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Evolution of multidimensional flat anisotropic cosmological models

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We study the dynamics of a flat multidimensional anisotropic cosmological model filled with an anisotropic fluidlike medium. By an appropriate choice of variables, the dynamical equations reduce to a two-dimensional dynamical system. We present a detailed analysis of the time evolution of this system and the conditions of the existence of spacetime singularities. We investigate the conditions under which violent, exponential, and power-law inflation is possible. We show that dimensional reduction cannot proceed by anti-inflation (rapid contraction of internal space). Our model indicates that it is very difficult to achieve dimensional reduction by classical means.

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I. INTRODUCTION

Recently considered theories of the unification of all elementary interactions including gravitational interactions are based on the concept of multidimensional spacetime. It is usually assumed that at very high energy, higher than the grand unified theory (GUT) energy scale $\sim 10^{15}$ GeV, spacetime is multidimensional and all interactions are indistinguishable. At lower energies additional dimensions are "compactified," and they form a compact manifold of a typical size comparable to the Planck length ($\sim 10^{-33}$ cm), and therefore at present, the additional dimensions are not observable [1, 2]. The process of reduction of multidimensional spacetime to the product of physical spacetime M and a compact manifold B of appropriate dimensionality is called spontaneous compactification.

In 1980 Chodos and Detweiler [3] pointed out that the dynamics of a gravitational field might provide a natural mechanism of dimensional reduction. This paper created a surge of interest in multidimensional cosmological models (see reviews and collections of papers [1, 2]). In this paper we consider a simple multidimensional model which has a topological structure of $M \times T^{k'}$, where M is a $(k+1)$-dimensional spacetime and $T^{k'}$ is a k'-dimensional torus. We study the simplest case when M is a homogeneous, isotropic and flat space. Some aspects of such models have been investigated before by several authors [3–5]. Here we would like to present the general properties of such models.

In Sec. II we present our model and describe its qualitative behavior. In Sec. III we describe the time evolution of our model. Section IV is devoted to the discussion of singular solutions. The special case of one-dimensional internal space is treated in Sec. V. In Sec. VI we discuss our results and present conclusions.

II. MULTIDIMENSIONAL COSMOLOGICAL MODEL, SPECIAL SOLUTIONS

Let us consider an $(n+1)$-dimensional spacetime which is a product of a $(k+1)$-dimensional spacetime M and a $(k' = n-k)$-dimensional torus. The line element of this spacetime assumes the form

$$ds^2 = dt^2 - g_{ab}(x,t)dx^a dx^b - g_{ij}(x,t)dx^i dx^j, \qquad (2.1)$$

where $a, b = 1, 2, \ldots, k,\ i, j = k+1, \ldots, n$, and we set the velocity of light $c = 1$. We take the Einstein field equations in the form

$$R_{\mu\nu} = 8\pi\bar{G}\left(T_{\mu\nu} - \frac{1}{n-1}g_{\mu\nu}T\right), \qquad (2.2)$$

where $\mu, \nu = 0, 1, 2, \ldots, n$,

$$T^{\mu}_{\nu} = \mathrm{Diag}|\varepsilon, -p, \ldots, -p, -p', \ldots, -p'| \qquad (2.3)$$

is the hydrodynamical energy-momentum tensor, ε is the energy density, p and p' are respectively pressures in the physical space and in the internal space, $\bar{G}$ is the generalized gravitational constant, and T is the trace of the energy-momentum tensor. We assume that matter filling the multidimensional spacetime is described by a simple equation of state

$$p = \alpha\varepsilon, \quad p' = \alpha'\varepsilon, \qquad (2.4)$$

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where α and α' are constant parameters such that $|\alpha| \leq 1$, $|\alpha'| \leq 1$.

We restrict our consideration to the simplest model, and we assume that the multidimensional spacetime is homogeneous and the line element reduces to

$$ds^2 = dt^2 - R^2(t)dL^2 - r^2(t)dl^2, \tag{2.5}$$

where dL^2 and dl^2 are Euclidean elements of appropriate dimensions. In general, this multidimensional spacetime is homogeneous but anisotropic.

The Einstein field equations are conveniently written down in terms of the Hubble parameters

$$H = \frac{\dot{R}}{R}, \quad h = \frac{\dot{r}}{r}, \tag{2.6}$$

and they assume the form [4]

$$\dot{H} + H(kH + k'h) = \frac{\kappa\varepsilon}{n-1}[\alpha(k'-1) + 1 - k'\alpha'], \tag{2.7a}$$

$$\dot{h} + h(kH + k'h) = \frac{\kappa\varepsilon}{n-1}[1 - k\alpha + (k-1)\alpha'], \tag{2.7b}$$

where $\kappa = 8\pi\bar{G}$. H, h and ε are related by the constraint equation

$$(kH + k'h)^2 - kH^2 - k'h^2 = 2\kappa\varepsilon. \tag{2.8}$$

If the expansion of the Universe is adiabatic (what is implied by our choice of the perfect fluid energy-momentum tensor) then

$$-\frac{\dot{\varepsilon}}{\varepsilon} = (1+\alpha)kH + (1+\alpha')k'h. \tag{2.9}$$

When α and α' are constant, which we assume, Eqs. (2.7a), (2.7b), and (2.8) describe a two-dimensional dynamical system. We are interested in the general properties of this dynamical system in the physical region of the parameter space, where the energy density ε is not negative.

