High-precision radiative corrections to the Dalitz plot in the semileptonic decays of neutral hyperons

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The radiative corrections to the Dalitz plot in the semileptonic decays of spin $\frac{1}{2}$ neutral hyperons are calculated by analytical means and compared with those previously obtained for the charged hyperon semileptonic decay. The completely integrated expressions are very accurate and include all the terms of the order α times the momentum transfer. The final results are appropriate for a model-independent experimental analysis and are suitable for high statistics decays of ordinary or heavy-quark baryons.

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I. INTRODUCTION

Because of the progress [1] of high-energy hyperon beams and to the improving precision of the measurements in the hyperon β-decay experiments the application of the radiative corrections in the analysis of the experimental data is indispensable in order to avoid systematic errors. At the present time, only numerical results [2] for the precise radiative corrections up to the order α for some semileptonic decays of neutral unpolarized baryons are available in the literature. We have obtained analytic expressions for the radiative corrections to the Dalitz plot of semileptonic decays of neutral hyperons (SDNH's). Those corrections are calculated within the approximation of considering terms of the order of $\alpha q/\pi M_1$ (with q the four-momentum transfer and M_1 the mass of the decaying baryon). The radiative corrections within this approximation allow for a reliable analysis of high-precision measurements, over most of the Dalitz plot especially when the four-momentum transfer is large and can no longer be neglected. This is the case of heavy-quark semileptonic decays such as in charm or flavored quark decays.

It is the purpose of this paper to describe the conditions under which we derive the new accurate analytical formulas. The theoretical framework underlying this calculation and the used techniques are set forth in Refs. [3–5]. In Ref. [3] all the terms of order of $\alpha q/\pi M_1$ in the radiative corrections to the Dalitz plot of semileptonic decays of charged hyperons (SDCH's) are included. These results [3] are taken as a basis to construct the useful formulas for the SDNH case. In this way we improve the precision of the results of Ref. [4] in which the terms of the order of $\alpha q/\pi M_1$ were ignored.

For accuracy, the model dependence of the radiative corrections has to be considered. In the virtual part we handle the model dependence by defining effective form factors in the uncorrected amplitude [6] (without radiative corrections). A functional dependence of the effective form factors on the energies of the emitted baryon and electron additional to the q^2 dependence of the original form factors has to be taken into account. The Low theorem [7–9] allows us to determine all the $\alpha q/\pi M_1$ terms in the bremsstrahlung part. There is no model dependence but the experimental values of the electromagnetic static parameters of the baryons have to be included. In the bremsstrahlung corrections we handle the infrared divergence by identifying carefully all the finite terms that come along with it and we integrate analytically over the photon variables to obtain a closed expression for this part of the Dalitz plot.

We organized this paper in the following way. In Sec. II, we briefly discuss the conditions under which the virtual radiative corrections up to the order of $\alpha q/\pi M_1$ to the Dalitz plot of semileptonic decays of neutral baryons with light or heavy quarks are derived. In Sec. III, we deduce in a covariant way the bremsstrahlung integrands for the SDNH; i.e., we consider a manifestly covariant expression for the photon polarization sum in order to incorporate unambiguously some of the results given in Ref. [10] by Ginsberg. In Sec. IV, we present the integration method and the integrated bremsstrahlung expressions which will constitute the easy to handle formula for the radiative corrections to the SDNH with contributions up to the first order in q. Section V contains our main results and conclusions.

Finally, and for the sake of completeness, we include in the appendixes the definitions of the effective form factors, the special form factor combinations we use, the formulas for the bremsstrahlung integrals, and the several coefficients and functions which appear in the final results.

II. VIRTUAL RADIATIVE CORRECTION OF THE DALITZ PLOT

The uncorrected transition amplitude for the process

$$A(p_1)\rightarrow B^+(p_2)+e^-(l)+\bar{\nu}_e(p_\nu) \tag{1}$$

in the $V-A$ theory is

$$M_0=\frac{G_V}{\sqrt{2}}\bar{u}_B W_\mu u_A \bar{u}_l O_\mu v_\nu \ , \tag{2}$$

where

$$W_\mu = f_1(q^2)\gamma_\mu + \frac{f_2(q^2)}{M_1}\sigma_{\mu\nu}q_\nu + \frac{f_3(q^2)}{M_1}q_\mu + \left[g_1(q^2)\gamma_\mu + \frac{g_2(q^2)}{M_1}\sigma_{\mu\nu}q_\nu + \frac{g_3(q^2)}{M_1}q_\mu \right]\gamma_5 . \quad (3)$$

In Eqs. (1)–(3), $O_\mu = \gamma_\mu(1+\gamma_5)$ and $q = p_1 - p_2$ is the four-momentum transfer. $G_V = G_\mu V_{ij}$ and G_μ is the muon decay coupling constant, V_{ij} is the corresponding Cabibbo-Kobayashi-Maskawa (CKM) matrix element. We shall adhere to the notation, metric, and γ-matrix conventions of Refs. [3,4]. We shall also assume that $m_\nu = 0$. p_1, p_2, l, and p_ν are the four-momenta of the particles involved, E_1, E_2, E, and E_ν^0 are their energies and M_1, M_2, m, and m_ν are their masses, respectively.

In the derivation of the virtual radiative corrections to the Dalitz plot of process in Eq. (1) we decompose [11] the order-α one-loop corrections into a model-independent part (MIP) and into a model-dependent part (MDP). We follow the procedure of Ref. [6] which is devoted to the decay of charged hyperons (Secs. I and II), and derive new results for the decay of neutral hyperons. These results for the SDNH case arise from the replacement of

$$\frac{\bar{u}_B W_\lambda(p_1,p_2)(2p_{1\mu} - k_\mu)u_A}{k^2 - 2p_1\cdot k + i\varepsilon} \quad (4)$$

in the amplitude M_1 of Eq. (3) in Ref. [6] by

$$\frac{\bar{u}_B(2p_{2\mu} + k_\mu)W_\lambda(p_1,p_2)u_A}{k^2 + 2p_2\cdot k + i\varepsilon} . \quad (5)$$

This new term comes from the Feynman graph where the photon is exchanged between the produced charged hadron and the electron and takes into account the Coulomb interaction between the final charged particles.

The MIP is gauge invariant and free of the ultraviolet divergence, and therefore, finite and calculable. The MDP contains the effects of the strong and weak interactions and can be absorbed into M_0 through the definition of the effective form factors. This was denoted by putting a prime on M_0 (see Appendix A).

The virtual radiative corrections in the decay amplitude are given by

$$M_V = M_0' + M_v , \quad (6)$$

where

$$M_v = \frac{\alpha}{2\pi}\left[M_0\Phi_N(E,E_2,\lambda) + M_{p_2}\Phi_N'(E,E_2) \right] . \quad (7)$$

The functions $\Phi_N(E,E_2,\lambda)$ and $\Phi_N'(E,E_2)$ depend on E, E_2, and λ, which are the energy of the β particle, the energy of the final baryon, and the infrared divergence cutoff, respectively, and contain terms up to the order of $\alpha q/\pi M_1$. The λ-dependent terms are canceled by their counterpart terms in the bremsstrahlung contribution.

The explicit forms of $\Phi_N(E,E_2,\lambda)$ and $\Phi_N'(E,E_2)$ are

$$\Phi_N(E,E_2,\lambda) = \Phi_N^{\rm IR}(E,E_2,\lambda) + \Phi_N^{\rm ND}(E,E_2) + \Phi_{\rm Coulomb}(E,E_2) , \quad (8)$$

where

$$\Phi_N^{\rm IR}(E,E_2,\lambda) = 2\left[\frac{1}{\beta_N}\,\mathrm{arctanh}\beta_N - 1\right]\ln\left[\frac{\lambda}{m}\right] + \frac{i\pi}{\beta_N}\left[\ln\left[\frac{(l+p_2)^2}{\lambda^2}\right]\right] ,$$

$$\begin{aligned}\Phi_N^{\rm ND}(E,E_2) = {}& -\frac{1}{\beta_N}(\mathrm{arctanh}\beta_N)^2 + \frac{1}{\beta_N}L\left[\frac{\Delta_V}{x_2^+}\right] - \frac{1}{\beta_N}L\left[\frac{-\Delta_V}{1-x_2^+}\right] + \frac{1}{\beta_N}\,\mathrm{arctanh}\beta_N\left[\frac{M_2^2 + (1+\beta_N^2)l\cdot p_2}{(l+p_2)^2}\right] \\ & + \frac{3}{2}\ln\left[\frac{M_2}{m}\right] - \frac{11}{8} - \frac{m^2}{(l+p_2)^2}\ln\left[\frac{M_2}{m}\right] - \frac{1}{\beta_N}\ln\left[1 + \frac{\Delta_V}{1-x_2^+}\right]\left[\ln\left[\frac{M_2}{m}\right] - \mathrm{arctanh}\beta_N\right] \\ & + \frac{i\pi}{\beta_N}\left[2\ln\Delta_V - \left[\frac{M_2^2 + (1+\beta_N^2)l\cdot p_2}{(l+p_2)^2}\right]\right] , \end{aligned} \quad (9)$$

$$\Phi_{\rm Coulomb}(E,E_2) = \pi^2/\beta_N ,$$

and

$$\Phi_N'(E,E_2) = \frac{1}{\beta_N}\left[1 - \beta_N^2\right]\left[-\mathrm{arctanh}\beta_N\left[1 + \frac{l\cdot p_2}{M_2^2}\right] - \frac{l\cdot p_2}{M_2^2}\beta_N \ln\left[\frac{M_2}{m}\right] + i\pi\left[1 + \frac{l\cdot p_2}{M_2^2}\right]\right] . \quad (10)$$

Above we use the following relations and definitions:

$$1-\beta_N^2=\frac{m^2M_2^2}{(l\cdot p_2)^2} ,$$

$$\Delta_V=x_2^+-x_2^-=\frac{2l\cdot p_2\beta_N}{(l+p_2)^2} ,$$

$$x_2^\pm=\left[\frac{m^2+(1\pm\beta_N)l\cdot p_2}{(l+p_2)^2}\right] ,$$

$$1-x_2^\pm=\left[\frac{M_2^2+(1\mp\beta_N)l\cdot p_2}{(l+p_2)^2}\right] ,$$

and

$$L\left[\frac{\Delta_V}{1-x_2^-}\right]=-L\left[\frac{-\Delta_V}{1-x_2^+}\right]+\frac{1}{2}\ln^2\left|\frac{1-x_2^-}{l-x_2^+}\right| , \quad (11)$$

where $L(x)$ is the Spence function, and the β_N is the velocity of the lepton in the rest frame of the charged hadron. The other matrix element in Eq. (7) is given by

$$M_{p_2}=\frac{G_V}{\sqrt{2}}\frac{l\cdot p_2}{m(l+p_2)^2}\bar{u}_B W_\lambda(p_1,p_2)u_A\bar{u}_l\not{p}_2O_\lambda v_\nu . \quad (12)$$

To calculate the Dalitz plot with virtual radiative corrections in terms of independent variables which are the energies E_2 and E of the emitted baryon and the electron, respectively, we use the standard trace method and the following definitions

$$\sum_{\text{spins}}|M_0|^2=\frac{G_V^2}{2}\frac{4M_1}{M_2mm_\nu}A_0' ,$$
$$\frac{1}{2}\sum_{\text{spins}}(M_0M_{P_2}^+ +M_{p_2}M_0^+)=\frac{G_V^2}{2}\frac{4M_1}{M_2mm_\nu}A_{1N}'' , \quad (13)$$

and

$$d\Gamma_V=\frac{dE_2dE\,d\Omega_l d\varphi_2 M_2mm_\nu}{(2\pi)^5}\frac{1}{2}\sum_{\text{spins}}|M_V|^2 .$$

