1, $u_3+d_2^c$, $u_2+d_3^c$; $\lambda_i = -\frac{1}{2}$ for $e - u_1^c$, $\nu^c - d_1$, $u_3 - d_2^c$, $u_2 - d_3^c$; and $\lambda_i = 0$ for all others. Since the $e + u^c$ and $e - u^c$ components have opposite (and fractional) eigenvalues, one can expect a pure e or u^c to turn into a mixture of e and u^c as it propagates around the string, producing baryon-number violation. From this point on, we will consider only the interesting cases where λ_i are fractional.

In the presence of an infinitely thin τ_1 string along the z axis, the gauge field A^1_μ takes the form $A^{1\,r} = A^{1\,z} = 0, A^{1\,\theta} = \frac{1}{er}$, where (r,θ) denote the usual polar coordinates with θ running counterclockwise from the positive x axis. Owing to the symmetry along the z axis, the matrix γ_3 in Eq. (47) drops out, and, with the choice for the γ matrices,

$$\gamma_0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix},$$
$$\gamma_2 = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{48}$$

Eq. (47) decouples into two independent equations for the upper and lower two-component spinors of $\tilde\psi_i$, where the two equations differ by the sign of the mass term. Writing the upper spinor of $\tilde\psi_i$ as

$$\begin{pmatrix} \chi_1(r) \\ \chi_2(r)e^{i\theta} \end{pmatrix} e^{in\theta - iEt}, \tag{49}$$

one can show

$$\begin{pmatrix} m-E & -i\left(\partial_r + \frac{n+\lambda_i+1}{r}\right) \\ -i\left(\partial_r - \frac{n+\lambda_i}{r}\right) & -m-E \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = 0\,, \tag{50}$$

and the solutions are Bessel functions of order $(n+\lambda_i)$ and $-(n+\lambda_i)$:

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} J_{\pm(n+\lambda_i)}(kr) \\ \pm\frac{ik}{E+m} J_{\pm(n+\lambda_i+1)}(kr) \end{pmatrix}, \quad k = \sqrt{E^2-m^2}\,. \tag{51}$$

The appropriate boundary conditions to impose, as pointed out in Ref. [14], are the square integrability of the wave functions near the origin and a self-adjoint Hamiltonian. The usual requirement that wave functions be regular at the origin is sometimes too strong and has to be relaxed. Since $J_\nu(r) \sim r^\nu/(2^\nu \nu!)$ for small r, one can see that the solutions above are square integrable if the $+$ sign is chosen for the modes $n+\lambda_i > 0$, and the $-$ sign for $n+\lambda_i < -1$. For the mode $-1 < n+\lambda_i < 0$, however, both choices are square integrable albeit neither is regular at the origin, and the solution takes the form

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} \sin\mu\, J_{n+\lambda_i} + \cos\mu\, J_{-(n+\lambda_i)} \\ \frac{ik}{E+m}(\sin\mu\, J_{n+\lambda_i+1} - \cos\mu\, J_{-(n+\lambda_i+1)}) \end{pmatrix} \text{ for } -1 < n+\lambda_i < 0\,, \tag{52}$$

where μ sometimes is referred to as the "self-adjoint parameter."

The scattering amplitude $f^{\lambda_i}(\theta)$ for the ith fermion in $\tilde{\psi}$ appears in the asymptotic wave function written as the sum of the incident plane wave and the scattered part:

$$\tilde{\psi}_i \sim u_E e^{-i\lambda_i(\pi-\theta)} e^{i(kx-Et)} + \sqrt{\frac{i}{r}}\, v_E e^{-i\lambda_i(\pi-\theta)} f^{\lambda_i}(\theta) e^{i(kr-Et)}\,, \tag{53}$$

where u_E and v_E are given by

$$u_E = \begin{pmatrix} 1 \\ \frac{k}{E+m} \end{pmatrix}, \quad v_E = \begin{pmatrix} 1 \\ \frac{k}{E+m} e^{i\theta} \end{pmatrix}. \tag{54}$$