The dynamical system (2.7a), (2.7b), and (2.8) is very simple, but because the pressures in the internal space and in the physical space could be different, it describes quite a large variety of interesting cases which are parametrized by α and α'. In particular, when $\alpha = -1, \alpha' = -1 + (k-1)/k'$, the system mimics curvature in the internal space, and when $\alpha = -1/k, \alpha' = -1$, it mimics curvature in the physical space.

In the physical region of the parameter space the dynamical system (2.7a), (2.7b), and (2.8) possesses critical points at $H = h = 0$, and when $\alpha = \alpha' = -1$ also at $H = h$.

In order to simplify the dynamical equations we introduce a new variable s by

$$h = sH. \tag{2.10}$$

The dynamical equations can be rewritten in the form [4] (we assume that $k' \neq 1$)

$$\frac{d\ln H}{d\ln R} + (k + k's) = \frac{k'(k'-1)}{2(n-1)}[1 + (k'-1)\alpha - k'\alpha'](s - s_+)(s - s_-), \tag{2.11}$$

$$\frac{ds}{d\ln R} + \frac{k'(k'-1)}{2(n-1)}[1 + (k'-1)\alpha - k'\alpha'](s - s_0)(s - s_+)(s - s_-) = 0, \tag{2.12}$$

and

$$2\kappa\varepsilon = H^2 k'(k'-1)(s - s_+)(s - s_-), \tag{2.13}$$

where

$$s_0 = \frac{1-k}{k'}\frac{\alpha' - \beta_2}{\alpha' - \beta_1}, \tag{2.14}$$

$$\beta_1 = \frac{1 + (k'-1)\alpha}{k'}, \quad \beta_2 = \frac{k\alpha - 1}{k-1}, \tag{2.15}$$

and

$$s_\pm = -\frac{k}{k'-1}(1 \mp \gamma), \tag{2.16}$$

where

$$\gamma = \sqrt{\frac{(n-1)}{kk'}}.$$

Let us note that the value of $s_\pm$ depends only on the dimensions of the physical space and the internal space, but does not depend on the equation of state while the value of s_0 depends also on the equation of state.

The region $s_- < s < s_+$ is unphysical since there $\varepsilon < 0$, and we exclude this region from our consideration.

The dynamical system (2.11) and (2.12) is almost explicitly integrable. From Eq. (2.12) we obtain

$$R = C|s - s_+|^{q_+}|s - s_-|^{q_-}|s - s_0|^{q_0}, \tag{2.17}$$

where

$$q_\pm = \frac{1 \pm \gamma}{2(\alpha' - \beta_\pm)}, \tag{2.18}$$

$$q_0 = -\frac{\alpha' - \beta_1}{(\alpha' - \beta_+)(\alpha' - \beta_-)}, \tag{2.19}$$

$$\beta_\pm = \frac{\gamma \mp \beta_1}{\gamma \mp 1}, \tag{2.20}$$

and C is a constant of integration. Let us observe that because $q_0 + q_+ + q_- = 0$, then when $s \to \pm\infty$, R tends to a finite value.

Combining (2.11) and (2.12) we obtain

$$H = \pm C' |s - s_0|^{p_0} |s - s_+|^{p_+} |s - s_-|^{p_-}, \qquad (2.21)$$

where C' is an integration constant and

$$p_\pm = \frac{(k'\gamma \mp 1)}{k'[\gamma(\alpha' - 1) \mp (\alpha' - \beta_1)]}, \qquad (2.22)$$

$$p_0 = -1 + \frac{2[k(1-\alpha) + k'(1-\alpha')]}{k(1-\alpha)^2 + k'(1-\alpha')^2 - kk'(\alpha - \alpha')^2}. \qquad (2.23)$$

From (2.22) and (2.23) it follows that $p_0 + p_+ + p_- = -1$, so when $s \to \pm\infty$, $H \to 1/s$.

One can easily check that the dynamical system (2.7a) and (2.7b) possesses a first integral

$$HV(s - s_0) = \text{const}, \qquad (2.24)$$

where $V = R^k r^{k'}$ is the volume element. From (2.24) it follows that when $HV \to \infty$, $s \to s_0$. Let us point out that even when the physical space expands ($H > 0$) the total volume element V contracts when $s < -k/k'$.

From the general form of Eq. (2.12) it is apparent that $s = s_\pm$ and $s = s_0$ are the only solutions of the form $s = \text{const}$. In fact, the special solutions $s = s_\pm$ and $s = s_0$ are separatrixes of the dynamical system (2.11) and (2.12). The solutions $s = s_\pm$ describe multidimensional generalization of the empty ($\varepsilon = 0$) Kasner universe. When $s = s_\pm$ the dynamical system is completely integrable and we obtain

$$H = \frac{a_\pm}{t - t_*}, \quad h = \frac{s_\pm a_\pm}{t - t_*}, \qquad (2.25)$$

where

$$a_\pm = \frac{1}{k + k' s_\pm} \qquad (2.26)$$

and t_* is a constant determined by the initial conditions. From (2.25) we have [5]

