Branching processes and multiparticle production

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The general theory of branching processes is used for establishing the relation between the parameters k and $\bar{n}$ of the negative binomial distribution. This relation gives the possibility to describe the overall data on multiplicity distributions in $pp(p\bar{p})$ collisions for energies up to 900 GeV and to make several interesting predictions for higher energies. This general approach is free from ambiguities associated with the extrapolation of the parameter k to unity.

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I. INTRODUCTION

A theoretical description of multiparticle production today is beyond the limits of QCD, and the natural approach is to look for empirical relations.

The most popular in this field was the Koba-Nielsen-Olesen (KNO) scaling for multiplicity distributions which was satisfied very well for hadronic collisions up to CERN Intersecting Storage Rings (ISR) energies and for e^+e^- annihilation. The evident violation of the KNO scaling at energies of the CERN collider [1] attracted much attention to the negative binomial distribution (NBD) which describes fairly well the overall features of the data on the multiplicity distribution (MD) of hadrons in different processes $[pp(\bar{p}p), e^+e^-, \nu p, AA, \ldots]$, in different ranges of rapidity and in a wide interval of energies [2,3]. It is especially relevant to the $pp(\bar{p}p)$ interaction.

Taking into account the special role of NBD in describing the multiplicity distribution at high energy, it seems important to consider the NBD on the basis of general assumptions about the character of the process of particle production in hadronic collisions without detailed specification of the dynamics. In Ref. [4] it was proposed as a basis for the NBD to consider the multiple production as a random stationary branching process which is a rather general probabilistic model for the processes of the multiplication and transformation of the active particles. In this approach the transformation of each particle is independent of the history of the process and of the transformation of other particles, obeying the general probabilistic laws of Markov processes. The same refers to the fate of the generation of each particle.

The branching processes may find their realization in terms of the quark-gluonic cascades, corresponding to the microscopic description of the nonequilibrium evolution of the partonic system, e.g., in the rapidity space [5,6]. It is important to stress that for us there is no need to know the details of the dynamical laws governing these cascades.

It was realized that the system of produced hadrons may be considered as a result of the contribution from coherent and chaotic components (so-called two-component model) and it is known that in $pp(\bar{p}p)$ collisions at high energy the chaotic component [7–10] dominates, which described by NBD, whereas in e^+e^- annihilation the coherent (Poisson) part is essential.

The observation of dynamical chaos in the dynamics of non-Abelian gauge fields (see, e.g., [11]) raises the question about the role and origin of the chaotic component in hadronic collisions.

There exists an interesting observation [7,9] that, in distinction to e^+e^- annihilation, where the addition of the small ($\approx 10-20\,\%$) chaotic (noise) amplitude essentially changes the multiplicity distribution, in $pp(\bar{p}p)$ collisions the addition of a small coherent component to the NBD does not change the shape of the distribution significantly.

Taking into consideration the above mentioned arguments and remarks we here consider the NBD as adequate for the description of the multiplicity distribution in $pp(\bar{p}p)$ collisions at high energy.[1] The NBD

$$P_n^{(k)} = \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)} \frac{(\bar{n}/k)^n}{(1+\bar{n}/k)^{n+k}} \tag{1}$$

has two parameters $\bar{n}$ and k. $\bar{n}$ is the average multiplicity. As for k, initially it was associated with the number of chaotically emitting cells. After the UA5 experiments [1,10,19,14] it is clear that such a meaning of k in general is not necessary, because data from the CERN Super Proton Synchrotron ($Sp\bar{p}S$) yield the empirical relation

[1]In principle on the basis of the quantum optics it is possible to generalize the NBD to take into account the chaotic as well as the coherent components [7,12].

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$$\frac{1}{k}=a+b\ln\sqrt{s} \quad (a\approx -0.1, b\approx 0.06)\ , \tag{2}$$

which is valid up to energy $\sqrt{s}=900$ GeV, not showing a tendency for saturation. So, at such energies the KNO-like scaling continues to be violated, which is more clearly expressed in the observed strong rise of the moments:

$$C_q=\frac{\langle n^q\rangle}{\langle n\rangle^q} \tag{3}$$

with $\sqrt{s}$ [14].