Expanding $e^{ikx} = e^{ikr\cos\theta}$ and e^{ikr} in Bessel functions using

$$e^{ikr\cos\theta} = \sum_{n=-\infty}^{\infty} i^n J_n(kr) e^{in\theta}\,, \tag{55}$$

and with

$$f^{\lambda_i}(\theta) = \sum_{n=-\infty}^{\infty} f_n^{\lambda_i} e^{in\theta}\,, \tag{56}$$

Eq. (53) can be matched to the solutions in Eq. (51) mode by mode at large r. Then the scattering amplitude can be calculated:

$$f^{\lambda_i}(\theta) = \frac{i}{\sqrt{2\pi k}} e^{-i([\lambda_i]+1)\theta} \left(\frac{\sin\left(\frac{\theta}{2} - \pi\lambda_i\right)}{\sin\frac{\theta}{2}} - e^{2i\delta} \right), \tag{57}$$

where $[\lambda_i]$ denotes the largest integer less than or equal to λ_i, and δ is related to λ_i and the self-adjoint parameter $\tan\mu$ by [14]

$$\tan\delta = \frac{(-1)^n - \tan\mu}{(-1)^n + \tan\mu} \tan\frac{\lambda_i \pi}{2}\,. \tag{58}$$

With the gauge transformation Eq. (46), one can easily see that $(\tilde{\psi}_0)_i$ in the "sheet" gauge is given by Eq. (53) without the phase $e^{-i\lambda_i(\pi-\theta)}$.

To illustrate the processes that violate the baryon number, we consider an incident beam of electrons propagating in the fields of the string. We will study the (e, u^c) subspace and ignore other fermions since e in ψ is mixed with u^c only. In the "sheet" gauge, the eigenstates of τ_1 can be written as

$$\frac{e+u^c}{\sqrt{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \frac{e-u^c}{\sqrt{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{59}$$

and the electron is simply given by

$$e = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{60}$$

An incident wave of electrons can be written as

$$\tilde{\psi}^e_{0\,\mathrm{inc}} = u_E \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} e^{i(kx-Et)}\,, \tag{61}$$

which scatters into

$$\tilde{\psi}_{0\mathrm{sca}} = \sqrt{\frac{i}{r}}\, v_E \left\{ f^{\frac{1}{2}}(\theta) \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} + f^{-\frac{1}{2}}(\theta) \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right\} e^{i(kr-Et)}\,. \tag{62}$$

Note that the suppressed index on the two-component spinors u_E and v_E should not be confused with the index associated with the two-component eigenvectors used here to label the $e+u^c$ and $e-u^c$ components of the Dirac field. Rewriting $\tilde{\psi}_{0\,\mathrm{sca}}$ above as

$$\tilde{\psi}_{0\,\mathrm{sca}} = \sqrt{\frac{i}{r}}\, v_E \left\{ \left(\frac{f^{\frac{1}{2}}(\theta) + f^{-\frac{1}{2}}(\theta)}{2} \right) \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} + \left(\frac{f^{\frac{1}{2}}(\theta) - f^{-\frac{1}{2}}(\theta)}{2} \right) \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \right\} e^{i(kr-Et)}\,, \tag{63}$$

one finds that the scattered wave consists of a mixture of electrons and u^c quarks if $f^{\frac{1}{2}} \neq f^{-\frac{1}{2}}$. The differential cross section per unit length for the production of a u quark is defined by

$$\frac{d\sigma}{d\theta} = \lim_{r\to\infty} \frac{\mathbf{J}^u_{\text{sca}} \cdot \mathbf{r}}{J_{\text{inc}}}, \tag{64}$$

where $J^i = \bar{\psi}\gamma^i\psi$. Substituting $\tilde{\psi}_{0\,\text{inc}}$ and $\tilde{\psi}_{0\,\text{sca}}$ into the currents, one obtains

$$\frac{d\sigma}{d\theta} = \frac{1}{4}\left|f^{\frac{1}{2}}(\theta) - f^{-\frac{1}{2}}(\theta)\right|^2 . \tag{65}$$