$$R \sim |t - t_*|^{a_\pm}, \quad r \sim |t - t_*|^{a_\pm s_\pm}, \qquad (2.27)$$

and the total volume element

$$V \sim |t - t_*|. \qquad (2.28)$$

The solution $s = s_0$ can be represented in the form

$$H \sim R^{-1/a_0}, \qquad (2.29)$$

where

$$a_0 = \frac{1}{(k + k's_0) - \frac{k'(k'-1)}{2(n-1)}[1 + (k'-1)\alpha - k'\alpha'](s_0 - s_+)(s_0 - s_-)} \qquad (2.30)$$

so

$$R \sim |t - t_*|^{a_0}, \quad r \sim |t - t_*|^{s_0 a_0}, \qquad (2.31)$$

and

$$V \sim |t - t_*|^{(k + k's_0)a_0}, \qquad (2.32)$$

where s_0 is given by (2.14). This solution is physically acceptable only when $s_0 > s_+$ or $s_0 < s_-$ (only in regions II, III, and IV of the parameter space, see Fig. 1).

III. TIME EVOLUTION OF THE MODEL

Let us now concentrate on the time evolution of our model described by two quantities s and H. The evolution equation for s has the form

$$\dot{s} = A\tilde{C}|s - s_0|^{(1+p_0)}|s - s_+|^{(1+p_+)}|s - s_-|^{(1+p_-)}, \qquad (3.1)$$

where $\tilde{C}$ is a constant and

$$A = \frac{k'^2(k'-1)}{2(n-1)}(\alpha' - \beta_1), \qquad (3.2)$$

and H is given by (2.21).

From Eq. (3.1) it follows that during the evolution $\dot{s}$ never changes sign.

The dynamical equations (2.7a) and (2.7b) possess a discrete symmetry

$$H \leftrightarrow h, \quad k \leftrightarrow k', \quad \alpha \leftrightarrow \alpha', \qquad (3.3)$$

which also replaces $s_0 \to 1/s_0$ and $s_\pm \to 1/s_\mp$. Equation (3.1) transforms into

$$\dot{\tilde{s}} \sim h\left(\tilde{s} - \frac{1}{s_0}\right)\left(\tilde{s} - \frac{1}{s_+}\right)\left(\tilde{s} - \frac{1}{s_-}\right). \qquad (3.4)$$

From this equation it follows that when $s_0 \neq \pm\infty$, $\tilde{s} = 0$ is not a solution. It means that $s = \pm\infty$ is not an asymptote for $s(t)$.

The general behavior of the dynamical system (2.11) and (2.12) depends on relations between s_0 and $s_\pm$ which are determined by relations between β_1, β_2, and $\beta_\pm$ [4]. From definitions of these parameters it follows that, for ($k' > 1$),

$$\beta_1 = \beta_2 = \beta_+ = \beta_- = 1 \quad \text{when } \alpha = 1, \qquad (3.5)$$

and, for all $|\alpha| \leq 1$,

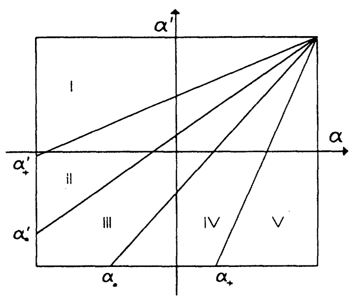

FIG. 1. Division of the physically acceptable parameter space $|\alpha| \leq 1$, $|\alpha'| \leq 1$ into regions corresponding to different behavior of the dynamical system (2.11), (2.12). The intersection points α_+, α_*, α'_*, and α'_+ are given by
$\alpha_+ = \frac{2(\frac{(n-1)k'}{k})^{1/2}-1-k'}{k'-1}$,
$\alpha_* = -\frac{k-2}{k}$, $\alpha'_* = -\frac{k'-2}{k'}$,
$\alpha'_+ = \frac{(\frac{(n-1)k'}{k})^{1/2}+2-k'}{(\frac{(n-1)k'}{k})^{1/2}+k'}$.
It turns out that for $k' > 1$ always $\alpha_+ > \alpha_*$ and $\alpha'_+ > \alpha'_*$.

$$\beta_1 - \beta_2 = \frac{n-1}{(k-1)k'}(1-\alpha) \geq 0, \tag{3.6}$$

$$\beta_1 - \beta_+ = \frac{\gamma}{1-\gamma}(1-\beta_1) \geq 0, \tag{3.7}$$

$$\beta_1 - \beta_- = -\frac{\gamma}{1+\gamma}(1-\beta_1) \geq 0, \tag{3.8}$$

$$\beta_2 - \beta_+ = \frac{k\gamma}{k-1}(1-\alpha) \geq 0, \tag{3.9}$$

so we have

$$\beta_+ \leq \beta_2 \leq \beta_1 \leq \beta_-. \tag{3.10}$$

Let us now discuss qualitative behavior of solutions of Eq. (2.12) for different values of α and α'. In region I in Fig. 1 ($\alpha' > \beta_-$),

$$s_- < s_0 < s_+ < 0, \tag{3.11}$$

and the behavior of solutions of Eq. (2.12) is shown in Fig. 2(a).