Of course, one cannot extrapolate the concrete form of the empirical relation (1) to higher energy since this would lead to a contradiction. For instance, from (1) it would follow that the peak of the distribution would be at $n=0$ at very high energy when $k=1$. This means that saturation of k must take place at ultrahigh energies at a value larger than unity (see also [15]). It indicates the necessity to establish the relation between k and $\sqrt{s}$ (or, at least, between k and $\bar{n}$) based on general theoretical considerations. We propose that such a basis could be a general theory of branching processes. As mentioned such an approach was developed in [4] where the idea of the *stationarity* of the branching process was used for establishing the relation between k and $\bar{n}$. The result

$$\frac{1}{k}=a+b\ln\frac{\bar{n}}{k} \quad (a\approx 0.12, b\approx 0.08) \tag{4}$$

gives the unconfined though weaker rise of $1/k$ with $\sqrt{s}$ leading to the above difficulty associated with the extrapolation of k to unity. Unfortunately, in deriving (4) the authors of [4] incorrectly used the conditions for stationarity of the branching process. In the paper [16] the condition for stationarity is also used though the author was inconsistent (see below) in deriving the relation between k and $\bar{n}$. This dependence of k on $\bar{n}$,

$$k=A\left[\frac{\bar{n}}{k}\right]^B \quad (A\simeq 11, B\approx -0.5) \tag{5}$$

again did not avoid the problem resulting from the extrapolation of k to unity.

II. RELATION BETWEEN k AND $\bar{n}$

Thus, we consider the NBD as a result of the stationary branching process with one sort of multiplied particles (pions) and continuous evolution parameter t. The generating function F for such a process satisfies the reverse Kolmogorov differential equation [17]:

$$\frac{dF}{dt}=-f(F,t)\ . \tag{6}$$

For the generating function of the NBD,

$$F(x,t)=\sum_n P_n^{(k)}x^n=\left[1+\frac{\bar{n}}{k}(1-x)\right]^{-k} \tag{7}$$

$-f(F,t)$ equals

$$-F\ln F\frac{\dot{k}}{k}+F(1-F^{1/k})k\frac{\dot{m}}{m}\ , \tag{8}$$

where $m=\bar{n}/k$ and $\dot{k}=dk/dt$, etc.

For a stationary branching process $f(F,t)$ is factorized, $f(F,t)=\varphi(F)\psi(t)$. Evidently the condition

$$\frac{\dot{k}}{k}=\text{const}\times k\frac{\dot{m}}{m}\ , \tag{9}$$

[4] which leads to the relation (4) does not give such a factorization. For $F\approx 1$ factorization takes place [16] if one changes only term $1-F^{1/k}$ in (8) by $(1-F)/k$, but of course, it is also necessary to consider the first term in (8) at $F\approx 1$. If one will do so, then it is easy to find that at the first order of the parameter $(1-F)/k$, no condition has arisen for the factorization: it fulfills automatically. So, the relation (14) used in [16] between k and $\bar{n}$ is not the result of the theory of the branching processes.

The adequate parameter here is

$$\delta=\frac{\ln F}{k}\ . \tag{10}$$

Expanding $1-F^{1/k}$ up to δ^2 in (8), it is easy to obtain the solution of the resulting differential equation

$$\frac{\dot{k}}{k}=\left[\frac{\alpha}{k}-1\right]\frac{\dot{m}}{m} \quad (\alpha=\text{const})\ , \tag{11}$$

which is a necessary and sufficient condition for factorization. (It is easy to see that at the first order in δ the factorization takes place automatically, without any condition as it must be in [16] too.)

Thus solving (11) we have

$$k=\frac{a\bar{n}}{\bar{n}-b}\ , \tag{12}$$

where a and b are the constants which one must find from comparison with experimental data. Thus it is possible to state that NBD with relation (12) between its parameters k and $\bar{n}$ is the consequence of a stationary branching process.