The scattering amplitude f^{λ_i} is given by Eq. (57) and depends on δ, which is related to the self-adjoint parameter μ by Eq. (58). To compute the cross section in Eq. (65), we first derive a simple expression relating μ for $\lambda_i = \frac{1}{2}$ to $\bar{\mu}$ for $\lambda_i = -\frac{1}{2}$. Recall that $\sin\mu$ and $\cos\mu$ are the coefficients in Eq. (52) for the special mode $-1 < n + \lambda_i < 0$. Here one is interested in $\lambda_i = \pm\frac{1}{2}$; thus the special mode takes the value $n + \lambda_i = -\frac{1}{2}$, where $n = -1$ for $\lambda_i = \frac{1}{2}$ and $n = 0$ for $\lambda_i = -\frac{1}{2}$. For $n + \lambda_i = -\frac{1}{2}$, the Bessel functions in Eq. (52) are simply $J_{\pm\frac{1}{2}}$, which have the analytic forms

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x\,, \quad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x\,. \tag{66}$$

For convenience, one can rescale χ_2 and r by

$$\begin{aligned} \chi_2 &\to i\frac{E+m}{k}\chi_2\,, \\ r &\to \sqrt{-v_1}\,r\,, \end{aligned} \tag{67}$$

where $v_1(<0)$ is the quadratic coupling in the Higgs potential in Eq. (16), and define a new parameter

$$\beta \equiv k/\sqrt{-v_1}\,. \tag{68}$$

Then Eq. (52) leads to the simple expression

$$\frac{\chi_1}{\chi_2} = \tan(\beta r + \mu) \quad \text{for } \lambda_i = \tfrac{1}{2} \tag{69}$$

and

$$\frac{\bar{\chi}_1}{\bar{\chi}_2} = \tan(\beta r + \bar{\mu}) \quad \text{for } \lambda_i = -\tfrac{1}{2}\,. \tag{70}$$

The bars over χ_1, χ_2, and μ are used to distinguish the solutions and the self-adjoint parameter of $\lambda_i = -\frac{1}{2}$ from those of $\lambda_i = \frac{1}{2}$.

Now let us examine Eq. (50) which χ_1, χ_2 and $\bar{\chi}_1, \bar{\chi}_2$ satisfy. For generality, we temporarily relax the assumption of zero string width and use the realistic form $\lambda_i g(r)/r$ for the gauge field in place of λ_i/r in Eq. (50). Rescaling $\chi_2, \bar{\chi}_2$ and r by Eq. (67), Eq. (50) can be rewritten as

$$\begin{aligned} \partial_r\chi_1 &= \frac{g(r)-2}{2r}\chi_1 + \beta\chi_2, \\ \partial_r\chi_2 &= -\frac{g(r)}{2r}\chi_2 - \beta\chi_1, \end{aligned} \tag{71}$$

for $\lambda_i = \frac{1}{2}, n = -1$, and

$$\begin{aligned} \partial_r\bar{\chi}_1 &= -\frac{g(r)}{2r}\bar{\chi}_1 + \beta\bar{\chi}_2, \\ \partial_r\bar{\chi}_2 &= \frac{g(r)-2}{2r}\bar{\chi}_2 - \beta\bar{\chi}_1, \end{aligned} \tag{72}$$

for $\lambda_i = -\frac{1}{2}, n = 0$. Upon closer inspection of the two sets of equations above, one finds that Eq. (72) is in fact identical to Eq. (71) if $\bar{\chi}_1$ is identified with χ_2 and $\bar{\chi}_2$ with $-\chi_1$. What about the boundary conditions at the origin? In Eq. (49), for $n = -1$, the upper component depends on θ but the lower component does not, and vice versa for $n = 0$. Therefore χ_1 and $\bar{\chi}_2$ must vanish at the origin for the solution to be continuous, but χ_2 and $\bar{\chi}_1$ can be nonzero at $r = 0$. One thus has

$$\bar{\chi}_1 = \chi_2\,, \quad \bar{\chi}_2 = -\chi_1\,. \tag{73}$$

Since Eq. (71) is linear, the value of $\chi_2(0)$ can be chosen arbitrarily when integrating the differential equations. From Eqs. (69), (70), and (73), one obtains