Let us consider a typical physically acceptable ($\varepsilon \geq 0$) solution of the dynamical equations. From the previous discussion it follows that this solution possesses two asymptotes: one in the future, when $s \to s_+$, and one in the past, when $s \to s_-$. At some intermediate moment $t = t_j$, s becomes discontinuous, and in the vicinity of t_j it tends to $\pm$ infinity. Let us investigate the behavior of $s(t)$ and $H(t)$ in the vicinity of $t = t_j$. When $s \to \pm\infty$ the dynamical equations (3.1) and (2.11) reduce to

$$\dot{s} = AHs^3, \quad \dot{H} = -AH^2s^2. \tag{3.12}$$

The general solution of these equations is given by

$$s = -\frac{a}{t-t_j}, \quad H = -\frac{1}{Aa^2}(t-t_j), \tag{3.13}$$

where a is a constant determined by initial conditions. Let us consider the case when $a > 0$ and note that in the region I, $A > 0$. When $t \to t_j$ but $t \leq t_j$, s grows to plus infinity, at $t = t_j$ it possesses typical $1/x$ discontinuity, and for $t > t_j$, s is negative. At $t = t_j$, H is zero and in the vicinity of t_j, it changes sign being positive for $t \leq t_j$. In the vicinity of t_j, h is positive and constant. When $a < 0$, for $t \to t_j$ ($t \leq t_j$), s tends to minus infinity with H decreasing to zero through positive values. For $t \geq t_j$, s decreases from plus infinity with H becoming more and more negative while h is positive and constant. So we have the following general picture: there are solutions $s(t)$ which in the past start from s_+, then s grows to infinity and at $t = t_j$ passes through discontinuity, and later starts to grow from minus infinity to s_- at the future. Another class of solutions starts from s_- in the past with $s(t)$ decreasing to minus infinity at $t = t_j$. At t_j, s jumps from minus infinity to plus infinity and later decreases asymptotically to s_+ in the future. It is clear that the second class can be obtained from the first by changing $t \to -t$ and $H \to -H$. Remembering about this possibility of obtaining the second class of solutions we will present only one class of solutions.

In region II ($\beta_1 < \alpha' < \beta_-$),

$$s_0 < s_- < s+ < 0, \tag{3.14}$$

and the qualitative behavior of Eq. (2.12) is shown in Fig. 2(b). In this region there are two different types of physically acceptable solutions. Solutions with discontinuous trajectories $s(t)$ which in the past start from s_+, and when $t \to t_j$ ($t \leq t_j$) grow to infinity, and at t_j jump from plus infinity to minus infinity, and in the future approach s_0. Before $t = t_j$, $H(t)$ is positive, $H(t_j) = 0$, and H is negative for $t > t_j$. In the vicinity of t_j, h is positive and constant. There are also smooth trajectories $s(t)$ which start from s_- in the past and asymptotically tend to s_0 in the future.

In region III ($\beta_2 < \alpha' < \beta_1$),

$$s_- < s_+ < 0 < s_0, \tag{3.15}$$

and the qualitative behavior of solutions of Eq. (2.12) is shown in Fig. 2(c). The general behavior of solutions is similar to the previous case but now the relative position of separatrixes is different and in this region the constant A appearing in Eq. (3.1) is negative. There are discontinuous trajectories $s(t)$ which in the past start from s_- and when $t \to t_j$ tend to minus infinity, becoming discontinuous at t_j. For $t > t_j$, $s(t)$ decreases from plus infinity and asymptotically tends to s_0 in the future. Along these discontinuous trajectories H is initially negative but decreases to zero at $t = t_j$, and becomes positive for $t > t_j$. In the vicinity of t_j, h is positive and constant. The continuous trajectory $s(t)$ begins in the past at s_+ and tends asymptotically to s_0 in the future.

Behavior of the dynamical system (3.1) and (2.21) in regions IV and V can be inferred from the above discussion and the already mentioned symmetry of the system.

IV. SINGULARITIES

Let us study the behavior of $s(t)$ close to separatrixes. From Eq. (3.1) it follows that

$$s(t) = s_* + C_* |t - t_*|^{-1/p_*}, \tag{4.1}$$

where $s_* = s_\pm, s_0$, C_*, and t_* are constants. It is clear that when $p_* > 0$, the separatrix is attained asymptotically for $t \to$ plus or minus infinity and when $p_* < 0$ the separatrix is attained after a finite time. From (3.1) and (2.10) we obtain