The decay rate is given by

$$d\Gamma_V=d\Omega\,\text{Re}\left[A_0'\left[1+\frac{\alpha}{\pi}\Phi_N^{\text{IR}}\right]+\frac{\alpha}{\pi}\left[A_{1N}'\left[\Phi_N^{\text{ND}}+\Phi_{\text{Coulomb}}\right]+A_{1N}''\Phi_N'\right]\right] , \quad (14)$$

where

$$d\Omega=\frac{G_V^2}{2}\frac{dE_2dE\,d\Omega_l d\varphi_2}{(2\pi)^5}2M_1 . \quad (15)$$

Let us recall here that the uncorrected rate is given by

$$d\Gamma_0(E,E_2)=A_0'd\Omega .$$

The explicit covariant form of the coefficients is

$$A_0'=\frac{1}{16M_1^2}\left[\Omega_1d_0p_1\cdot l+\Omega_2\frac{a}{2}b_0+\Omega_3b_0p_1\cdot l+\Omega_4\frac{a}{2}d_0-\Omega_5f_0\right] ; \quad (16)$$

the coefficient A_{1N}' is obtained from A_0' by neglecting second-order terms in q/M_1, and

$$A_{1N}''=\frac{l\cdot p_2}{16M_1^2(l+p_2)^2}[(\Omega_1p_1\cdot p_2+\Omega_4p_2^2-\Omega_5)d_0+(\Omega_2p_2^2+\Omega_3p_1\cdot p_2)b_0] . \quad (17)$$

Here we have introduced the following definitions for the entailed invariants and their values in the $\mathbf{p}_1=0$ frame:

$$\begin{aligned} a&=2p_2\cdot l=2(EE_2-|\mathbf{p}_2||\boldsymbol{l}|y_0)=2mM_2/(1-\beta_N^2)^{1/2} ,\\ b_0&=2p_1\cdot p_\nu=2M_1E_\nu^0 ,\\ d_0&=2p_2\cdot p_\nu=2M_1(E_m-E) ,\\ f_0&=2p_\nu\cdot l=2EE_\nu^0+2|\boldsymbol{l}|(|\boldsymbol{l}|+|\mathbf{p}_2|y_0) , \end{aligned} \quad (18)$$

where

$$\begin{aligned} E_\nu^0&=M_1-E_2-E ,\\ E_m&=(M_1^2-M_2^2+m^2)/2M_1 ,\\ E_\nu^0x_0&=-(|\boldsymbol{l}|+|\mathbf{p}_2|y_0) ,\\ y_0&=\frac{(E_\nu^0)^2-\mathbf{p}_2^2-\boldsymbol{l}^2}{2|\mathbf{p}_2||\boldsymbol{l}|}=\cos\theta_{lB} , \end{aligned} \quad (19)$$

and θ_{lB} is the angle between the electron and the baryon B three-momenta.

It is easy to show by the conservation of energy and momentum that

$$b_0=d_0+f_0 .$$

The Ω_i's ($i=1,\ldots,5$) in Eqs. (16) and (17) are bilinear combinations of the form factors. Previous results are given in terms of functions Q_i ($i=1,\ldots,5$) which are also bilinear combinations of the form factors with known functions of E and E_2 as coefficients. They are presented explicitly in Ref. [4] and in the interest of completeness we reproduce them here in Appendix B. The Q_i's are very useful relations because through them one can easily distinguish the different orders of approximation in a given expression, and readily detect the terms which do or do not contribute to the required order of approximation. It is not difficult to see that terms such as EQ_i and $E_\nu^0Q_i$ with $i=2$ or 4 are one order in q higher than Q_i when $i=1$ or 3, and E^2Q_5 is two orders higher than Q_i when $i=1$ or 3.

In terms of the Q_i's, the Ω_i's of Eqs. (16) and (17) are given as

$$\Omega_1=-8M_1(Q_2+E_2Q_5)\ ,$$
$$\Omega_2=-8M_1(Q_4+E_2Q_5)\ ,$$
$$\Omega_3=8(Q_1+Q_3+(Q_2+Q_4)E_2+E_2^2Q_5)\ , \tag{20}$$
$$\Omega_4=8M_1^2Q_5\ ,$$
$$\Omega_5=8M_1^2Q_3\ ,$$

and the coefficients that appear in Eq. (14) are given as

$$A'_0=A'_{1N}-Q_5\mathbf{p}_2^2|l|y_0(|\mathbf{p}_2|+|l|y_0)\ ,$$
$$A'_{1N}=Q_1EE_\nu^0-Q_2E|\mathbf{p}_2|(|\mathbf{p}_2|+|l|y_0)-Q_3|l|(|\mathbf{p}_2|y_0+|l|)+Q_4E_\nu^0|\mathbf{p}_2||l|y_0\ ,$$
$$A''_{1N}=\frac{a}{2M_1^2(1-2E_\nu^0/M_1)}\times[E_\nu^0(Q_1E_2+Q_4\mathbf{p}_2^2)-(\mathbf{p}_2^2+|\mathbf{p}_2||l|y_0)(Q_2E_2+Q_3)]\ . \tag{21}$$

The imaginary parts of $\Phi_N(E,\lambda)$ and $\Phi'_N(E)$ do not contribute to this physical process (to this order in α). We also kept the infrared divergent terms, though they cancel out in the final result.

The Coulomb part of the final-state interaction is important when one of the charged particles is moving slowly relative to the other, corresponding to $\beta_N\to 0$ in $\pi^2/2\beta_N$.

III. BREMSSTRAHLUNG CORRECTION OF THE DALITZ PLOT

The emission of real photons must be added to Eq. (1). Hence we have to consider the process

$$A(p_1)\to B^+(p_2)+e^-(l)+\bar{\nu}_e(p_\nu)+\gamma(k) \tag{22}$$

to obtain a complete expression for the Dalitz plot with radiative corrections, which includes all the $\alpha q/\pi M_1$ terms. The amplitude for this process is obtained in a model-independent fashion from the Low theorem [7–9], and is given by

$$M_{BN}=M_{1N}+M_{2N}+M_{3N}\ ,$$

with

$$M_{1N}=eM_0\left[\frac{\varepsilon\cdot l}{l\cdot k}-\frac{\varepsilon\cdot p_2}{p_2\cdot k}\right]\ ,$$
$$M_{2N}=\frac{eG_V}{\sqrt{2}}\varepsilon_\mu\bar{u}_BW_\lambda u_A\bar{u}_l\frac{\gamma_\mu k\!\!\!/}{2l\cdot k}O_\lambda v_\nu\ , \tag{23}$$
$$M_{3N}=\frac{-G_V}{\sqrt{2}}\bar{u}_lO_\lambda v_\nu\varepsilon_\mu\bar{u}_B\left\{\frac{e\gamma_\mu k\!\!\!/W_\lambda}{2p_2\cdot k}+\kappa_1W_\lambda\frac{p\!\!\!/_1+M_1}{2p_1\cdot k}\sigma_{\mu\nu}k_\nu-\kappa_2\sigma_{\mu\nu}k_\nu\frac{p\!\!\!/_2+M_2}{2p_2\cdot k}W_\lambda -e\left[\frac{p_{2_\mu}k_\rho}{p_2\cdot k}-g_{\mu\rho}\right]\left[\left[\frac{f_2+g_2\gamma_5}{M_1}\right]\sigma_{\lambda\rho}+g_{\lambda\rho}\left[\frac{f_3+g_3\gamma_5}{M_1}\right]\right]\right\}u_A\ .$$

This amplitude depends on κ_1 and κ_2 which are the anomalous magnetic moments of A and B, on the photon polarization four-vector ε_μ, and on the photon four-momentum k_ν; k_0 is its energy.

The differential decay rate of Eq. (22) is obtained from

$$\sum_{\text{spins}}|M_{BN}|^2=\sum_{\text{spins}}|M_{1N}|^2+\sum_{\text{spins}}(2M_{1N}M_{2N}+M_{2N}^2)+\sum_{\text{spins}}2(M_{1N}M_{3N}+M_{2N}M_{3N})\ . \tag{24}$$

In Eq. (24) we neglected the term $|M_{3N}|^2$ which contributes to the order of $\alpha q^2/\pi M_1^2$. We will also consistently neglect the terms of the same order and higher order in further analysis.

In order to correctly handle the infrared divergent terms which arise from the first summand in Eq. (24) we separate them from the convergent ones. For this purpose we write the differential decay rate of Eq. (22) as

$$d\Gamma_{BN}=d\Gamma_{BN}^{\rm IR}+d\Gamma_{BN}^{\rm I}+d\Gamma_{BN}^{\rm II}+d\Gamma_{BN}^{\rm III}\ , \tag{25}$$

$d\Gamma_{BN}^{\rm IR}$ and $d\Gamma_{BN}^{\rm I}$ contain the infrared divergent and convergent terms of $\sum_{\rm spins}|M_{1N}|^2$, respectively. $d\Gamma_{BN}^{\rm II}$ and $d\Gamma_{BN}^{\rm III}$ contain the contributions from the second and third summands in Eq. (24).

We shall follow the approach of Ref. [10] and adapt some of their results to our case. According to this,

$$d\Gamma_{BN}^{\rm IR}=\frac{\alpha}{N}d\Gamma_0(E,E_2)I_0^N(E,E_2)\ , \tag{26}$$

with

$$d\Gamma_0(E,E_2)=A'_0d\Omega\ , \tag{27}$$

where A'_0 is given in Eq. (21), and

$$I_0^N(E,E_2)=\lim_{\lambda\to 0}\frac{1}{4}\int_{\lambda^2}^{x_{\max}^g}dx^g[2l\cdot p_2I_{1,1}(l,p_2)-m^2I_{2,0}(l,p_2)-M_2^2I_{0,2}(l,p_2)]\ , \tag{28}$$

where $k^2=\lambda^2$, λ being the infrared cutoff. The x^g is the invariant mass of the undetected particles and is given by

$$x^g=(p_\nu+k)^2$$

with

$$x^g_{\max}=(E^0_\nu-|\mathbf{p}_2|+|\boldsymbol{l}|)(E^0_\nu+|\mathbf{p}_2|-|\boldsymbol{l}|)$$

as defined in Ref. [10], and the invariant integrals $I_{n,m}(l,p_2)$ are defined by

$$I_{n,m}(l,p_2)=\frac{1}{2\pi}\int\frac{d^3p_\nu}{E_\nu}\frac{d^3k}{k_0}\delta^4(p_1-p_2-l-p_\nu-k)\times\frac{1}{(l\cdot k)^n}\frac{1}{(p_2\cdot k)^m}\ . \tag{29}$$

We decompose the result of Eq. (28) into two terms to separate the infrared-divergent terms from the convergent ones:

$$I_0^N(E,E_2)=I_0^{\mathrm{IR}}(\lambda,E,E_2)+I_0^{ND}(E,E_2)\ , \tag{30}$$

and obtain

$$I_0^{\mathrm{IR}}(\lambda,E,E_2)=\frac{a}{\Delta}\ln\frac{\Delta}{\lambda^2}\ln\frac{a+\Delta}{2mM_2}-\ln\frac{mM_2}{\lambda^2}\ , \tag{31}$$

and

$$I_0^{ND}(E,E_2)=\frac{a}{\Delta}\left[2\ln\frac{w_{\max}}{w_{\min}}\ln\frac{a+\Delta}{2mM_2}-\ln\frac{w_{\max}+2\Delta}{2\Delta}\ln\frac{(w_{\max}+2\Delta)\Delta}{2m^2M^2}-\ln\frac{(a+\Delta)}{2mM_2}\ln\frac{w_0}{2mM_2}+2L_{i2}\left[-\frac{a-\Delta}{2\Delta}\right]\right.$$
$$\left.-2L_{i2}\left[-\frac{w_{\max}}{a+\Delta}\right]-2L_{i2}\left[-\frac{a-\Delta}{w_{\max}+2\Delta}\right]+\frac{1}{2}L_{i2}\left[-\frac{a+\Delta}{w_0}\right]-\frac{1}{2}L_{i2}\left[-\frac{a-\Delta}{w_0}\right]\right]$$
$$+\ln\frac{(H^2-M_2^2)(q^2-m^2)}{(x^g_{\max})^2}+\left[\ln\frac{a+\Delta+w_{\max}}{2mM_2}\right]^2-\left[\ln\frac{a+\Delta}{2mM_2}\right]^2\ , \tag{32}$$

where

$$\begin{aligned}
&w_{\max}=x^g_{\max}-\Delta+[(a+x^g_{\max})^2-4m^2M_2^2]^{1/2}\ ,\\
&w_{\min}=\Delta^{-3/2}(a+\Delta)[a(H^2-M_2^2)(q^2-m^2)-M_2^2(q^2-m^2)^2-m^2(H^2-M_2^2)^2]^{1/2}\ ,\\
&w_1=\frac{aM_2^2(q^2-m^2)^2+am^2(H^2-M_2^2)^2-4m^2M_2^2(H^2-M_2^2)(q^2-m^2)}{a(H^2-M_2^2)(q^2-m^2)-M_2^2(q^2-m^2)^2-m^2(H^2-M_2^2)^2}\ ,\\
&w_0=w_1+(w_1^2-4m^2M_2^2)^{1/2}\ ,\quad a=M_1^2-H^2-q^2\ ,\\
&\Delta=(a^2-4m^2M_2^2)^{1/2}\ ,\quad \Delta=a\beta_N\ ,\quad \text{and } H^2=(p_1-l)^2\ ,\\
&q^2=(p_1-p_2)^2\ .
\end{aligned} \tag{33}$$

The relation between the function $L_{i2}(x)$ given in Ref. [10] and the Spence function is $L_{i2}(x)=-L(x)$.