$$\bar{\mu} = \mu + \frac{\pi}{2} + n_1\pi\,, \tag{74}$$

and consequently

$$\bar{\delta} = \delta + n_2\pi \tag{75}$$

from Eq. (58); $n_1, n_2 \in Z$. Finally, one can compute $f^{\frac{1}{2}} - f^{-\frac{1}{2}}$ and write out the baryon-number-violating differential cross section in Eq. (65) as

$$\frac{d\sigma}{d\theta} = \frac{1}{2\pi k}\left\{\frac{\cos^4\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} + \sin^2\frac{\theta}{2} + 2\cos^2\frac{\theta}{2}\sin 2\delta\right\}\,, \tag{76}$$

where δ is the self-adjoint parameter for $\lambda_i = \frac{1}{2}$.

Although the scattering problem above was analyzed in the limit of zero string width, the structure of the string core is "encoded" in the parameter δ which appears in the differential cross section above. In general the self-adjoint parameter is determined either from physical properties at the origin or sometimes by symmetry arguments. Since the string solutions have already been obtained in the previous section, one can first solve for $\chi_1(r), \chi_2(r)$ by integrating Eq. (71) numerically, using the function $g(r)$ for the gauge field computed earlier. Then Eq. (69) can be inverted to give μ at a given r. The true value of μ is given in the limit $r \to \infty$. Since $g(r)$ depends on the quartic couplings λ_2, λ_3 in the Higgs potential, one can see from Eq. (71) that χ_1 and χ_2 depend on β, λ_2, and λ_3. Defined in Eq. (68), β measures the ratio of the incident fermion momentum k to the Higgs boson mass parameter $\sqrt{-v_1}$, which is of the order of the grand-unified theory (GUT) energy scale. To put it another way, β measures the string width relative to the wavelength of the incident fermion. In Fig. 5, we set $\beta = 1$ and plot μ computed at a given r for three sets of λ_2 and λ_3. At large r, μ approaches different asymptotic values due to the different core structures corresponding to the three sets of λ_2 and λ_3. In Fig. 6, we choose the

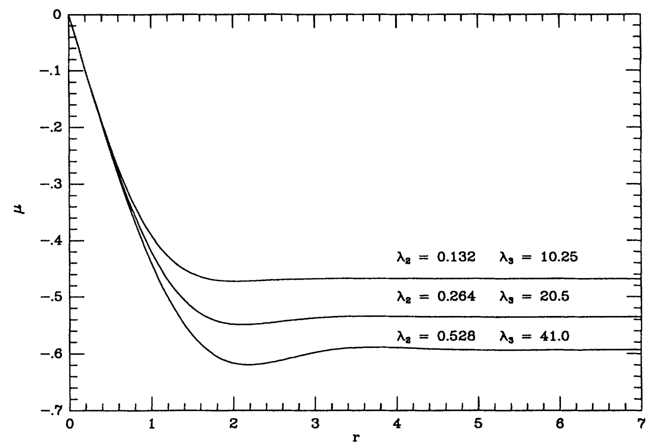

FIG. 5. The self-adjoint parameter μ computed from Eqs. (69) and (71) at a given r, for three sets of (λ_2, λ_3): (0.132, 10.25), (0.264, 20.50), and (0.528, 41.0), where $\beta = 1$. The true value of μ is given in the limit $r \to \infty$.

same set of parameter as in Figs. 1–3: $\lambda_2 = 0.132$ and $\lambda_3 = 10.25$, and plot μ for five values of β ranging from 0.1 to 2.0. One can see that as β decreases, i.e., when the wavelength of the fermion becomes large compared to the string width, μ decreases.

VI. CONCLUSIONS

We constructed two types of strings, string τ_{all} and string τ_1, in the SO(10) grand-unified theory. They are topologically equivalent but dynamically different strings, produced during the phase transition Spin(10) $\to$ SU(5) $\times Z_2$ in the early universe. String τ_{all} is effectively Abelian, and can catalyze baryon-number violation with a strong cross section via grand-unified processes inside the string. It has been the subject of study in several recent papers. The richer Higgs structure of string τ_1, on the other hand, has been shown in this paper to induce baryon catalysis by mixing components in the fermion multiplet, turning leptons into quarks as they travel around the string. The underlying B-violating mechanism is the "twisting" of the scalar field, which leads to different unbroken SU(5) subgroups around the string. This mechanism is distinct from the grand-unified processes which can only occur inside the string core where the GUT symmetry is restored.