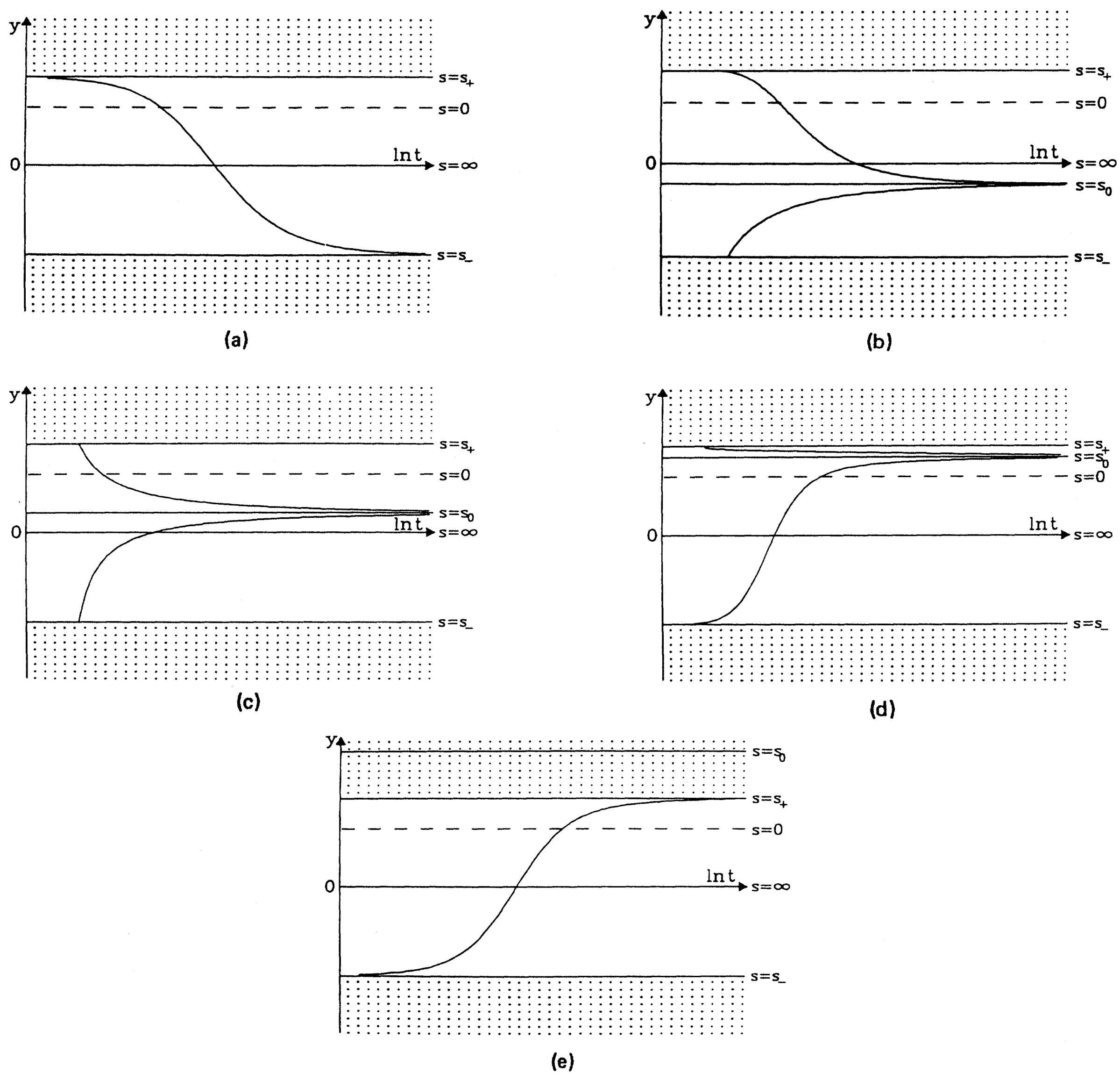

FIG. 2. Trajectories $y(t) = 1/[s(t) - s_m]$ where $s_m = (s_+ + s_-)/2$ for a multidimensional model with $k = 3$, $k' = 7$ and different fixed equation of state. We show only trajectories representing solutions without singularities in the future. Such trajectories can start from s_+ with $H > 0$ or from s_- with $H < 0$. In the dotted region $\varepsilon \leq 0$. (a) α and α' are confined to region I in Fig. 1. $s(t)$ is at a certain time discontinuous. Dimensional reduction is not possible. (b) α and α' are confined to region II. There are two different types of trajectories. The trajectory starting from s_+ is discontinuous and asymptotically tends to s_0. The other trajectory begins from s_- and smoothly tends to s_0. (c) α and α' are confined to region III. In this region $s_0 > s_+$ and $s_0 > 0$. We show typical trajectories. (d) α and α' are confined to region IV. The behavior of trajectories is similar to the previous case but now $s_+ < s_0 < 0$. In this region there are trajectories representing dimensional reduction. (e) α and α' are confined to region V. Behavior of trajectories depends on the equation of state. Drawing these curves we used the following equations of state: (a) $\alpha = -1$, $\alpha' = 1$; (b) $\alpha = -1/2$, $\alpha' = -1$; (c) $\alpha = \alpha' = 0$; (d) $\alpha = 0$, $\alpha' = -1$; (e) $\alpha = 1$, $\alpha' = -1$.

$$H(t) = \frac{a_*}{t - t_*}, \quad h(t) = \frac{s_* a_*}{t - t_*}, \tag{4.2}$$

where $a_* = a_\pm, a_0$ and $s_* = s_\pm, s_0$. When the separatrix is attained after an infinite time the Hubble parameters H and h tend to zero. However, when the separatrix is attained, after a finite time H and h become infinite and the Riemann tensor tends to infinity (see Appendix). Since $s_\pm < 0$, when $H \to +\infty$, $h \to -\infty$ and vice versa. After Hawking and Ellis [6] we call this type of singularity cigarlike.