The a given in Eq. (33) is equal to the invariant a defined in Eq. (18). In the rest frame of the decaying hadron the infrared-divergent function Eq. (31) takes the form

$$I_0^{\mathrm{IR}}(\lambda,E,E_2)=2\ln\left[\frac{m}{\lambda}\right]\left[\frac{\operatorname{arctanh}\beta_N}{\beta_N}-1\right]-\frac{\operatorname{arctanh}\beta_N}{\beta_N}\ln\left[\frac{m^2}{a\beta_N}\right]-\ln\left[\frac{M_2}{m}\right]\ . \tag{31a}$$

The evaluation of the other nondivergent inner-bremsstrahlung contributions by analytic means is obtained following the procedure of Refs. [3,4]. Let us recall here that the integrations for the nondivergent terms ($\lambda\to 0$) in Refs. [3,4] are taken over the azimuthal angle of the photon φ_k, the cosine of the angle between the electron and final baryon y, and over the cosine of the angle between the electron and photon x.

The relation between the variables x^g and y is $x^g=F(y)$, where $F(y)$ is given in Eq. (45) of Ref. [4],

$$F(y)=2|\mathbf{p}_2||\boldsymbol{l}|(y_0-y)\ ,\quad -1\le y\le y_0\ , \tag{34}$$

and $x^g_{\max}$ in Eq. (28) corresponds to

$$F(-1)=2|\mathbf{p}_2||\boldsymbol{l}|(y_0+1)\ .$$

Let us consider now the contributions of the second summand in Eq. (24). The trace calculation yields

$$d\Gamma^{\mathrm{I}}_{BN}=\frac{\alpha}{\pi}d\Omega\frac{|\mathbf{p}_2||\boldsymbol{l}|}{2\pi}\int_{-1}^{1}dx\int_{-1}^{y_0}dy\int_0^{2\pi}d\varphi_k\frac{F}{4D^2}\frac{1}{M_1^2}\sum_{n,m=0}^{2}C^{\mathrm{I}}_{n,m}\frac{1}{(l\cdot k)^n}\frac{1}{(p_2\cdot k)^m}\ , \tag{35}$$

where the nonvanishing coefficients $C^{\rm I}_{n,m}$ are

$$C^{\rm I}_{2,0}=\frac{1}{2}m^2\left\{2p_1\cdot k\Omega_{14}+F\left[\Omega_{24}\left[p_1\cdot k-\frac{b_0}{2}\right]-\Omega_5+\frac{f_0}{2}\Omega_4\right]\right\},$$

$$C^{\rm I}_{1,1}=-\frac{1}{2}F\left[\left[\Omega_{14}-\Omega_5-\frac{b_0}{2}\Omega_{24}+\frac{f_0}{2}\Omega_4\right](a+F)+F\left[\frac{a}{2}+\frac{F}{2}\right]\Omega_{24}\right],$$

$$C^{\rm I}_{0,2}=\frac{1}{2}M_2^2\left[l\cdot k(\Omega_{25}+F\Omega_2)+F\left[\Omega_{14}-\Omega_5+\frac{f_0}{2}\Omega_4-\frac{b_0}{2}\Omega_{24}+\frac{F}{2}\Omega_{24}\right]\right],\tag{36}$$

$$C^{\rm I}_{1,0}=-\left\{(a+F)\Omega_{14}+m^2\left[\Omega_5+\left[\frac{a+F}{2}\right]\Omega_4+p_1\cdot l\Omega_1\right]+\frac{F}{2}(a+F)\Omega_{24}\right\},$$

$$C^{\rm I}_{0,1}=\left[M_2^2\Omega_{14}-\left[\frac{a+F}{2}\right]\Omega_{25}+\frac{F}{2}[M_2^2\Omega_{24}-(a+F)\Omega_2]\right],$$

and

$$\Omega_{14}=\left[p_1\cdot l(\Omega_1+\Omega_3)+\frac{a}{2}(\Omega_2+\Omega_4)\right],$$
$$\Omega_{24}=(\Omega_2+\Omega_4),\tag{37}$$
$$\Omega_{25}=(a\Omega_2+2p_1\cdot l\Omega_3-2\Omega_5).$$

The D can be written as

$$D=p_\nu\cdot k/k_0 .\tag{38}$$

The following relations have been also used:

$$p_\nu\cdot k=\frac{F-\lambda^2}{2}$$
$$p_2\cdot k=p_1\cdot k-l\cdot k-\frac{F+\lambda^2}{2},$$
$$p_2\cdot l=\tfrac{1}{2}(a+F),$$
$$p_1\cdot p_\nu=\tfrac{1}{2}(b_0-F-\lambda^2)-p_2\cdot k-l\cdot k,\tag{39}$$
$$p_2\cdot p_\nu=\frac{(d_0-F)}{2}-p_2\cdot k,$$
$$p_\nu\cdot l=\tfrac{1}{2}(f_0-F-2l\cdot k),$$
$$k_0=F/2D,$$

where a, b_0, d_0, and f_0 are given in Eq. (18).

The complete expressions for the squared matrix elements in Eq. (24) are rather long. In Refs. [3,4] we have already developed part of this calculation and evaluated some of the terms that will appear here. Let us take advantage of this and consider only the additional contributions to the ones already obtained.

The new contributions for the second summand in Eq. (24) will come from the matrix element

$$|\Delta M_B^{\rm II}|^2=\frac{e^2G_V^2}{2}\frac{1}{M_1M_2mm_\nu}\frac{1}{l\cdot k}{\rm Tr}W_\lambda(\not p_1+M_1)\overline{W}_\rho(\not p_2+M_2)\times\left[\left[\frac{p_2\cdot l}{p_2\cdot k}-\frac{p_1\cdot l}{p_1\cdot k}\right]{\rm Tr}O_\lambda\not p_\nu O_\rho\not k+\frac{l\cdot k}{p_1\cdot k}{\rm Tr}O_\lambda\not p_\nu O_\rho\not p_1-\frac{l\cdot k}{p_2\cdot k}{\rm Tr}O_\lambda\not p_\nu O_\mu\not p_2\right].\tag{40}$$

For consistency we evaluate these complementary results in terms of the same form factors as used in Ref. [3] and we obtain

$$\Delta d\Gamma^{\rm II}_{BN}=\frac{\alpha}{\pi}d\Omega\frac{|\mathbf{p}_2||l|}{2\pi}\int_{-1}^{1}dx\int_{-1}^{y_0}dy\int_0^{2\pi}d\varphi_k\frac{F}{4D^2}\frac{1}{4M_1^2}\sum_{n,m=0}^{1}C^{\rm II}_{n,m}\frac{1}{(l\cdot k)^n}\frac{1}{(p_2\cdot k)^m},\tag{41}$$

where the nonvanishing coefficients $C^{\rm II}_{n,m}\ (l,p_2)$ are

$$C^{\rm II}_{1,1}=M_1^2Fp_2\cdot l\left[R^{\rm II}_{1,1}+M_1Q_2\frac{F}{M_1^2}-\frac{1}{2}N_3\frac{p_1\cdot k}{M_1^2}\right],$$

$$C^{\rm II}_{1,0}=M_1^2\left\{(p_2\cdot l-p_1\cdot l)\left[R^{\rm II}_{1,0}+\frac{p_1\cdot k}{M_1^2}[4M_1(Q_2+Q_4)-N_3]\right]+2F\left[M_1Q_2\frac{p_2\cdot l}{M_1^2}+M_1Q_4\frac{p_1\cdot l}{M_1^2}+\frac{p_1\cdot l}{p_1\cdot k}\left[Q_3-\frac{M_1Q_4b_0}{2M_1^2}\right]\right]\right\},\tag{42}$$

$$C^{\rm II}_{0,1}=M_1^2\left\{R^{\rm II}_{0,1}+p_2\cdot l\left[2M_1Q_2\frac{f_0}{M_1^2}+\frac{b_0}{M_1^2}\left[\frac{N_3}{2}-2M_1Q_2\right]-N_3\frac{p_1\cdot k}{M_1^2}\right]\right.$$
$$\left.+2l\cdot k(2Q_3+2E_2Q_2-\rho^{\rm II}_{0,1})+F\left[2M_1Q_2\frac{p_2\cdot l}{M_1^2}-\rho^{\rm II}_{0,1}\right]\right\},$$
$$C^{\rm II}_{0,0}=4M_1^2\left[R^{\rm II}_{0,0}+\frac{b_0}{2p_1\cdot k}[Q_1+(E_2-M_1)Q_2]+\frac{f_0M_1Q_2}{2p_1\cdot k}-\frac{M_1Q_2l\cdot k}{p_1\cdot k}-\frac{p_1\cdot l}{p_1\cdot k}\frac{b_0}{2M_1^2}M_1Q_4\right],$$

and

$$R^{\rm II}_{1,1}=\left[\frac{N_3}{4}-M_1Q_2\right]\frac{b_0}{M_1^2}+M_1Q_2\frac{f_0}{M_1^2}-2Q_3\ ,$$
$$R^{\rm II}_{1,0}=\frac{b_0}{M_1^2}\left[\frac{N_3}{2}-2M_1(Q_2+Q_4)\right]+\frac{2f_0M_1Q_2}{M_1^2}\ ,$$
$$R^{\rm II}_{0,1}=b_0\rho^{\rm II}_{0,1}-2f_0(Q_3+E_2Q_2)\ , \tag{43}$$
$$R^{\rm II}_{0,0}=\frac{p_1\cdot l}{M_1^2}M_1(Q_2+Q_4)-\frac{1}{2}\rho^{\rm II}_{0,1}-[Q_1+(E_2-M_1)Q_2]\ ,$$
$$\rho^{\rm II}_{0,1}=2Q_3+2E_2Q_2+2\left[\frac{M_2}{M_1}\right]^2M_1Q_4-\frac{N_3}{2}\frac{p_1\cdot p_2}{M_1^2}\ ,$$

with

$$N_3=4[Q_1+Q_3+(Q_2+Q_4)E_2]\ .$$

Then the $d\Gamma^{\rm II}_{BN}$ that appears in Eq. (25) is

$$d\Gamma^{\rm II}_{BN}=d\Gamma^{\rm II}_{B}+\Delta\, d\Gamma^{\rm II}_{BN}\ ,$$

where $d\Gamma^{\rm II}_B$ is given in Ref. [3], Eq. (29).