The corresponding string solutions have been calculated numerically with both the "shooting" and the relaxation methods. The energy of both strings was computed. With an additional upper-bound argument, we found string τ_1 to have lower energy than string τ_{all} in a wide range of parameters: $10^{-4} < \lambda_2 < 2.5 \times 10^3$ and all λ_3. The ratio of the upper bound on τ_1 energy to the τ_{all} energy increases as λ_2 decreases, and possibly approaches one from below as $\lambda_2 \to 0$. Scattering of fermions in the fields of string τ_1 has also been analyzed, and the B-violating cross section is given by Eq. (76). We conclude that string τ_1 is more stable than string τ_{all}, and can catalyze baryon decay with strong cross sections via the interesting mechanism of Higgs field twisting.

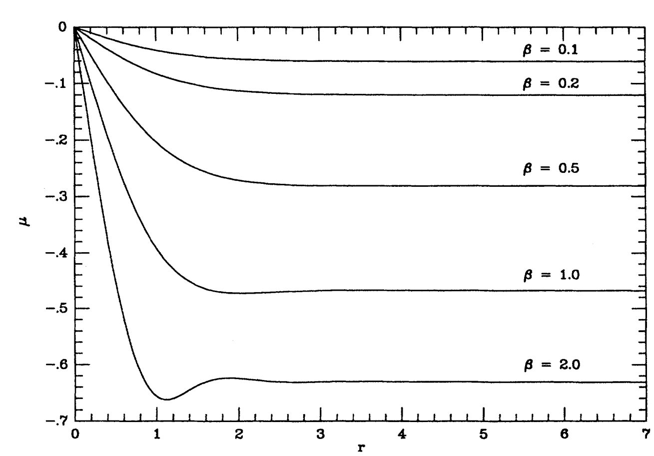

FIG. 6. The self-adjoint parameter μ computed from Eqs. (69) and (71) at a given r for different ratios of β, where $\lambda_2 = 0.132, \lambda_3 = 10.25$.

ACKNOWLEDGMENTS

I wish to thank Alan Guth for many valuable suggestions on this work and a critical reading of the manuscript. I am also grateful for advice from Ed Bertschinger, Robert Brandenberger, Martin Bucher, Jeffrey Goldstone, Roman Jackiw, and Leandros Perivolaropoulos, and assistance from Roger Gilson.

APPENDIX

The generators of SO($2n$) in the spinor representation can be constructed from a set of $2^n \times 2^n$ Hermitian matrices $\Gamma_a^{(n)}, a = 1, \ldots, 2n$, which satisfy the Clifford algebra

$$\{\Gamma_a^{(n)}, \Gamma_b^{(n)}\} = 2\delta_{ab}\,. \tag{A1}$$

Starting with the two Pauli matrices for $n = 1$,

$$\Gamma_1^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma_2^{(1)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{A2}$$

one can iteratively build the higher-dimensional $\Gamma_a^{(n+1)}$ from $\Gamma_a^{(n)}$ by

$$\begin{aligned} \Gamma_a^{(n+1)} &= \begin{pmatrix} \Gamma_a^{(n)} & 0 \\ 0 & -\Gamma_a^{(n)} \end{pmatrix}, \quad a = 1, \ldots, 2n \\ \Gamma_{2n+1}^{(n+1)} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \Gamma_{2n+2}^{(n+1)} &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \end{aligned} \tag{A3}$$

One can check that these Γ matrices satisfy the Clifford algebra. The $\frac{2n(2n-1)}{2}$ generators of SO($2n$) are constructed by

$$M_{ab} = \frac{1}{4i}[\Gamma_a, \Gamma_b]\,, \quad a,b = 1,\ldots,2n, \tag{A4}$$

where M_{ab} satisfy the SO($2n$) commutation relations

$$[M_{ab}, M_{cd}] = -i(\delta_{bc}M_{ad} + \delta_{ad}M_{bc} - \delta_{ac}M_{bd} - \delta_{bd}M_{ac})\,. \tag{A5}$$