The function $k(\bar{n})$ is very simple. At $ab>0$ k is decreasing from a to $-\infty$ and from $+\infty$ to a. But *experimentally* (at least for $10<\sqrt{s}<900$ GeV) k is decreasing with $\sqrt{s}$ [1,14,10], so physically interesting is a case $ab>0$. But at the same time the case $a<0, b<0$ is also unphysical, because it corresponds to $\bar{n}<0$, or to $k<0$. By the same reason, if we do not want to have negative k, we must discard the lower branch of (12) with $ab>0$ corresponding to decreasing k in $(a,-\infty)$ interval.[2]

[2]In this connection there is an interesting observation in [18] that the multiplicity data for small ($\sqrt{s}<10$ GeV) energies is possible to describe well with negative k. From this point of view it may be said that two branches of the hyperbola (12) naturally divide the large energy region ($\bar{n}>b>0$) from small energies ($\bar{n}<b$). From our fit (see below) $b\approx 7$, so it means that we must not consider energies below $\sqrt{s}\approx 14$ GeV.

Thus this simple analysis has shown that in the region of high energy the relation between k and $\bar{n}$ is given by (12) with positive a,b, and it is necessary to confine to the branch of the hyperbola (12) in the first quadrant.

III. CONSEQUENCES AND PREDICTIONS OF THE MODEL

The relation (12) between k and $\bar{n}$, in spite of its simplicity, is rather rich in content. Let us stress once more that this relation must be only used for $\bar{n} > b \approx 7$, i.e., for $\sqrt{s} > 14$ GeV; smaller energies should not be considered here. Our model ensures that $k > a > 0$ (from the fit follows $a \approx 3$), implying that the limit $k=1$ never is achieved. Thus there does not exist the difficulty associated with $k=1$ at very high energy that is characteristic of some other ansatz [1,4,16].

From (12) it follows that asymptotically, when k goes to the saturation, KNO scaling (or the dependence of $\bar{n}P_n^{(k)}$ on $\sqrt{s}$ only through $\bar{n}$) is restored. The asymptotic distribution function at $\sqrt{s} \gg 14$ GeV and $\bar{n} \gg k = a \approx 3.06$ has a form of Γ distribution:

$$\psi(z) \equiv \bar{n}P_n^{(k)} \approx \frac{k^k z^{k-1} e^{-kz}}{\Gamma(k)} = 14.49 z^{2.06} e^{-3.06z} \quad \left[z = \frac{n}{\bar{n}} \right] . \tag{13}$$

Our model give rather clear predictions for C_q moments. In particular, Wroblewski's relation here takes place well at high energies:

$$\frac{D_2}{\bar{n}} \equiv \frac{\sqrt{\overline{n^2} - \bar{n}^2}}{\bar{n}} = (C_2 - 1)^{1/2} = \left[\frac{1}{k} + \frac{1}{\bar{n}} \right]^{1/2} \approx 0.57 \left[1 - \frac{1.92}{\bar{n}} \right] . \tag{14}$$

Now, when always $k > 1$ the second-order correlation $g^{(2)} = \overline{n(n-1)}/\bar{n}^2$ which is increasing slowly and asymptotically equals 1.33 indicates not necessarily the presence of a coherent component as sometimes stated but just the fact that k is always larger than unity. All high order moments C_q are rising and saturate asymptotically, as is easy to understand from the relation $(q > 2)$

$$C_q = 1 + \sum_{m=0}^{q-2} P_m^{(q)} \left[\frac{1}{k} \right] (C_2 - 1)^{q-m-1} , \tag{15}$$

where $P_m^q(1/k)$ are polynomials of order m with positive coefficients $(P_0^{(q)} = 1)$ and $1/k = (1/a) - (b/a)(1/\bar{n})$ is increasing. Asymptotically we have

$$C_q \approx \frac{\Gamma(k+q)}{k^{q-1}\Gamma(k+1)} , \tag{16}$$

i.e. (up to $1/\bar{n}^2$),

$$\begin{aligned} C_2 &\approx 1.33 - \frac{1.27}{\bar{n}} , \\ C_3 &\approx 2.21 - \frac{5.82}{\bar{n}} , \\ C_4 &\approx 4.39 - \frac{21.3}{\bar{n}} , \\ C_5 &\approx 10.19 - \frac{76.11}{\bar{n}} . \end{aligned} \tag{17}$$

The scaled peak of the multiplicity distributions is moving to the left toward its asymptotic value:

$$z_{\text{peak}}(\sqrt{s}) = \frac{n_{\text{peak}}}{\bar{n}} = 1 - \frac{1}{a} + \frac{b}{a}\frac{1}{\bar{n}} = 0.67 + \frac{2.27}{\bar{n}} . \tag{18}$$