From (4.2) it follows that if for a given solution there exists an epoch when $Ha_* < 0$, then at that epoch $t < t_*$, so H and h tend to infinity while $t \to t_*$. Similarly, when there exists an epoch when $Ha_* > 0$, then $t > t_*$ and there is a singularity in the past (when $t \to t_*$). To investigate singularities of the Riemann tensor for different separatrixes we note that it is sufficient to consider only singularities appearing in the future since singularities in the past can be obtained by changing $t \to -t$ and $H \to -H$.

The Riemann tensor is regular when $s \to s_0$. This conclusion follows from the fact that $a_0 > 0$ in regions III and IV, and $a_0 < 0$ in region II, but, for all solutions which tend to s_0 in regions III and IV, $H > 0$, while $H < 0$ in region II. Therefore, in regions II–IV, $Ha_0 > 0$ for all solutions which tend to s_0 in the future. This also follows from the fact that $p_0 > 0$ in regions II, III, and IV. In regions I and V there are no physically acceptable solutions with $s \to s_0$.

From the definition of a_+ it follows that $a_+ > 0$ for all k and k'. Thus all the solutions approaching s_+ with $H < 0$ are singular in the future. Such solutions exist in regions I–IV, since $p_+ < 0$ in these regions. Similarly, a_- is negative for all k and k' therefore all solutions approaching s_- with $H > 0$ are singular in the future [7]. These singular solutions exist in the regions II–V since $p_- < 0$ there. In these regions there are of course corresponding singularities in the past.

The Riemann tensor becomes singular only when $s \to s_\pm$, therefore all singular solutions are vacuum like $\Omega \sim \varepsilon/H^2 \to 0$. However the energy density can grow to infinity too. From Eqs. (2.13) and (2.21) we obtain

$$\varepsilon = C|s - s_0|^{2p_0}|s - s_+|^{2p_+ + 1}|s - s_-|^{2p_- + 1}, \tag{4.3}$$

where C is a constant. For $s \to s_\pm$ the energy density ε tends to infinity when $p_\pm < -\frac{1}{2}$ and to zero when $p_\pm > -\frac{1}{2}$. There exists a region in the parameter space describing a situation when components of the Riemann tensor and also energy density tend to infinity with $s \to s_\pm$ [see Figs. 3(a) and 3(b)].

We identify spacetime singularities with places where at least one of the invariants of the Riemann tensor becomes infinite. We have explicitly calculated $\mathcal{R} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ and

$$\mathcal{R} = 4\{k(\dot{H} + H^2)^2 + k'(\dot{h} + h^2)^2 + kk'H^2h^2 + k(k-1)H^4 + k'(k'-1)h^4\}. \tag{4.4}$$

It is clear that $\mathcal{R}$ becomes singular in all the cases listed above when H and h tend to infinity and $k' > 1$.

Our discussion of the singularities can be summarized in the following way: (1) any physically acceptable solution of our model is singular either in the past and then it is regular in the future or vice versa; (2) though $\Omega \to 0$ for all singular solutions the behavior of energy density ε depends on α and α'. Also the behavior of $\dot{s}$ depends on α and α'. From (3.1) it follows that $\dot{s}$ tends to infinity only when $p_\pm < -1$ [see Figs. 4(a) and 4(b)].

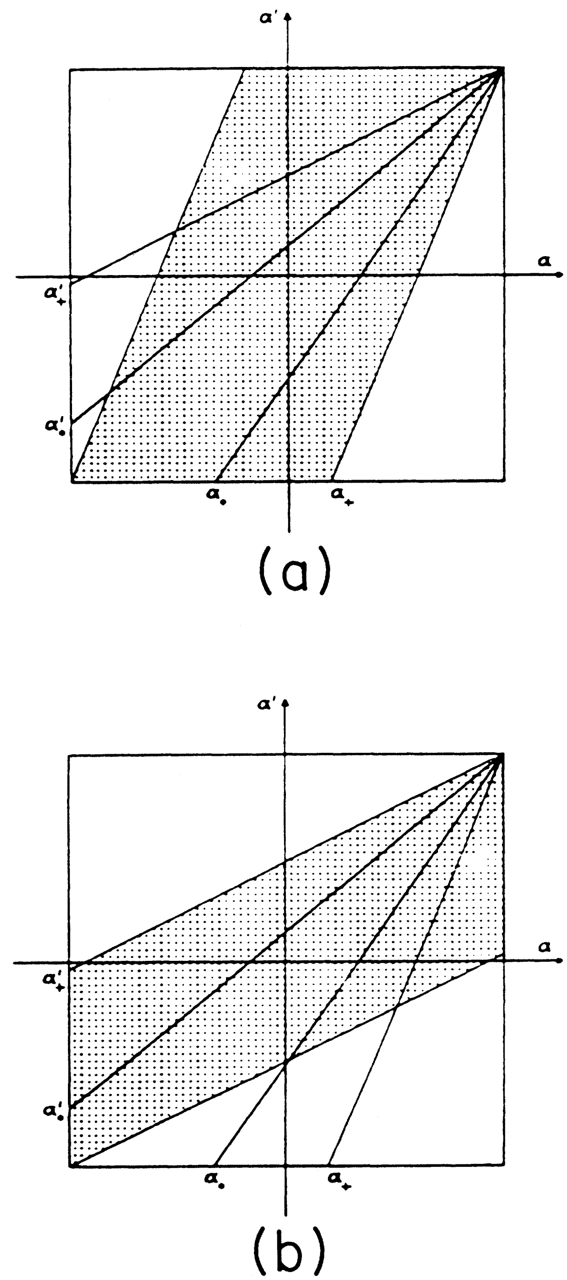

FIG. 3. Dotted region represents equations of state which allow singularities of the energy density in a multidimensional space with $k = 3$, $k' = 7$; (a) when $s \to s_+$, (b) when $s \to s_-$.