Thus far, we have used the explicit matrix notation to construct Γ and M. For convenience, however, we will use an alternative notation in which each of the $2^n \times 2^n$ matrices is written as a tensor product of n independent Pauli matrices, each acting on a different two-dimensional space. We choose the convention that the first matrix from the right in the tensor product acts on the largest 2×2 block in the matrix notation, while the second from the right acts on the next, and so on, with the matrix on the left acting on the smallest 2×2 block. In this notation, the 10 Γ's of SO(10) given by Eq. (A3) become

$$\begin{aligned}
\Gamma_1 &= \sigma_1\sigma_3\sigma_3\sigma_3\sigma_3\,,\ \Gamma_2 = \sigma_2\sigma_3\sigma_3\sigma_3\sigma_3\,,\\
\Gamma_3 &= I\ \sigma_1\sigma_3\sigma_3\sigma_3\,,\ \Gamma_4 = I\ \sigma_2\sigma_3\sigma_3\sigma_3\,,\\
\Gamma_5 &= I\ I\ \sigma_1\sigma_3\sigma_3\,,\ \Gamma_6 = I\ I\ \sigma_2\sigma_3\sigma_3\,,\\
\Gamma_7 &= I\ I\ I\ \sigma_1\sigma_3\,,\ \Gamma_8 = I\ I\ I\ \sigma_2\sigma_3\,,\\
\Gamma_9 &= I\ I\ I\ I\ \sigma_1\,,\ \Gamma_{10} = I\ I\ I\ I\ \sigma_2\,,
\end{aligned} \tag{A6}$$

and the 45 generators M can be found accordingly. Furthermore, one can write down the five diagonal M's that generate the Cartan subalgebra:

$$\begin{aligned}
M_{12} &= \frac{1}{2}\ \sigma_3 IIII\,,\\
M_{34} &= \frac{1}{2}\ I\sigma_3 III\,,\\
M_{56} &= \frac{1}{2}\ II\sigma_3 II\,,\\
M_{78} &= \frac{1}{2}\ III\sigma_3 I\,,\\
M_{9\,10} &= \frac{1}{2}\ IIII\sigma_3\,.
\end{aligned} \tag{A7}$$

The eigenvalues of the five generators above can be used to label the states in the spinor representation. Let $\frac{1}{2}\epsilon_1, \ldots, \frac{1}{2}\epsilon_5$ be the eigenvalues of $M_{12}, \ldots, M_{9\,10}$, respectively with $\epsilon_i = +1$ or -1, and denote the states by

$$|\,\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5\,\rangle\,. \tag{A8}$$

This 32-dimensional representation is reducible to two 16-dimensional irreducible representations because there exists a chirality operator

$$\begin{aligned}
\chi &\equiv (-i)^5\Gamma_1\Gamma_2\cdots\Gamma_{10}\\
&= \sigma_3\sigma_3\sigma_3\sigma_3\sigma_3\,,
\end{aligned} \tag{A9}$$

which satisfies the commutation relations

$$\{\,\chi, \Gamma_i\,\} = 0\,, \quad [\,\chi, M_{ab}\,] = 0\,. \tag{A10}$$

Moreover,

$$\chi|\,\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5\,\rangle = \prod_i \epsilon_i|\,\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5\,\rangle\,, \tag{A11}$$

where the eigenvalue $\prod_i \epsilon_i$ is $+1$ or -1 depending on whether the number of spins that are down ($\epsilon_i = -1$) is even or odd.