Finally, before going into the comparison with experimental data, let us make one comment. By no means do we consider the limiting value of $k = a$ as an indication that corresponds to asymptotic value of the number of clusters, fireballs, minijets, etc., in the multiple production.[3]

Let us stress only that the often used value of $k=1$ is meaningless.[4] In particular, in connection with this value of k in [16] a very strong and unusual statement was made that in the process of the multiple production the information entropy achieves its maximal value for $k=1$ and as a result $\bar{n}$ achieves its maximal value and thus does not depend on the energy at all. This statement is derived from the fact that this entropy for the NBD near $k=1$ behaves as $\ln\bar{n} + 1 - (\pi^2/6 - 1)(k-1)^2$. But the lower bound on $k \geq a$ in our model shows that such a statement is a result of the unphysical interpolation of k to unity.

IV. COMPARISON WITH EXPERIMENTS

To obtain numerical values of constants a and b in (12) we used the results of the fit of parameters $\bar{n}$ and k of NBD by experimental distributions of charged particles multiplicity in the range from $\sqrt{s} = 19.5$ GeV to $\sqrt{s} = 900$ GeV [10,14,19] (nonsingle diffractive events). The results of the fit of a and b in (12) on the basis of these data gives

[3]Note that some experiments (see [20]) are indicating that at sufficiently high energies ($E_L \approx 400$ GeV) the number of the clusters produced in pp interactions is 4.2 ± 1.7. If we continue such an interpretation of k, then k^{-1} may be considered as the ratio of the probability for two particles to be emitted from one cluster to the probability of emission of these particles by two different clusters [6]. So the asymptotic "aggregation" degree is $a^{-1} \approx 0.3$.

[4]The $k=1$ in our model corresponds to the negative $\bar{n}$ [lower branch of (12) from a to $-\infty$]. Notice that $k > a \simeq 3$ shows that our expansion parameter of $\delta = k^{-1}\ln F$ is adequate and self-consistent. This also means that the factorization condition $f(F,t) = \varphi(F)\psi(t)$, sometimes expressed as "stationarity," is a good first approximation.

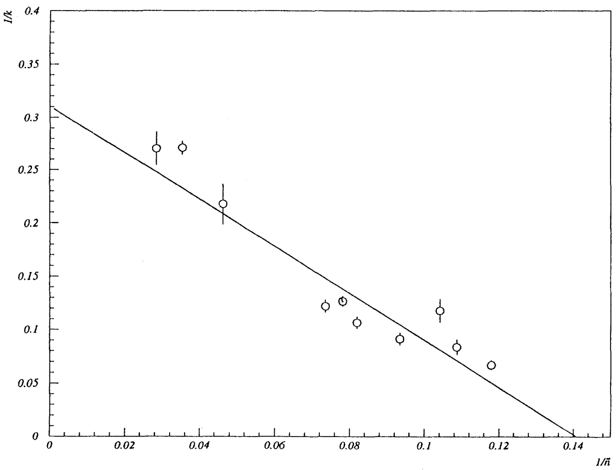

FIG. 1. The dependence of $1/k$ on $1/\bar{n}$ from Eq. (12) with coefficients $a=3.06$, $b=6.95$ obtained by fit.

$$a=3.06\pm0.06\ ,\quad b=6.95\pm0.08\ . \tag{19}$$

In Fig. 1 is shown the function (12) plotted in $1/k$ versus $1/\bar{n}$ with these values for a and b. We did not consider points corresponding to low energy (see footnote 2). Figure 2 gives the curves for C_q ($q=2-5$) for our model (solid line) and compares them with experimental data [10,14,19] for nonsingle diffractive component of $pp(\bar{p}p)$ reactions. It is seen that higher moments ($q=4,5$) have not yet achieved their asymptotic values ($C_q=4.39$ and $C_5=10.19$ at $\sqrt{s}=900$ GeV).