Finally, the $d\Gamma^{\rm III}_{BN}$ in Eq. (25) contains the contributions from M_{3N} given in Eq. (23). This amplitude is one order of magnitude higher in q/M_1 than M_{1N} and M_{2N} and it is clear that it will yield terms of order of $\alpha q/\pi M_1$, which means that we can neglect terms which are proportional to f_i and g_i with $i=2,3$ in the W_λ given in Eq. (3). These are the same form factors as used in Ref. [3] and we can therefore directly compare the respective results. The only source of additional terms that contribute to the required order in the third summand in Eq. (24) is the combination of

$$\Delta M_{3N}=\frac{-G_V}{\sqrt{2}}\bar{u}_lO_\lambda v_\nu\varepsilon_\mu\bar{u}_Be\left[\frac{\gamma_\mu\not k W_\lambda}{2p_2\cdot k}+\frac{W_\lambda\not k\gamma_\mu}{2p_1\cdot k}\right]u_A \tag{44}$$

with the M_{1N} and the M_{2N} given in Eq. (23). We also have

$$\Delta\, d\Gamma^{\rm III}_{BN}=\frac{\alpha}{\pi}d\Omega\frac{|\mathbf{p}_2||\boldsymbol{l}|}{2\pi}\int_{-1}^{1}dx\int_{-1}^{y_0}dy\int_0^{2\pi}d\varphi_k\frac{F}{4D^2}\frac{2}{M_1^2}\sum_{n,m=0}^{2}C^{\rm III}_{n,m}\frac{1}{(l\cdot k)^n}\frac{1}{(p_1\cdot k)^m}\ , \tag{45}$$

where the nonvanishing coefficients are

$$C^{\rm III}_{1,1}=-[p_1\cdot l(p_1\cdot kl\cdot p_\nu-p_1\cdot lk\cdot p_\nu)+m^2p_1\cdot p_\nu p_1\cdot k]R^+\ ,$$
$$C^{\rm III}_{0,1}=3p_1\cdot lp_1\cdot p_\nu R^+-M_1M_2p_\nu\cdot kR^-\ , \tag{46}$$
$$C^{\rm III}_{0,2}=-p_1^2(p_1\cdot p_\nu l\cdot k+p_1\cdot lk\cdot p_\nu)R^+\ ,$$

and

$$R^+=|f_1|^2+|g_1|^2\ ,$$
$$R^-=|f_1|^2-|g_1|^2\ . \tag{47}$$

Then the $d\Gamma^{\rm III}_{BN}$ to be substituted in Eq. (25) is

$$d\Gamma^{\rm III}_{BN}=d\Gamma^{\rm III}_{B}+\Delta\, d\Gamma^{\rm III}_{BN}\ ,$$

where $d\Gamma^{\rm III}_B$ is given in Ref. [3], Eq. (30). In the simplification procedure previous to the integration and evaluation of $\Delta\, d\Gamma^{\rm III}_{BN}$ the following relations are used:

$$\mathbf{p}_2\cdot\mathbf{p}_\nu=-|\mathbf{p}_2|(|\mathbf{p}_2|+|\boldsymbol{l}|y_0)+\frac{F}{2D}(E\beta x+E^0_\nu)\ ,$$
$$\boldsymbol{l}\cdot\mathbf{p}_\nu=-|\boldsymbol{l}|(|\boldsymbol{l}|+|\mathbf{p}_2|y_0)+\frac{F}{2}-\frac{|\boldsymbol{l}|x}{2D}F\ ,$$
$$\hat{\mathbf{k}}\cdot\mathbf{p}_\nu=E^0_\nu-\frac{F}{2D}-D\ ,$$
$$\mathbf{p}_2\cdot\boldsymbol{l}=E_2E-\frac{1}{2}(a+F)=|\boldsymbol{l}||\mathbf{p}_2|y_0-\frac{F}{2}\ ,$$
$$\mathbf{p}_2\cdot\hat{\mathbf{k}}=D-E^0_\nu-E\beta x\ ,$$

where $\hat{\mathbf{k}}=\mathbf{k}/k_0$ and $E_\nu=E_\nu^0-k_0$ is the energy of the neutrino in Eq. (22).

We wish to remark that despite its apparent length, the bremsstrahlung decay probability is organized in a rather easy to handle compact expression through Eq. (25). As it stands now it is ready for numerical or analytical integration.

Let us emphasize that the infrared divergence is completely extracted. The coefficient of this divergence is opposite to that of the virtual part, as required for the cancellation of such a divergence. This fact is evident when the divergent term in Eq. (31a) is substituted in Eq. (30), then Eq. (30) and Eq. (27) are substituted in Eq. (26), and this expression is compared with the real part of the virtual divergent term ϕ_N^{IR} in Eq. (8) substituted in Eq. (14). The finite terms that accompany the divergent ones can be integrated without infrared-divergence ambiguities.

IV. NONDIVERGENT-BREMSSTRAHLUNG ANALYTICAL INTEGRATION

Since we are interested in the radiative corrections to the process in Eq. (1) and not in the process in Eq. (22), we restrict ourselves to the three-body region of the Dalitz plot defined by

$$E_2^{\min}\le E_2\le E_2^{\max}$$

and

$$m\le E\le E_m \ ,$$

with

$$E_2^{\max\atop\min}=\frac{1}{2}(M_1-E\pm l)+\frac{M_2^2}{2(M_1-E^{\pm}l)}$$

and

$$E_m=(M_1^2-M_2^2+m^2)/2M_1 \ .$$

We use a coordinate frame in the rest system of A with the z axis along the electron three-momentum and the x axis oriented so that the final baryon three-momentum is in the first or fourth quadrant of the x-y plane.

The order of integration which is followed for the nondivergent terms is, first, over the azimuth angle of the photon φ_k, second, over the cosine of the polar angle of the final baryon y, and, third, over the cosine of the polar angle of the photon x.

There are five integrals over φ_k which were evaluated previously and are given by Eqs. (64)–(67) of Ref. [4]. These are

$$\begin{aligned}K_0&=\int_0^{2\pi}d\varphi_k=2\pi \ ,\\ K_1(y,x)&=\int_0^{2\pi}\frac{d\varphi_k}{D(x,y,\varphi_k)} \ ,\\ K_2(y,x)&=\int_0^{2\pi}\frac{d\varphi_k}{D^2(x,y,\varphi_k)} \ ,\\ K_3(y,x)&=\int_0^{2\pi}\frac{d\varphi_k}{D^3(x,y,\varphi_k)} \ .\end{aligned}\tag{48}$$

In addition to those integrals, and due to the term $1/(p_2\cdot k)^m$ in Eqs. (35) and (41), where

$$\frac{1}{(p_2\cdot k)}=\frac{2D}{F(r-D)}\tag{49}$$

and

$$r-D=E_2-\mathbf{p}_2\cdot\hat{\mathbf{k}}\tag{50}$$

with

$$r=M_1-E(1-\beta x) \ ,$$

we obtain new integrals defined by

$$K_n'(y,x)=\int_0^{2\pi}\frac{d\varphi_k}{[r-D(x,y,\varphi_k)]^n}\tag{51}$$

and

$$K_{nm}'(y,x)=\int_0^{2\pi}\frac{d\varphi_k}{[r-D(x,y,\varphi_k)]^nD^m(x,y,\varphi_k)} \ .$$

The expressions which involve K_1' are given in Appendix C. The other integrals can be approximated in the following way:

$$K_{nm}'(y,x)=\frac{1}{[M_1-E(1-\beta x)]^n}\int_0^{2\pi}\frac{d\varphi_k}{[1-D(x,y,\varphi)/r]^nD^m(x,y,\varphi)}\simeq\frac{1}{M_1^n}K_m(y,x) \ .$$

When we carry out the integration with respect to the variable y, we get five integrals; three of them are given in Ref. [4], Eqs. (68)–(70), and the other two are in Ref. [3], Eqs. (33)–(34); the procedure of integration is described in detail in Ref. [4].

The long part of the analytical calculation presents itself with the integration over x, partly because of the number of integrals, and partly because of the splitting of the range of x explained in Ref. [4]. These integrals, which appear in the final result, are θ_i ($i=0,\dots,17$) and θ_{1N}. Nine θ_i's are given by Eqs. (87) and (80)–(83) in Ref. [4], and the other seven are given in Ref. [3], by Eqs. (37)–(41), while the θ_{1N} and θ_{17} are given in this paper by

$$\begin{aligned}\theta_{1N}&=I_0^N(E,E_2) \ ,\\ \theta_{17}&=(1+y_0)I_1 \ ,\\ I_1&=\frac{2}{\beta}\,\mathrm{arctanh}\beta \ .\end{aligned}\tag{52}$$

Once we have identified all the different integrals, we have to put them in the corresponding places.

In order to simplify this laborious task, and to avoid cumbersome calculations, we have constructed a new table, given in Appendix C, in which the results for several double integrals are shown. Therefore, by consulting this table one can obtain the final integrated expression in just one step.

After some straightforward but long algebraic calculations, we obtain the following expressions for the nondivergent elements of the Dalitz plot complementary to those given in Ref. [3].

The second term of Eq. (25) becomes

$$d\Gamma^{\rm I}_{BN}=d\Gamma^{\rm I}_{B}+\Delta\, d\Gamma^{\rm I}_{BN}\ , \tag{53}$$

where

$$d\Gamma^{\rm I}_{B}=\frac{\alpha}{\pi}\,d\Omega\left[H_0\theta_0+\sum_{i=2}^{13}H_i\theta_i\right]$$

and

$$\Delta\, d\Gamma^{\rm I}_{BN}=\frac{\alpha}{\pi}\,d\Omega\sum_{i=0}^{17}N_i^{\rm I}\theta_i\ .$$

The H_i's ($i=0,\ldots,13$) are given in Ref. [3]. The $N_i^{\rm I}$'s that contribute to the required order are

$$N_0^{\rm I}=-|\mathbf{p}_2||\boldsymbol{l}|\frac{E}{M_1}(Q_1+Q_3)+|\mathbf{p}_2||\boldsymbol{l}|\frac{1}{2}\left[\frac{E_2}{M_1}-1\right]Q_3\ ,$$

$$N_{12}^{\rm I}=\frac{\mathbf{p}_2^2\boldsymbol{l}^2}{M_1}(Q_1+Q_3)\ ,$$

$$N_{13}^{\rm I}=-\frac{\mathbf{p}_2^2\boldsymbol{l}^2}{M_1}Q_3\ ,$$

$$N_{17}^{\rm I}=\frac{|\mathbf{p}_2||\boldsymbol{l}|}{2}Q_3\left[\left[\frac{E_2}{M_1}-1\right]+\left[1-y_0\right]\frac{|\mathbf{p}_2|\beta}{M_2}\right]\ .$$

Let us emphasize here that terms of order of $\alpha q^2/M_1^2$ contribute beyond our approximation and therefore have been neglected all along our calculation.