We assign the 16 left-handed fermions to the states of positive chirality, i.e., states with even number of $\epsilon_i = -1$. The explicit identification of states to fermions can be achieved by first breaking the SO(10) 10×10 representation into an upper 6×6 and a lower 4×4 blocks for the subgroups SO(6) and SO(4), and then embedding SU(3) in SO(6) and SU(2) in SO(4). The generators for SO(4) are $M_{ab}, a, b = 7, 8, 9, 10$, and with the choice [13]

$$\tau_i = \frac{1}{2}\epsilon_{ijk}M_{jk} - M_{i\,10}\,, \quad i,j,k = 7,8,9 \tag{A12}$$

for the generators of SU(2), one can easily verify that the last two spins in $|\,\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5\,\rangle$ label the SU(2) states with $|+-\rangle, |-+\rangle$ labeling the doublets and $|++\rangle, |--\rangle$ the singlets. Similarly, the first three spins in $|\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5\rangle$ label the SU(3) states with $|+++\rangle, |---\rangle$ labeling the singlets, and $|++-\rangle, |-++\rangle$ with their permutations labeling the SU(3) triplets. One also needs the charge operator Q to make the assignment unique. In SU(5), $Q = \text{diag}(1/3, 1/3, 1/3, 0, -1)$, which takes the form

$$Q = \frac{1}{3}(M_{12} + M_{34} + M_{56}) - M_{9\,10}\,. \tag{A13}$$

In the SO(10) spinor representation,

$$Q|\,\epsilon_1\cdots\epsilon_5\,\rangle = \left\{\frac{1}{6}(\epsilon_1 + \epsilon_2 + \epsilon_3) - \frac{\epsilon_5}{2}\right\}|\,\epsilon_1\cdots\epsilon_5\,\rangle\,. \tag{A14}$$

Putting all the above together one obtains

$$\begin{aligned}
&|+++++\rangle = \nu^c\,,\ |+++--\rangle = e^+,\\
&|--+++\rangle = u_1^c\,,\ |--+--\rangle = d_1^c,\\
&|-+-++\rangle = u_2^c\,,\ |-+---\rangle = d_2^c,\\
&|+--++\rangle = u_3^c\,,\ |+----\rangle = d_3^c,\\
&\\
&|---+-\rangle = \nu\,,\ |----+\rangle = e^-,\\
&|++-+-\rangle = u_1\,,\ |++--+\rangle = d_1,\\
&|+-++-\rangle = u_2\,,\ |+-+-+\rangle = d_2,\\
&|-+++-\rangle = u_3\,,\ |-++-+\rangle = d_3\,.
\end{aligned} \tag{A15}$$

Since we already know how to express the generators M_{ab} as matrices, we can write the states as a single 32-dimensional column vector which is projected into two 16-dimensional vectors of positive and negative chirality by the operator $P_\pm \equiv \frac{1}{2}(1 \pm \chi)$. We find

$$\psi_0 = (\nu^c\ u_1^c\ u_2^c\ u_3^c\ d_3\ d_2\ d_1\ e^-\ u_3\ u_2\ u_1\ \nu\ e^+\ d_1^c\ d_2^c\ d_3^c)_L\,. \tag{A16}$$

In this paper, we studied two types of strings: string τ_{all}, where τ_{all} is given by Eq. (10), and string τ_1, where τ_1 can be any of the generators in Eq. (12). It is easy to see that in terms of $M_{ab}, \tau_{\rm all}$ is written as

$$\tau_{\rm all} = \frac{1}{5}(M_{12} + M_{34} + M_{56} + M_{78} + M_{9\,10})\,, \tag{A17}$$

and $|\,\epsilon_1 \cdots \epsilon_5\,\rangle$ is an eigenstate of $\tau_{\rm all}$ with eigenvalue $\frac{1}{10}\sum_i \epsilon_i$. For the left-handed fermions above, $\frac{1}{10}\sum_i \epsilon_i = \frac{1}{2}$ for ν^c, $\frac{1}{10}$ for e^+, u, d, u^c, and $-\frac{3}{10}$ for ν, e^-, d^c.