In Figs. 3 and 4 are shown the distribution $\bar{n}P_n^{(k)}$ for the Fermilab Tevatron and Superconducting Super Collider (SSC) as a function of $z=n/\bar{n}$. Solid lines show the

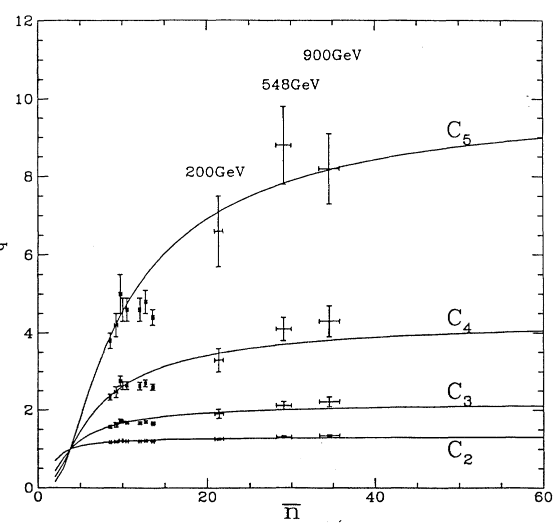

FIG. 2. The C_q moments ($q=2-5$) as a function of $\bar{n}$ from (12) (solid lines) compared with experimental data on inelastic, nonsingle-diffractive component of $pp(\bar{p}p)$ reactions (see Table 2 from [27] and [10,14]).

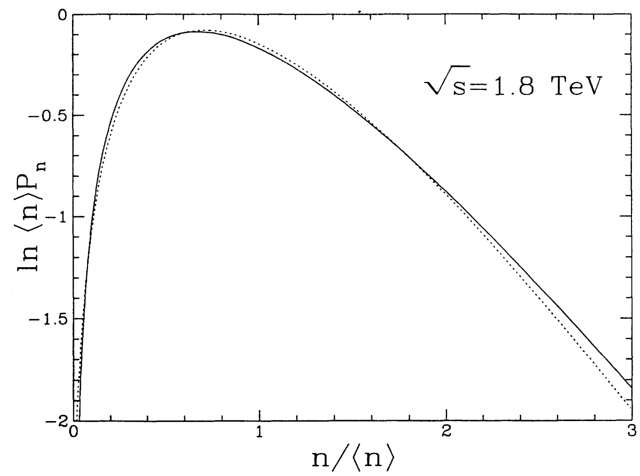

FIG 3. The dependence of $\bar{n}P_n$ on $z=n/\bar{n}$. Dashed line for $\sqrt{s}=1.8$ TeV, solid line is an asymptotic distribution.

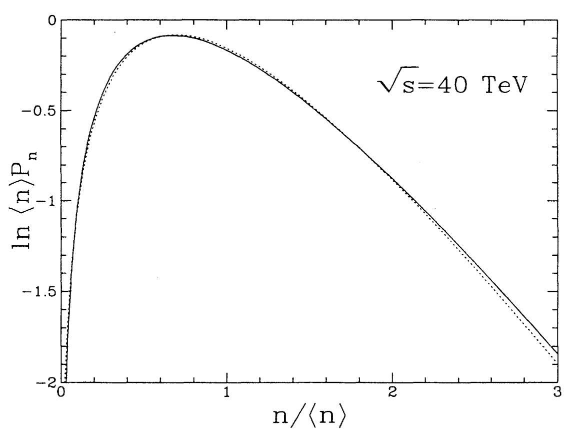

FIG. 4. Same for $\sqrt{s}=40$ TeV.

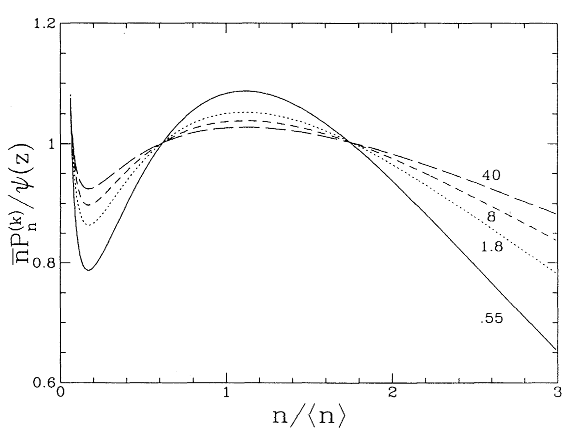

FIG. 5. The dependence of $\bar{n}P_n^{(k)}/\psi(z)$ on $z=n/\bar{n}$ for energies 0.55, 1.8, 8, and 40 TeV.