The third term of Eq. (25) becomes

$$d\Gamma^{\rm II}_{BN}=d\Gamma^{\rm II}_{B}+\Delta\, d\Gamma^{\rm II}_{BN}\ , \tag{54}$$

where $d\Gamma^{\rm II}_{B}$ is given in Ref. [3], Eq. (44),

$$d\Gamma^{\rm II}_{B}=\frac{\alpha}{\pi}d\Omega\left[\sum_{i=2}^{15}H_{i+14}\theta_i\right]\ ,$$

and

$$\Delta\, d\Gamma^{\rm II}_{BN}=\frac{\alpha}{\pi}\,d\Omega\sum_{i=0}^{17}N_i^{\rm II}\theta_i\ ,$$

with

$$N_0^{\rm II}=N_1^{\rm II}=N_2^{\rm II}=N_6^{\rm II}=N_9^{\rm II}=N_{11}^{\rm II}=N_{15}^{\rm II}=0\ ,$$

$$N_3^{\rm II}=\frac{1}{2}|\mathbf{p}_2||\boldsymbol{l}|\frac{E}{M_1}(1-\beta^2)(E+E_\nu^0)(Q_1+Q_3)\ ,$$

$$N_4^{\rm II}=-\frac{|\mathbf{p}_2||\boldsymbol{l}|}{2M_1}[m^2(Q_1+Q_3)+Q_3(E_\nu^{02}+EE_\nu^0-\mathbf{p}_2^2)]\ ,$$

$$N_5^{\rm II}=-\frac{1}{2}\frac{|\mathbf{p}_2||\boldsymbol{l}|^2}{M_1}[(E+2E_\nu^0)(Q_1+Q_3)+E_\nu^0Q_3]\ ,$$

$$N_7^{\rm II}=\frac{|\mathbf{p}_2||\boldsymbol{l}|}{4M_1}(E+E_\nu^0+|\mathbf{p}_2|\beta y_0)(Q_1+Q_3)\ ,$$

$$N_8^{\rm II}=\frac{|\mathbf{p}_2||\boldsymbol{l}|}{4}\left[\frac{E_2}{M_1}-1\right](Q_1+Q_3)\ ,$$

$$N_{10}^{\rm II}=-\frac{1}{2}\frac{|\mathbf{p}_2||\boldsymbol{l}|^3}{M_1}(Q_1+2Q_3)\ ,$$

$$N_{12}^{\rm II}=\frac{1}{2}\frac{|\mathbf{p}_2|^2|\boldsymbol{l}|\beta}{M_1}(E-E_\nu^0)(Q_1+Q_3)\ ,$$

$$N_{13}^{\rm II}=-\frac{1}{2}\frac{|\mathbf{p}_2|^2|\boldsymbol{l}|^2}{M_1}Q_1\ ,$$

$$N_{14}^{\rm II}=-\frac{|\mathbf{p}_2||\boldsymbol{l}|^2}{4M_1}(Q_1+Q_3)\ ,$$

$$N_{16}^{\rm II}=-\frac{|\mathbf{p}_2||\boldsymbol{l}|}{8M_1E}(Q_1+Q_3)\ ,$$

$$N_{17}^{\rm II}=\frac{|\mathbf{p}_2||\boldsymbol{l}|}{2}\left\{\left[\left[1-\frac{E_2}{M_2}-(1-y_0)\frac{|\mathbf{p}_2|\beta}{2M_2}\right]Q_3+\frac{E_\nu^0}{M_1}\left[Q_1+Q_3\right]\right]\right\}.$$

And, finally, the fourth term of Eq. (25) becomes

$$d\Gamma^{\rm III}_{BN}=d\Gamma^{\rm III}_{B}+\Delta\, d\Gamma^{\rm III}_{BN}\ , \tag{55}$$

where $d\Gamma^{\rm III}_{B}$ is given in Eq. (45) of Ref. [3],

$$d\Gamma^{\rm III}_{B}=\frac{\alpha}{\pi}\,d\Omega\left[\sum_{i=0}^{16}H_{i+30}\theta_i\right]$$

and

$$\Delta\, d\Gamma^{\rm III}_{BN}=\frac{\alpha}{N}\,d\Omega\sum_{i=0}^{14}N_i^{\rm III}\theta_i\ ,$$

with

$$N_1^{\rm III}=N_2^{\rm III}=N_6^{\rm III}=N_9^{\rm III}=N_{11}^{\rm III}=0\ ,$$

$$N_0^{\rm III}=\frac{|\mathbf{p}_2||\boldsymbol{l}|E}{M_1}R^+\ ,$$

$$N_3^{\rm III}=-E_\nu^0(1-\beta^2)N_0^{\rm III}\ ,$$

$$N_4^{\rm III}=\frac{|\mathbf{p}_2||\boldsymbol{l}|}{2M_1}\left[-2|\mathbf{p}_2||\boldsymbol{l}|y_0R^-+2E(E_\nu^0-\beta^2E)R^+\right]\ ,$$

$$N_5^{\rm III}=|\mathbf{p}_2||\boldsymbol{l}|^2\frac{E_\nu^0}{M_1}(R^+-R^-)\ ,$$

$$N_7^{\rm III}=\frac{(1-\beta^2)}{2}N_0^{\rm III}\ ,$$

$$N_8^{\rm III}=-N_0^{\rm III}/2\ ,$$

$$N_{10}^{\rm III}=|\mathbf{p}_2||\boldsymbol{l}|^3(R^+-R^-)/M_1\ ,$$

$$N_{12}^{\rm III}=-|\mathbf{p}_2|\beta N_0^{\rm III}\ ,$$

$$N_{13}^{\rm III}=\frac{|\mathbf{p}_2|^2|\mathbf{l}|^2}{M_1}R^- ,$$

$$N_{14}^{\rm III}=-\frac{\beta}{2}N_0^{\rm III} .$$

One can easily notice that

$$\Delta\, d\Gamma_{BN}=\Delta\, d\Gamma_{BN}^{\rm I}+\Delta\, d\Gamma_{BN}^{\rm II}+\Delta\, d\Gamma_{BN}^{\rm III}$$

contains only terms of order of $\alpha q/\pi M_1$; it is therefore sufficient to consider the Q's up to the zeroth-order form factors combinations. This simplifies the final result in which Q_2, Q_4, and Q_5 no longer appear, and the following approximations have to be taken:

$$Q_1\to 2R^+-R^- \qquad Q_3\to R^- , \tag{56}$$

where $R^\pm$ are given in Eqs. (47).

By summing Eqs. (26) and (53)–(55), the full analytic result for the bremsstrahlung part of the Dalitz plot of the semileptonic decay of neutral hyperons which is model independent and contains all the $\alpha q/\pi M$ contributions is obtained and given by

$$d\Gamma_{BN}(A\to B^+e^-\bar\nu_e)=d\Gamma_B(A\to B^+e^-\bar\nu_e)+\Delta\, d\Gamma_{BN}(A\to B^+e^-\bar\nu_e) , \tag{57}$$

where

$$d\Gamma_B(A\to B^+e^-\bar\nu_e)=\frac{\alpha}{\pi}\left[H_0'\theta_0+\sum_{i=2}^{16}H_i'\theta_i\right]d\Omega \tag{58}$$

and

$$\Delta\, d\Gamma_{BN}(A\to B^+e^-\bar\nu_e)=\frac{\alpha}{\pi}\,d\Omega\left[\Delta N_0\theta_0+\Delta N_1\theta_{1N}+\sum_{i=2}^{17}(\Delta N)_i\theta_i\right] . \tag{59}$$

The H_i' $(i=0,2,\ldots,16)$, are explicitly given in Appendix D and implicitly in Ref. [3] [Eq. (48)]. They are quadratic functions of the Dirac form factors, the $\kappa_i(i=1,2)$ and of the kinematical variables E_2 and E.

The ΔN_i are

$$\Delta N_0=\frac{|\mathbf{p}_2||\mathbf{l}|}{2}\left[\left[\frac{E_2}{M_2}-1\right]R^- -2\frac{E}{M_1}R^+\right] ,$$

$$\Delta N_1=A'_{1N} ,$$

$$\Delta N_2=\Delta N_6=\Delta N_9=\Delta N_{11}=\Delta N_{15}=0 ,$$

$$\Delta N_3=|\mathbf{p}_2||\mathbf{l}|\frac{m^2}{M_1}R^+ ,$$

$$\Delta N_4=\frac{-|\mathbf{p}_2||\mathbf{l}|E}{2M_1}\left[2(E-E_\nu^0)R^+ \right.$$

$$\Delta N_5=-\frac{|\mathbf{p}_2||\mathbf{l}|^2}{2M_1}\left[2(E+E_\nu^0)R^+ +3E_\nu^0R^-\right] ,$$

$$\Delta N_7=\frac{|\mathbf{p}_2|\beta}{2M_1}\left[2m^2+EE_\nu^0(1-\beta x_0)\right]R^+ , \tag{60}$$

$$\Delta N_8=-\frac{|\mathbf{p}_2||\mathbf{l}|}{2M_1}(E_\nu^0+2E)R^+ ,$$

$$\Delta N_{10}=-\frac{3}{2}\frac{|\mathbf{p}_2||\mathbf{l}|^3}{M_1}R^- ,$$

$$\Delta N_{12}=\frac{|\mathbf{p}_2|^2|\mathbf{l}|\beta}{M_1}(2E-E_\nu^0)R^+ ,$$

$$\Delta N_{13}=-\frac{|\mathbf{p}_2|^2|\mathbf{l}|^2}{2M_1}(2R^+-R^-) ,$$

$$\Delta N_{14}=-\frac{|\mathbf{p}_2||\mathbf{l}|^2}{M_1}R^+ ,$$

$$\Delta N_{16}=-\frac{|\mathbf{p}_2|\beta}{4M_1}R^+ ,$$

$$\Delta N_{17}=\frac{|\mathbf{p}_2||\mathbf{l}|}{2}\left[(1-y_0)\frac{|\mathbf{p}_2|\beta}{2M_2}R^- +\frac{2E_\nu^0}{M_1}R^+\right] .$$

Let us remark here that the term $H_1'\theta_1$ is excluded in Eq. (58); instead the corresponding term $\Delta N_1\theta_{1N}$ is included in Eq. (59). This is due to the fact that the θ_1 given in Eq. (88) of Ref. [4] contains essentially the infrared contribution for the SDCH case while the θ_{1N} given in Eq. (52) corresponds to the infrared contribution for the SDNH case.

V. FINAL RESULTS AND CONCLUSIONS

Let us now present the Dalitz plot of process in Eq. (1) with radiative corrections up to order of $\alpha q/\pi M_1$. Our result for the complete radiatively corrected Dalitz plot for the SDNH which arises from the sum of Eqs. (14) and (25) is compactly summarized in the expression

$$d\Gamma(A\to B^+e^-\bar\nu_e)=\left[A_0'+\frac{\alpha}{\pi}\left[A'_{1N}(\Phi_N+\theta_{1N})+A''_{1N}\Phi'_N+(H_0'+\Delta N_0)\theta_0+\sum_{i=2}^{17}\Delta N_i\theta_i+\sum_{i=2}^{16}H_i'\theta_i\right]\right]d\Omega , \tag{61}$$

where the, Φ_N, Φ'_N, A_0', A'_{1N}, A''_{1N}, and $d\Omega$ are given in Eqs. (9), (10), (21), and (15), respectively. The H_i''s are given in an explicit form in Appendix D, and the integrated functions θ_i are given in Refs. [3,4], and also in Appendix E. The θ_0, θ_1, and θ_i $(i=2,\ldots,9)$, are given in Eqs. (95), (96), and (99) of Ref. [4] and $\theta_{10},\ldots,\theta_{16}$ are given in Eq. (46) of Ref. [3]. The θ_{1N} and θ_{17} which are appropriate for the SDNH are given in Eq. (52).

To present the results for both cases, i.e., the neutral and charged semileptonic decays let us also reproduce here the result for the charged semileptonic decay given in Eq. (48) of Ref. [3]:

$$d\Gamma(A^-\to Be^-\bar{\nu}_e)=\left[A_0'+\frac{\alpha}{\pi}\left[H_1'(\Phi+\theta_1)+B_1''\Phi'+H_0'\theta_0+\sum_{i=2}^{16}H_i'\theta_i\right]\right]d\Omega\ , \qquad (62)$$

where the Φ, Φ', A_0', B_1'', and $d\Omega$ are given in Ref. [3], Eqs. (6), (7), (10), (12), and (13), respectively, θ_1 is given in Ref. [4], Eq. (96), and H_1' is equal to A_{1N}' in Eq. (21). For the sake of completeness, all the coefficients and functions that appear in Eq. (62) are presented in the appendixes.