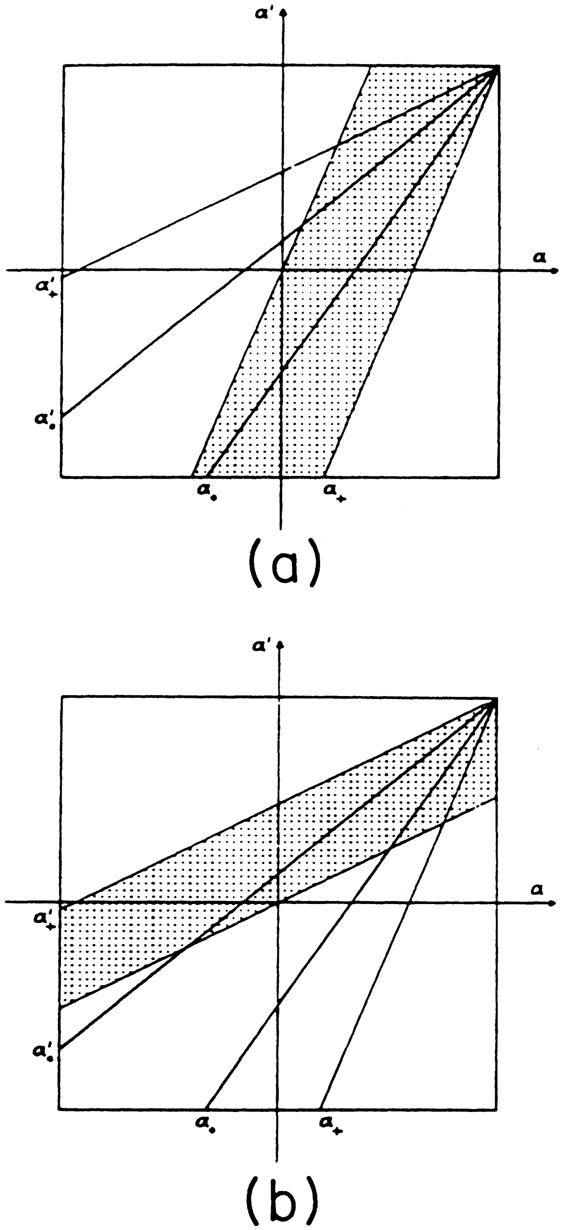

FIG. 4. Dotted region represents equations of state which allow singularities of $\dot{s}$ ($k = 3$, $k' = 7$); (a) when $s \to s_+$, (b) when $s \to s_-$.

V. ONE-DIMENSIONAL INTERNAL SPACE

The case $k' = 1$ is degenerate and requires special consideration. When $k' = 1$ the dynamical equations (2.7a), (2.7b), and (2.8) reduce to

$$\dot{H} + H(kH + h) = \kappa\varepsilon\left(\alpha + \frac{1 - k\alpha - \alpha'}{k}\right), \quad (5.1)$$

$$\dot{h} + h(kH + h) = \kappa\varepsilon\left(\alpha' + \frac{1 - k\alpha - \alpha'}{k}\right), \quad (5.2)$$

$$2\kappa\varepsilon = (kH + h)^2 - kH^2 - h^2. \quad (5.3)$$

As in the previous case we introduce a new variable $s = h/H$ and with this substitution the dynamical equations assume the form

$$\dot{H} = -H^2[k + s + (\alpha' - 1)(s - s_+)], \quad (5.4)$$

$$\dot{s} = (\alpha' - 1)H(s - s_+)(s - s_0), \quad (5.5)$$

where $s_+ = (1 - k)/2$, $s_0 = \frac{k\alpha - 1 - (k-1)\alpha'}{\alpha' - 1}$, and

$$\kappa\varepsilon = kH^2(s - s_+). \quad (5.6)$$

The dynamical equations can be integrated and we obtain

$$H = C|s - s_0|^{p_0}|s - s_+|^{p_+} \quad (5.7)$$

where

$$p_0 = -1 - \frac{s_0 + k}{(\alpha' - 1)(s_0 - s_+)}, \quad (5.8)$$

$$p_+ = \frac{s_+ + k}{(\alpha' - 1)(s_0 - s_+)}, \quad (5.9)$$

and C is constant.

The most important difference between the case $k' = 1$ and $k' > 1$ is the absence of the third separatrix. From (5.6) we see that the energy density is negative for $s < s_+$. The general behavior of solutions of the dynamical system (5.4), (5.5) depends on relations between α and α'. The parameter space $|\alpha| \leq 1$, $|\alpha'| \leq 1$ can be divided into three regions.