To study how τ_1 act on the fermions, we write τ_{1+} and τ_{1-} defined in Eq. (23) as a product of five Pauli matrices using Eqs. (A4) and (A6), and then replace the matrices σ_1 and σ_2 by the usual raising and lowering operators $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$. One obtains

$$\begin{aligned}\tau_{1+} &= \frac{1}{2}(\tau^{2\alpha-1,2\beta} + \tau^{2\alpha,2\beta-1}) \\ &= I \cdots I\sigma_+\sigma_3 \cdots \sigma_3\sigma_+ I \cdots I \\ &\quad + I \cdots I\sigma_-\sigma_3 \cdots \sigma_3\sigma_- I \cdots I\end{aligned} \tag{A18}$$

and

$$\begin{aligned}\tau_{1-} &= \frac{1}{2}(\tau^{2\alpha-1,2\beta-1} - \tau^{2\alpha,2\beta}) \\ &= I \cdots I\sigma_+\sigma_3 \cdots \sigma_3\sigma_- I \cdots I \\ &\quad - I \cdots I\sigma_-\sigma_3 \cdots \sigma_3\sigma_+ I \cdots I\end{aligned} \tag{A19}$$

where $\alpha, \beta = 1, \ldots, 5, \alpha < \beta$, and the two $\sigma_\pm$ matrices in each term occupy the αth and βth positions from the left. Now one can read off from the list of fermions above which particles are mixed by a given τ_1. For generators of the form τ_{1+}, one immediately finds that except for the case $\alpha = 4, \beta = 5$, all mix leptons with quarks; when $\alpha = 4, \beta = 5$, the generator mixes $(e^+, \nu^c), (u_1^c, d_1^c), (u_2^c, d_2^c)$, and (u_3^c, d_3^c). For generators of the form τ_{1-}, leptons are mixed with quarks when $\alpha =$ 1, 2, or 3 and $\beta =$ 4 or 5.

[1] V. Rubakov, Nucl. Phys. **B203**, 311 (1982); Pis'ma Zh. Eksp. Teor. Fiz. **33**, 658 (1981) [JETP Lett. **33**, 644 (1981)]; C. Callan, Phys. Rev. D **25**, 2141 (1982); **26**, 2058 (1982); Nucl. Phys. **B212**, 391 (1982).

[2] M. Alford, J. March-Russell, and F. Wilczek, Nucl. Phys. **B328**, 140 (1989).

[3] W. Perkin, L. Perivolaropoulos, A.C. Davis, R. Brandenberger, and A. Matheson, Nucl. Phys. **B353**, 237 (1991), and the papers referenced within.

[4] R. Brandenberger, A.C. Davis, and A. Matheson, Phys. Lett. B **218**, 304 (1989).

[5] L. Perivolaropoulos, A. Matheson, A.C. Davis, and R. Brandenberger, Phys. Lett. B **245**, 556 (1990).

[6] H. Georgi, in *Particles and Fields—1974 (APS/DPF Williamsburg)*, Proceedings of the 1974 Meeting of the APS Division of Particles and Fields, edited by C. A. Carlson, AIP Conf. Proc. No. 23 (AIP, New York, 1975); H. Fritzsch and P. Minkowski, Ann. Phys. (N.Y.) **93**, 193 (1975).

[7] T.W.B. Kibble, G. Lazarides, and Q. Shafi, Phys. Lett. **113B**, 237 (1982).

[8] M. Aryal and A. Everett, Phys. Rev. D **35**, 3105 (1987).

[9] E. Witten, Nucl. Phys. **B249**, 557 (1985).

[10] J.C. Pati and A. Salam, Phys. Rev. D **10**, 275 (1974).

[11] R. Slansky, Phys. Rep. **79**, 1 (1975); S. Rajpoot, Phys. Rev. D **22**, 2244 (1980).

[12] W. Press, B. Flannery, S. Teukolsky, and W. Vetterling, *Numerical Recipes* (Cambridge University Press, Cambridge, England, 1989).

[13] Some examples are F. Wilczek and A. Zee, Phys. Rev. D **25**, 553 (1982); G. Ross, *Grand Unified Theories* (Benjamin Cummings, New York, 1985); H. Georgi, *Lie Algebras in Particle Physics* (Benjamin Cummings, New York, 1982).

[14] P. Gerbert, Phys. Rev. D **40**, 1346 (1989); P. Gerbert and R. Jackiw, Commun. Math. Phys. **124**, 229 (1989).