Even though we have limited our calculations of the bremsstrahlung part to the Dalitz plot of the nonradiative semileptonic decay, our results can be used safely on the boundary of the Dalitz plot. It is at this boundary that the infrared divergence appears [13]. A detailed discussion of this fact can be found in the Appendix of Ref. [3].

Equations (61) and (62) are derived under the same assumptions since all the terms of the order of $\alpha q/\pi M_1$ were included in both of them. These equations have the property that they have no infrared divergences; they do not contain an ultraviolet cutoff and are not compromised by any model dependence of radiative corrections. They are also applicable for the processes where the momentun transfer is large and where it cannot be neglected. To the first order in q this leads to terms of order of $\alpha q/\pi M_1$ in the radiative corrections. The expected error by the omission of higher-order terms is of the order of $\alpha q^2/\pi M_1^2\approx 0.0006$ in charm decay. If the accompanying factors amount to one order of magnitude increase, then we can estimate an upper bound to the theoretical uncertainty of 0.6%. This is acceptable even with an experimental precision of 2–3 %. This precision will not be attained experimentally in the immediate future in the case of heavy-quark baryon decays. We may then expect our result to be useful for the forthcoming experiments.

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APPENDIX A

As it was pointed our before, if we consider contributions up to first order in q, the model dependence can be handled by defining [6] effective form factors in M_0. Only the effective form factors f_1' and g_1' are functions of $p_+\cdot l=(p_1+p_2)\cdot l$. These quantities are the only energy-dependent contributions of the model dependence to the virtual radiative corrections. In the rest frame of A, $p_+\cdot l$ takes the form

$$p_+\cdot l=(M_1+E_2)E-|\mathbf{p}_2||l|y_0\ ,$$

which shows the direct dependence on E_2 and E, and an indirect dependence through y_0. The primes in Eq. (6) will remind us that the above primed form factors are the ones that appear in it:

$$f_1'(q^2,p_+\cdot l)=f_1(q^2)+\frac{\alpha}{\pi}a(p_+\cdot l)\ ,$$

$$g_1'(q^2,p_+\cdot l)=g_1(q^2)+\frac{\alpha}{\pi}a'(p_+\cdot l)\ ,$$

$$f_2'(q^2,p_+\cdot l)=f_2(q^2)+\frac{\alpha}{\pi}b\ ,$$

$$g_2'(q^2,p_+\cdot l)=g_2(q^2)+\frac{\alpha}{\pi}b'\ ,$$

$$f_3'(q^2,p_+\cdot l)=f_3(q^2)+\frac{\alpha}{\pi}c\ ,$$

$$g_3'(q^2,p_+\cdot l)=g_3(q^2)+\frac{\alpha}{\pi}c'\ .$$

In our approximations b, b', c, and c' are constant. Only a and a' are functions of $p_+\cdot l$.

APPENDIX B

The coefficients Q_i $(i=1,\ldots,5)$ depend on the form factor [12] as

$$Q_1=F_1^2\left[\frac{2E_2-M_2}{M_1}\right]+\frac{1}{2}F_2^2\left[\frac{M_2+E_2}{M_1}\right]+F_1F_2\left[\frac{M_2+E_2}{M_1}\right]+F_1F_3\left[1+\frac{M_2}{M_1}\right]\left[1-\frac{E_2}{M_1}\right]$$
$$+F_2F_3\left[\frac{M_2+E_2}{M_1}\right]\left[1-\frac{E_2}{M_1}\right]+G_1^2\left[\frac{2E_2+M_2}{M_1}\right]-\frac{1}{2}G_2^2\left[\frac{M_2-E_2}{M_1}\right]+G_1G_2\left[\frac{M_2-E_2}{M_1}\right]$$
$$+G_1G_3\left[\frac{M_2}{M_1}-1\right]\left[1-\frac{E_2}{M_1}\right]-G_2G_3\left[\frac{M_2-E_2}{M_1}\right]\left[1-\frac{E_2}{M_1}\right]+M_1^2Q_5\left[\left[\frac{M_1-E_2}{M_1}\right]^2-\frac{1}{2}\frac{q^2}{M_1^2}\right]\ ,$$

$$Q_2=-\frac{F_1^2}{M_1}-\frac{F_1F_2}{M_1}+\frac{F_1F_3}{M_1}\left[1+\frac{M_2}{M_1}\right]+\frac{F_2F_3}{M_1}\left[\frac{M_2+E_2}{M_1}\right]-\frac{G_1^2}{M_1}+\frac{G_1G_2}{M_1}$$
$$+\frac{G_1G_3}{M_1}\left[\frac{M_2}{M_1}-1\right]-\frac{G_2G_3}{M_1}\left[\frac{M_2-E_2}{M_1}\right]+\frac{2F_1G_1}{M_1}+M_1Q_5\left[\frac{M_1-E_2}{M_1}\right],$$
$$Q_3=Q_1-2\left[\frac{E_2-M_2}{M_1}\right]F_1^2-2\left[\frac{E_2+M_2}{M_1}\right]G_1^2-M_1^2Q_5\left[\left[1-\frac{E_2}{M_1}\right]^2-\frac{q^2}{M_1^2}\right],$$
$$Q_4=Q_2-4\frac{F_1G_1}{M_1},$$
$$Q_5=\frac{F_3^2}{M_1^2}\left[\frac{M_2+E_2}{M_1}\right]-\frac{G_3^2}{M_1^2}\left[\frac{M_2-E_2}{M_1}\right]-2\frac{F_1F_3}{M_1^2}+2\frac{G_1G_3}{M_1^2},$$

where

$$F_1=f_1'+\left[1+\frac{M_2}{M_1}\right]f_2,\quad F_2=-2f_2,\quad F_3=f_2+f_3,\quad G_1=g_1'-\left[1-\frac{M_2}{M_1}\right]g_2,\quad G_2=-2g_2,\quad G_3=g_2+g_3.$$

APPENDIX C

In this appendix we give the results for the double integrals that appear in the real inner bremsstrahlung contributions to the radiative corrections to the neutral and charged semileptonic decays.

The limits of integration in all the integrals are $-1\le x\le 1$ and $-1\le y\le y_0$. The arguments of the functions $F(y)$ and $K_i(x,y)$ are suppressed in the integrands:

$$\int\int K_0dy\,dx=4\pi(1+y_0),\quad \int\int FK_0dy\,dx=4\pi(1+y_0)^2|\mathbf{p}_2||\boldsymbol{l}|,$$

$$\int\int(1-\beta x)K_0dy\,dx=4\pi(1+y_0),\quad \int\int\frac{K_0}{1-\beta x}dy\,dx=2\pi(1+y_0)I_1,$$

$$\int\int\frac{xK_0}{1-\beta x}dy\,dx=\frac{2\pi}{\beta}\theta_0,\quad \int\int\frac{K_0}{(1-\beta x)^2}dy\,dx=2\pi(1+y_0)I_4,$$

$$\int\int\frac{FK_0}{(1-\beta x)^2}dy\,dx=2\pi|\mathbf{p}_2||\boldsymbol{l}|(1+y_0)^2I_4,\quad \int\int K_1dy\,dx=2\pi\theta_4,$$

$$\int\int xK_1dy\,dx=2\pi\theta_5,\quad \int\int FK_1dy\,dx=4\pi|\boldsymbol{l}|(|\mathbf{p}_2|y_0\theta_4+E_\nu^0\theta_5+|\boldsymbol{l}|\theta_{10}-|\mathbf{p}_2|\theta_{13}),$$

$$\int\int x^2K_1dy\,dx=2\pi\theta_{10},\quad \int\int\frac{K_1}{1-\beta x}dy\,dx=2\pi\theta_3,$$

$$\int\int\frac{xK_1}{1-\beta x}dy\,dx=\frac{2\pi}{\beta}(\theta_3-\theta_4),\quad \int\int\frac{(1-x^2)K_1}{1-\beta x}dy\,dx=\frac{2\pi}{\beta^2}[(\beta^2-1)\theta_3+\theta_4+\beta\theta_5],$$

$$\int\int\frac{yK_1}{1-\beta x}dy\,dx=\frac{2\pi}{|\mathbf{p}_2|\beta}[-(E+E_\nu^0)(\theta_3-\theta_4)+|\boldsymbol{l}|\theta_5+|\mathbf{p}_2|\beta\theta_{12}],$$

$$\int\int\frac{FK_1}{1-\beta x}dy\,dx=4\pi E[|\mathbf{p}_2|\beta y_0\theta_3+(E+E_\nu^0)(\theta_3-\theta_4)-|\boldsymbol{l}|\theta_5-|\mathbf{p}_2|\beta\theta_{12}],$$

$$\int\int\frac{xFK_1}{1-\beta x}dy\,dx=4\pi E\left[\frac{1}{\beta}(|\mathbf{p}_2|\beta y_0+E+E_\nu^0)(\theta_3-\theta_4)-(E+E_\nu^0)\theta_5-|\boldsymbol{l}|\theta_{10}+|\mathbf{p}_2|(\theta_{13}-\theta_{12})\right],$$

$$\int\int\frac{K_1}{(1-\beta x)^2}dy\,dx=2\pi\theta_2,\quad \int\int\frac{xK_1}{(1-\beta x)^2}dy\,dx=\frac{2\pi}{\beta}(\theta_2-\theta_3),$$

$$\int\int\frac{FK_1}{(1-\beta x)^2}dy\,dx=4\pi E[(|\mathbf{p}_2|\beta y_0+E+E_\nu^0)\theta_2-(2E+E_\nu^0)\theta_3+E\theta_4-|\mathbf{p}_2|\beta\theta_{11}],$$

$$\int\int FK_2dy\,dx=2\pi(-2|\boldsymbol{l}|\theta_5+\theta_8),\quad \int\int xFK_2dy\,dx=2\pi(\theta_{14}-2|\boldsymbol{l}|\theta_{10}),$$

$$\int\int \frac{FK_2}{1-\beta x}\,dy\,dx = 2\pi\left[-2E(\theta_3-\theta_4)+\theta_7\right], \quad \int\int \frac{xFK_2}{1-\beta x}\,dy\,dx = \frac{2\pi}{\beta}\left[-2E(\theta_3-\theta_4)+2|\boldsymbol{l}|\theta_5+\theta_7-\theta_8\right],$$

$$\int\int \frac{x^2FK_2}{1-\beta x}\,dy\,dx = \frac{2\pi}{\beta^2}[-2E(\theta_3-\theta_4-\beta\theta_5-\beta^2\theta_{10})+\theta_7-\theta_8-\beta\theta_{14}],$$

$$\int\int \frac{(1-x^2)FK_2}{1-\beta x}\,dy\,dx = \frac{2\pi}{\beta^2}[-2E(\beta^2-1)(\theta_3-\theta_4)-2|\boldsymbol{l}|\theta_5-2E\beta^2\theta_{10}+(\beta^2-1)\theta_7+\theta_8+\beta\theta_{14}],$$

$$\int\int \frac{F^2K_2}{1-\beta x}\,dy\,dx = 2\pi\left\{\theta_{16}+4E^2\left[E_\nu^0\beta^2\theta_3+(\theta_4-\theta_3)[3E_\nu^0+(3-\beta^2)E+2|\mathbf{p}_2|\beta y_0]+3\beta(E+E_\nu^0)\theta_5+3E\beta^2\theta_{10}\right]\right\},$$

$$\int\int \frac{FK_2}{(1-\beta x)^2}\,dy\,dx = 2\pi\left[-2E(\theta_2-\theta_3)+\theta_6\right], \quad \int\int \frac{xFK_2}{(1-\beta x)^2}\,dy\,dx = \frac{2\pi}{\beta}[-2E(\theta_2-2\theta_3+\theta_4)+\theta_6-\theta_7],$$

$$\int\int F^2K_3\,dy\,dx = 2\pi[-2|\boldsymbol{l}|^2(\theta_4-3\theta_{10})+\theta_{15}], \quad \int\int \frac{F^2K_3}{1-\beta x}\,dy\,dx = 2\pi[2E^2(3-\beta^2)\theta_3-6E^2(\theta_4+\beta\theta_5)+\theta_9],$$

$$\int\int \frac{xF^2K_3}{1-\beta x}\,dy\,dx = \frac{2\pi}{\beta}[2E^2(3-\beta^2)(\theta_3-\theta_4)-6E^2\beta\theta_5+\theta_9-6E^2\beta^2\theta_{10}-\theta_{15}],$$

$$\int\int \frac{K_1'}{1-\beta x}\,dy\,dx = \frac{2\pi(1+y_0)I_1}{M_2}, \quad \int\int \frac{FK_1'}{1-\beta x}\,dy\,dx = \frac{2\pi(1+y_0)^2I_1}{M_2}|\mathbf{p}_2||\boldsymbol{l}|,$$

$$I_1=\operatorname{arctanh}\beta/\beta, \quad \theta_0=(1+y_0)(I_1-2), \quad I_4=2/(1-\beta^2).$$