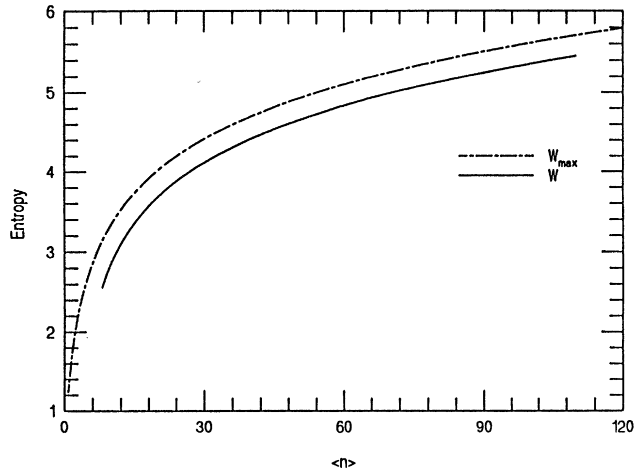

FIG. 6. The information entropy as a function of $\bar{n}$. Solid line is for our model with k dependence of (12). Dashed-dotted line corresponds to the "maximal" entropy ($k=1$).

asymptotic distribution (13). These curves show the systematic shift to the left of the peaks of $\psi(z,k)$ with increasing $\sqrt{s}$.

The character of the approaching of $\bar{n}P^{(k)}$ to the "scaling"-like asymptotic law (13) is clear from Fig. 5, were the ratios $\bar{n}P_n^{(k)}/\psi(z)$ at different energies (0.55; 1.8; 8; 40 TeV) are plotted against $z=n/\bar{n}$. This dependence seems quite interesting, especially for z in interval from 0.9 to ≈ 1.8 and for large z.

Finally, Fig. 6 shows the information entropy

$$w=-\sum_n P_n \ln P_n \approx \ln \bar{n} - \int_0^\infty \psi(z,k) \ln \psi(z,k) dz \qquad (20)$$

which is defined by the chaotic component only for k from (12). The figure also shows the "maximal" entropy $w_{\max}$ corresponding to $k=1$ which is meaningless in our model.

V. CONCLUSIONS

In the present paper we have attempted to establish the relation between parameters of k and $\bar{n}$ of NBD on the basis of the general theory of random branching processes. This relation seems to be rather interesting, self-consistent, and has a predictive power. It removes some contradictions which occurred in the use of NBD for description of multiplicity distributions for high energy hadronic collisions.

On the whole the agreement of our model with the existing experimental data is good enough which, of course, is not surprising because of the coefficients a and b in (12) were derived from a fit to the experimental data for k and $\bar{n}$. More important are the predictions for the behavior of $C_q(D_2/\bar{n})$ and $\psi(z,k)$ at higher energies which may be checked at the CERN Large Hadron Collider (LHC) and SSC: the restoration of the KNO scaling in the multi-TeV region, asymptotic constant values of C_q, depending on q, "explanation" of the Wroblewski rule at high energy, and the asymptotic value of the peak of $\psi(z)$.

It is interesting to compare qualitatively our model of general branching processes with the detailed models of quark gluonic branching processes. If one neglects the quark branching the resulting parton distribution looks very similar to NBD and their conclusions qualitatively coincide with ours (limit of the widening of the distribution shape, increase and final saturation of C_q, etc. [21]). The dominant role of gluonic branching in comparison with quark branching is the characteristic feature of the detailed study of corresponding processes from the point of view of dynamical chaos [22], or from the approach based on the detailed consideration of branching of quarks and gluons at the formation of quark-gluon plasma [23,24].

There is at least one aspect which apparently necessitates the quark branching: the observed small oscillations in the high-multiplicity tail of P_n distribution at Tevatron energy [25]. If we recall the very old prediction of such oscillations in the Regge-pole approach [26] which is connected with Pomeron cuts, then it seems reasonable that quarks may be responsible for these phenomena. [The "explanation" of these oscillations by the addition of two binomial distributions (five-parameter fit [25]) may also be the reflection of this two-Reggeon cut.]

Finally in connection with the meaning of parameter of k of NBD and its asymptotic limit in our model ($k_{\min}\approx 3$) it would be very interesting to apply our model to the multiplicity distribution of hadrons in πp, as well in e^+e^-, νp, and ep collisions at high energies.

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