APPENDIX D

In this appendix we summarize in a compact form the H_i' and B_1'' that appear in the results for the real inner bremsstrahlung contributions to the radiative corrections to the SDNH and SDCH and in the final results, Eqs. (58), (61), and (62):

$$H_0' = |\mathbf{p}_2||\boldsymbol{l}|\left[\frac{1}{2}(Q_3-Q_4E_\nu^0)-E\left[\frac{B^+}{2M_1}+\frac{A^+}{M_1}\right]\right],$$

$$H_1' = Q_1EE_\nu^0-Q_2E|\mathbf{p}_2|(|\mathbf{p}_2|+|\boldsymbol{l}|y_0)-Q_3|\boldsymbol{l}|(|\mathbf{p}_2|y_0+|\boldsymbol{l}|)+Q_4E_\nu^0|\mathbf{p}_2||\boldsymbol{l}|y_0,$$

$$H_2' = \frac{|\mathbf{p}_2||\boldsymbol{l}|(1-\beta^2)}{2}\left[-\left[Q_1+Q_3\right]E_\nu^0+Q_4\left[E_\nu^0+E\right]^2+(Q_2+Q_4)|\mathbf{p}_2||\boldsymbol{l}|y_0+Q_2\mathbf{p}_2^2\right],$$

$$\begin{aligned}H_3' = \frac{|\mathbf{p}_2||\boldsymbol{l}|}{2}\Bigg\{&-\left\{2(Q_1-Q_2E_\nu^0+Q_4E)+(3-\beta^2)[Q_3-Q_2E-Q_4(E_\nu^0+2E)]\right\}E+(Q_1+Q_3)\left[\frac{1-\beta^2}{2}E+E_\nu^0+E\right]\\&-Q_3E_\nu^0(1-\beta x_0)-Q_2\left[|\mathbf{p}_2||\boldsymbol{l}|y_0+\mathbf{p}_2^2+E(E_\nu^0+E)+\frac{1-\beta^2}{2}E(E_\nu^0+3E)\right]\\&+Q_4\left[-E(E_\nu^0+E)+|\mathbf{p}_2|\beta E_\nu^0y_0-\frac{1-\beta^2}{2}E(5E_\nu^0+7E)\right]\\&+\frac{E(1-\beta^2)}{M_1}\Bigg[[(E_\nu^0+E)B^- -EB^+]+2\Bigg[E_\nu^0[A^-+g_1(f_2-g_2)]-2Eg_1(f_2+g_2)\\&\qquad+M_1\frac{h^+}{e}(E_\nu^0+2E)\Bigg]\Bigg]\Bigg\},\end{aligned}$$

$$H'_4=\frac{|\mathbf{p}_2||\boldsymbol{l}|}{2}\left[[Q_1+2Q_3-(Q_2+2Q_4)(E^0_\nu+2E)+Q_4E(1+\beta^2)]E-Q_1\frac{E}{2}+Q_3\left[E^0_\nu-\frac{E}{2}\right]\right.$$

$$+Q_2\left[\frac{E(E^0_\nu+E)}{2}+\frac{(2-\beta^2)E^2}{2}\right]+Q_4\left[-|\mathbf{p}_2||\boldsymbol{l}|y_0+\frac{E(E^0_\nu+E)}{2}+\frac{(4-3\beta^2)E^2}{2}-E^{02}_\nu\right]$$

$$+\frac{E}{M_1}\left[-(E^0_\nu+E)B^-+EB^++|\mathbf{p}_2|\beta y_0C^-\right]$$

$$+\frac{2E}{M_1}\left[-M_1\frac{h^+}{e}[E^0_\nu+2E(1-\beta^2)]+M_1\frac{h^-}{e}\beta^2E+E^0_\nu[g_1(f_2+3g_2)-(f_1f_2+g_1g_2)(1+\beta x_0)]\right.$$

$$\left.\left.+E[(1-\beta^2)(f_1f_2+4g_1g_2-g_1f_2)-f_1f_2+g_1(3f_2-2g_2)]\right]\right],$$

$$H'_5=\frac{|\mathbf{p}_2|\boldsymbol{l}^2}{4}\left[-2[Q_2E-Q_3+Q_4(E^0_\nu+2E)]+[Q_1+Q_3+Q_2(E-E^0_\nu)+Q_4(E-5E^0_\nu)]\right.$$

$$+\frac{2}{M_1}[(E^0_\nu-E)B^-+EB^++E^0_\nu C^-]$$

$$\left.+\frac{4}{M_1}\left[-M_1\frac{h^+}{e}(E^0_\nu+2E)-M_1\frac{h^-}{e}2E^0_\nu+E^0_\nu(2f_1f_2+3g_1g_2-g_1f_2)+2Eg_1(f_2+g_2)\right]\right],$$

$$H'_6=\frac{|\mathbf{p}_2||\boldsymbol{l}|(1-\beta^2)}{4}[Q_1+Q_3-(Q_2+Q_4)(E^0_\nu+E)],$$

$$H'_7=-\frac{|\mathbf{p}_2||\boldsymbol{l}|}{4}\left[(Q_1+Q_3)\frac{2E-E^0_\nu}{E}+(Q_2+Q_4)[|\mathbf{p}_2|\beta y_0-2(E^0_\nu+E)-E(1-\beta^2)]\right.$$

$$+\frac{Q_2\mathbf{p}_2^2}{E}+Q_4\frac{(E^0+E)^2}{E}+\frac{E}{M_1}(1-\beta^2)B^-$$

$$\left.-\frac{2}{M_1}\left[M_1\frac{h^+}{e}(E^0_\nu+\beta^2E+|\mathbf{p}_2|\beta y_0)-E^0_\nu(1-\beta x_0)g_1(g_2-f_2)-E(1-\beta^2)A^-\right]\right],$$

$$H'_8=\frac{|\mathbf{p}_2||\boldsymbol{l}|}{4}\left[Q_1+Q_3-Q_2(E^0_\nu+2E)+Q_4(E^0_\nu-E)+\frac{(E-2E^0_\nu)B^-}{M_1}\right.$$

$$\left.+\frac{2}{M_1}\left[\frac{E^0_\nu}{e}M_1(2h^--h^+)-E^0_\nu[2f_1f_2-g_1(f_2+g_2)]+EA^-\right]\right],$$

$$H'_9=\frac{|\mathbf{p}_2|\beta}{8}[-Q_1-Q_3+(Q_2+Q_4)(E^0_\nu+E)],$$

$$H'_{10}=\frac{|\mathbf{p}_2||\boldsymbol{l}|^3}{4}\left\{-(Q_2+5Q_4)+\frac{2}{M_1}\left[2B^-+C^-+2\left[-2M_1\frac{h^+}{e}-3M_1\frac{h^-}{e}+3f_1f_2+4g_1g_2-3g_1f_2\right]\right]\right\},$$

$$H'_{11}=0,$$

$$H'_{12}=|\mathbf{p}_2|^2|\boldsymbol{l}|^2\left[-\frac{Q_4}{2}+\frac{1}{2M_1}B^++\frac{1}{M_1}\left[M_1\frac{h^+}{e}+A^+-g_1(g_2-f_2)\right]\right],$$

$$H'_{13}=\frac{|\mathbf{p}_2||^2\boldsymbol{l}|^2}{2}\left[Q_4-\frac{C^-}{M_1}-\frac{2}{M_1}\left[M_1\frac{h^+}{e}+f_2(f_2+g_1)\right]\right],$$

$$H'_{14}=-\frac{|\mathbf{p}_2||\boldsymbol{l}|^2}{4}\left[Q_2-2\left[\frac{B^-}{2M_1}+\frac{A^-}{M_1}\right]\right],$$

$$H'_{15}=|\mathbf{p}_2||\boldsymbol{l}|\left[-\frac{1}{8}(Q_2+Q_4)+\frac{1}{4M_1}B^-+\frac{1}{2M_1}\left[-M_1\frac{h^-}{e}+f_2(f_1-g_1)\right]\right],$$

$$H'_{16}=\frac{|\mathbf{p}_2|}{4M_1}\beta\left[-M_1\frac{h^+}{e}+g_1(g_2-f_2)\right].$$

We have used the definitions

$$A^{\pm}=f_1f_2-g_1g_2\pm 2g_1f_2\ ,$$

$$h^{\pm}=-g_1^2(\kappa_1+\kappa_2)\pm f_1g_1(\kappa_2-\kappa_1)\ ,$$

$$B^{\pm}=(f_1^{\mp}g_1)^2+2f_1(f_3-f_2)\ ,$$

$$C^-=f_1^2-g_1^2+2f_1(f_3-f_2)\ ,$$

and

$$B_1''=Q_1EE_\nu^0-Q_2E|\mathbf{p}_2|(|\mathbf{p}_2|+|\boldsymbol{l}|y_0)\ .$$

APPENDIX E

For completeness and handiness, we reproduce here all of the θ_i functions that appear in the final results Eqs. (61) and (62), which are scattered in several references:

$$\theta_0=2(1+y_0)\left[\frac{\mathrm{arctanh}\beta}{\beta}-1\right],$$

$$\theta_1=I_0\left[\frac{M_1(1+y_0)}{\lambda}\right]+C+C_1\ ,$$

and

$$\theta_i=\frac{1}{|\mathbf{p}_2|}\left[T_i^++T_i^-\right],\quad i=2,\ldots,16\ ,$$

with

$$T_2^{\pm}=\pm\frac{1\mp a^{\pm}}{(1\pm\beta)(1+\beta a^{\pm})}\ln\left[\frac{1\mp\beta}{1-\beta x_0}\right]\pm\frac{(1\pm x_0)\ln(1\pm x_0)}{(1\pm\beta)(1-\beta x_0)}\pm\frac{1\pm a^{\pm}}{(1\mp\beta)(1+\beta a^{\pm})}\ln(1\pm a^{\pm})-\frac{(x_0+a^{\pm})\ln(\pm x_0\pm a^{\pm})}{(1+\beta a^{\pm})(1-\beta x_0)}\ ,$$

$$T_3^{\pm}=\frac{1}{2\beta}\left[L\left[\frac{1-\beta}{1-\beta x_0}\right]-L\left[\frac{1-\beta x_0}{1+\beta}\right]-L\left[\frac{1+\beta a^-}{1+\beta x_0}\right]+L\left[\frac{1+\beta a^-}{1+\beta}\right]\right.$$
$$\left.+L\left[\frac{1-\beta x_0}{1+\beta a^+}\right]-L\left[\frac{1-\beta}{1+\beta a^+}\right]+\ln\left[\frac{1-\beta x_0}{1-\beta}\right]\ln\left[\frac{1+\beta a^+}{1+\beta}\right]\right],$$

$$T_4^{\pm}=(x_0\pm 1)\ln(1\pm x_0)\pm(1\pm a^{\pm})\ln(1\pm a^{\pm})-(x_0+a^{\pm})\ln(\pm x_0\pm a^{\pm})\ ,$$

$$T_5^{\pm}-\frac{1}{2}\left\{(1-x_0^2)\ln(1\pm x_0)+(x_0\mp 1)a^{\pm}+1-(1-a^{\pm 2})\ln(1\pm a^{\pm})+(x_0^2-a^{\pm 2})\ln[\pm(x_0+a^{\pm})]\right\},$$

$$T_6^{\mp}=\left[-|\boldsymbol{l}|+\mathbf{p}_2|\pm\frac{\beta E_\nu^0(x_0+a^{\mp})}{1+\beta a^{\mp}}\right]I_4\pm\frac{\beta E_\nu^0(x_0+a^{\mp})}{(1+\beta a^{\mp})^2}I_1+\left[E_\nu^0-\frac{\beta E_\nu^0(x_0+a^{\mp})}{1+\beta a^{\mp}}\right]J_4$$
$$-\frac{\beta E_\nu^0(x_0+a^{\mp})}{(1+\beta a^{\mp})^2}J_1\pm\frac{E_\nu^0(x_0+a^{\mp})}{(1+\beta a^{\mp})^2}I_2^{\mp}-\frac{E_\nu^0(x_0+a^{\mp})}{(1+\beta a^{\pm})^2}J_2^{\mp}\ ,$$

$$T_7^{\pm}=\left[-|\boldsymbol{l}|+\mathbf{p}_2|\mp\frac{\beta E_\nu^0(x_0+a^{\pm})}{1+\beta a^{\pm}}\right]I_1\mp\frac{E_\nu^0(x_0+a^{\pm})}{1+\beta a^{\pm}}I_2^{\pm}+\left[E_\nu^0-\frac{\beta E_\nu^0(x_0+a^{\pm})}{1+\beta a^{\pm}}\right]J_1-\frac{E_\nu^0(x_0+a^{\pm})}{1+\beta a^{\pm}}J_2^{\pm}\ ,$$

$$T_8^{\pm}=-2(|\boldsymbol{l}|-|\mathbf{p}_2|+E_\nu^0x_0)\mp E_\nu^0(x_0+a^{\pm})I_2^{\pm}-E_\nu^0(x_0+a^{\pm})J_2^{\pm}\ ,$$

$$\frac{T_9^{\pm}}{4l^2}=-\frac{3E}{2l^2}(|\boldsymbol{l}|-|\mathbf{p}_2|+E_\nu^0x_0)+\left[\frac{3(|\boldsymbol{l}|-|\mathbf{p}_2|)}{4\beta|\boldsymbol{l}|}+\frac{3E_\nu^0|\mathbf{p}_2|}{4l^2}+\beta G^{\pm}\right]I_1$$
$$\mp\frac{E_\nu^{0^2}(x_0+a^{\pm})^2}{4l^2(1+\beta a^{\pm})}I_3^{\pm}-\frac{E_\nu^{0^2}(x_0+a^{\pm})^2}{4l^2(1+\beta a^{\pm})}J_3^{\pm}+\left[-\frac{3E_\nu^0}{4\beta|\boldsymbol{l}|}+\frac{3E_\nu^0(E_\nu^0+|\boldsymbol{l}|x_0)}{4l^2}\pm\beta G^{\pm}\right]J_1\pm G^{\pm}J_2^{\pm}+G^{\pm}I_2^{\pm}\ ,$$

$$T_{10}^{\mp}=\tfrac{1}{3}(x_0^3\mp 1)\ln(1\mp x_0)+\tfrac{1}{3}(a^{\mp 3}\mp 1)\ln(1\mp a^{\mp})-\tfrac{1}{3}(x_0^3+a^{\mp 3})\ln[\mp(x_0+a^{\mp})]$$
$$+\tfrac{1}{6}(1-x_0^2)(a^{\mp}+1)-\tfrac{1}{3}(x_0\pm 1)(1-a^{\mp 2}) ,$$

$$T_{11}^{\pm}=\frac{1}{2|\mathbf{p}_2|\beta}\left\{E_\nu^0[(1-\beta x_0)J_4-J_1]-(\beta E_\nu^0+|\boldsymbol{l}|-|\mathbf{p}_2|)I_4+(|\boldsymbol{l}|-|\mathbf{p}_2|)I_1\right\} ,$$

$$T_{12}^{\pm}=\frac{1}{2|\mathbf{p}_2|\beta}[E_\nu^0(1-\beta x_0)J_1+2E_\nu^0 x_0+2(|\boldsymbol{l}|-|\mathbf{p}_2|)-(\beta E_\nu^0+|\boldsymbol{l}|-|\mathbf{p}_2|)I_1] ,$$

$$T_{13}^{\pm}=-\frac{1}{2|\mathbf{p}_2|}E_\nu^0(1-x_0^2) ,$$

$$T_{14}^{\pm}=E_\nu^0[1+x_0^2+2a^{\pm}(x_0\mp 1)\pm a^{\pm}(x_0+a^{\pm})(I_2^{\pm}\pm J_2^{\pm})] ,$$

$$T_{15}^{\pm}=3E_\nu^0[2|\mathbf{p}_2|(1+y_0)+|\boldsymbol{l}|(1-x_0^2)]-E_\nu^{0^2}(x_0+a^{\pm})^2(J_3^{\pm}\pm I_3^{\pm})-2|\boldsymbol{l}|E_\nu^0(x_0+a^{\pm})a^{\pm}(J_2^{\pm}\pm I_2^{\pm}) ,$$

$$T_{16}^{\pm}=4l^2\left[\frac{3}{2\beta^2}[2(|\boldsymbol{l}|-|\mathbf{p}_2|+E_\nu^0 x_0)+\beta E_\nu^0(1-x_0^2)]\right.$$
$$+\left[-\frac{3(|\boldsymbol{l}|-|\mathbf{p}|+\beta E_\nu^0)}{2\beta^2}-|\mathbf{p}_2|(1+y_0)+\frac{|\mathbf{p}_2|E_\nu^{0^2}}{2l^2}\right]I_1$$
$$\left.-\frac{E_\nu^{0^2}(x_0+a^{\pm})^2}{2|\boldsymbol{l}|(1+\beta a^{\pm})}\left[\beta J_1+J_2^{\pm}\pm\beta I_1\pm I_2^{\pm}\right]+\left[\frac{3E_\nu^0(1-\beta x_0)}{2\beta^2}+\frac{E_\nu^{0^2}(E_\nu^0+|\boldsymbol{l}|x_0)}{2l^2}\right]J_1\right] ,$$

where

$$x_0=-\frac{|\mathbf{p}_2|y_0+|\boldsymbol{l}|}{E_\nu^0} ,\quad a^{\pm}=\frac{E_\nu^0\pm|\mathbf{p}_2|}{|\boldsymbol{l}|} ,$$

$I_1, I_2^{\pm}, I_3^{\pm}, I_4, J_1, J_2^{\pm}, J_3^{\pm}$, and J_4 are given by

$$G^{\pm}=\mp\frac{\beta E_\nu^{0^2}(x_0+a^{\pm})^2}{4l^2(1+\beta a^{\pm})^2}\mp\frac{a^{\pm}(a^{\pm 2}-1)}{4(1+\beta a^{\pm})} ,$$

$$I_1=\frac{2}{\beta}\,\text{arctanh}\beta ,\quad I_2^{\pm}=\ln\left|\frac{a^{\pm}+1}{a^{\pm}-1}\right| ,$$

$$I_3^{\pm}=\frac{2}{a^{\pm 2}-1} ,\quad I_4=\frac{2}{1-\beta^2} ,$$

$$J_1=-\frac{1}{\beta}\left[\ln\left[\frac{1+\beta}{1-\beta x_0}\right]+\ln\left[\frac{1-\beta}{1-\beta x_0}\right]\right] ,$$

$$J_2^{\pm}=\ln\left|\frac{a^{\pm}-1}{a^{\pm}+x_0}\right|+\ln\left|\frac{a^{\pm}+1}{a^{\pm}+x_0}\right| ,$$

$$J_3^{\pm}=-2\left[\frac{a^{\pm}}{a^{\pm 2}-1}-\frac{1}{a^{\pm}+x_0}\right] ,$$

and

$$J_4=\frac{2}{\beta}\left[\frac{1}{1-\beta^2}-\frac{1}{1-\beta x_0}\right] .$$

The θ_1 in Eq. (62) is given in terms of

$$I_0[M_1(1+y_0)/\lambda]=2\ln[M_1(1+y_0)/\lambda]\times(\text{arctanh}\beta/\beta-1) ,$$

$$C=2\ln 2\left[\frac{\text{arctanh}\beta}{\beta}-1\right]+1+\frac{\text{arctanh}\beta}{2\beta}\left[2+\ln\left[\frac{1-\beta^2}{4}\right]\right]+\frac{1}{\beta}L\left[\frac{2\beta}{1+\beta}\right]-\ln\left[\frac{1+\beta}{2}\right]\frac{\text{arctanh}\beta}{\beta} ,$$

and

$$C_1=-\frac{1}{2}\left\{\ln\left|\frac{(a^+-1)(1-a^-)}{4|\mathbf{p}_2|/M_1}\right|\left[\frac{\beta^2-1}{\beta(1-\beta x_0)}-\frac{\beta-1}{\beta}-\frac{2}{\beta}\ln\left[\frac{1-\beta x_0}{1+\beta}\right]-(1+x_0)\right]\right.$$
$$-\ln\left|\frac{(a^++1)(a^-+1)}{4|\mathbf{p}_2|/M_1}\right|\left[\frac{\beta^2-1}{\beta(1-\beta x_0)}+\frac{1+\beta}{\beta}+\frac{2}{\beta}\ln\left[\frac{1-\beta}{1-\beta x_0}\right]+(1-x_0)\right]$$
$$+\left[-8\ln 2+\frac{1+\beta}{\beta}\ln(1+\beta)-\frac{1-\beta}{\beta}\ln(1-\beta)+2+\frac{2+\beta(1-x_0)}{1-\beta x_0}(1-x_0)\ln(1-x_0)\right.$$
$$\left.-2\ln(1-\beta x_0)+\frac{2-\beta(1+x_0)}{1-\beta x_0}(1+x_0)\ln(1+x_0)\right]$$
$$\left.+\frac{2}{\beta}\left[L\left[\frac{1-\beta}{1-\beta x_0}\right]-2L\left[\frac{1-\beta}{1+\beta}\right]+L\left[\frac{1-\beta x_0}{1+\beta}\right]+\ln\left[\frac{\beta}{1+\beta}\right]\ln\left[\frac{1-\beta}{1+\beta}\right]-\frac{1}{2}\ln^2\left[\frac{1-\beta x_0}{1+\beta}\right]\right]\right\}.$$

Finally the model-independent functions $\Phi(E)$ and $\Phi'(E)$ containing terms up to order $\alpha q/\pi M_1$ are explicitly given by

$$\Phi(E)=2\left[\frac{1}{\beta}\operatorname{arctanh}\beta-1\right]\ln\left[\frac{\lambda}{m}\right]-\frac{1}{\beta}(\operatorname{arctanh}\beta)^2+\frac{1}{\beta}L\left[\frac{2\beta}{1+\beta}\right]$$
$$-\frac{1}{\beta}L\left[\frac{2\beta}{M_1/E-1+\beta}\right]+\frac{1}{\beta}\operatorname{arctanh}\beta\left[1+\frac{E(1-\beta^2)}{M_1-2E}\right]+\frac{3}{2}\ln\left[\frac{M_1}{m}\right]-\frac{11}{8}$$
$$-\frac{1}{\beta}\ln\left[1-\frac{2\beta}{M_1/E-1+\beta}\right]\left[\ln\left[\frac{M_1}{m}\right]-\operatorname{arctanh}\beta\right],$$

and

$$\Phi'(E)=\frac{1-\beta^2}{\beta}\left[-\operatorname{arctanh}\beta\left[1+\frac{E}{M_1-2E}\right]+\frac{\beta E}{M_1-2E}\ln\left[\frac{M_1}{m}\right]\right].$$

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