From (5.5) we see that when $s_0 > 0$ and $H > 0$ then s_0 is an attractive and s_+ is a repulsive separatrix. When $0 > s_0 > s_+$ the general behavior is similar but now with $s \to s_0$ the internal space contracts. In both cases s_0 is an attractive separatrix and general trajectories in the past tend to s_+ or to infinity. When $s_+ > s_0$ we have only one type of solutions with positive energy density. Corresponding trajectories tend to s_+ in the future and to s_0 in the past. (Cf. Fig. 5.)

After the transformation $H \to -H$ attractive asymptotes become repulsive and vice versa.

While s tends to s_0 or s_+, H and h assume the asymptotic form

$$H = \frac{a_*}{t - t_*}, \quad h = \frac{s_* a_*}{t - t_*}, \quad (5.10)$$

where $s_* = s_+, s_0$ and $a_0 = 1/[k + s_0 + (\alpha' - 1)(s_0 - s_+)]$, $a_+ = 2/(k + 1)$. From (5.10) we see that the behavior of solutions $s \to s_+$ and $s \to s_0$ is analogous to the case $k' > 1$.

If s tends to infinity the situation is quite different than in the case $k' > 1$. To see this let us make a conformal transformation $\tilde{s} = 1/s$, then (5.5) becomes

$$\dot{\tilde{s}} \sim \tilde{s}. \quad (5.11)$$

We see that s never changes sign in the course of evolution; therefore, s cannot cross infinity and H never changes sign as well. We see from (2.16) that $s_- \to \infty$ when $k' \to 1$, and we may formally consider infinity as a third asymptote of the dynamical system (5.1)–(5.3). When $s \to \infty$ we get

$$|\dot{s}| \sim |s|^{p_0+p_++2}, \tag{5.12}$$

$$|s| \sim (t-t_*)^{-1/(p_0+p_++1)}. \tag{5.13}$$

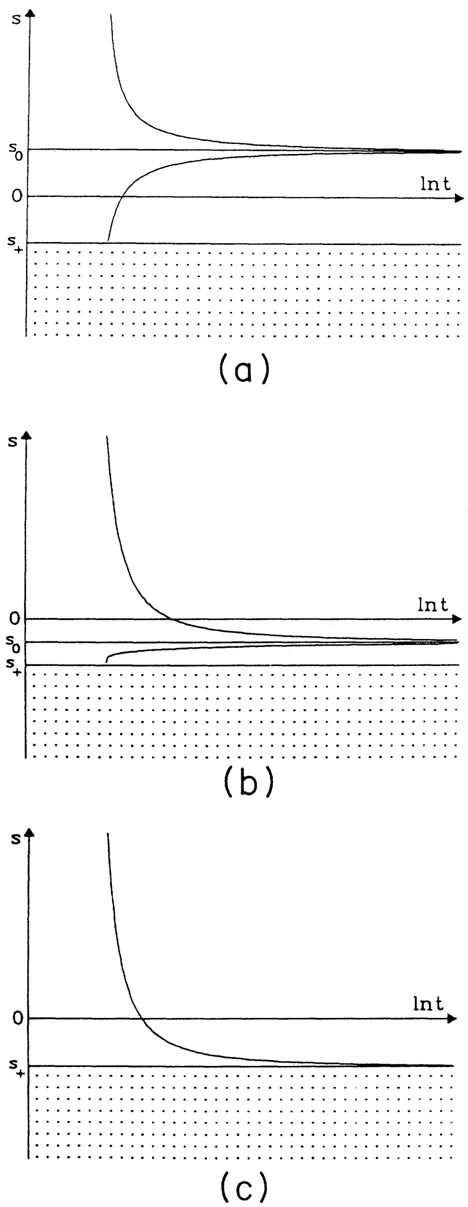

FIG. 5. Trajectories $s(t)$ for a multidimensional model with $k = 3$ and $k' = 1$. The physically acceptable parameter space divides in this case into three regions. We show only trajectories which do not represent future singularities. (a) $s \to s_0 > 0$, (b) $s \to s_0 < 0$, (c) $s \to s_+$. We used the following equations of state (a) $\alpha = \alpha' = 0$; (b) $\alpha = 0$, $\alpha' = -1$; (c) $\alpha = 1$, $\alpha' = -1$.

From (5.12) and (5.13) we see that s and $\dot{s} \to \infty$ when $t \to t_*$, since $-p_0 - p_+ - 1 = 1/(\alpha' - 1) < 0$. s reaches infinity in a finite time with infinite derivative, while

$$H \sim |t-t_*|^{-\alpha'}, \tag{5.14}$$

$$h = sH \sim |t-t_*|^{-1}, \tag{5.15}$$

and the energy density

$$\varepsilon \sim |t-t_*|^{-\alpha'-1}. \tag{5.16}$$

When $t \to t_*$ the energy density tends to infinity and the metric tends to degenerate Kasner-type metric. Thus when s reaches infinity we obtain strong singularity.

Let us emphasize the peculiarities of the $k' = 1$ case: (1) s inevitably tends to infinity in the past or in the future creating a pancakelike singularity [7] (when $\alpha' \leq 0$) or balloonlike singularity (when $\alpha' > 0$), or $s \to s_+$ creating a cigarlike singularity; (2) H never changes sign during the evolution.

Other properties of the system are similar to the previous case $k' > 1$.

VI. DISCUSSION AND CONCLUSIONS

It is interesting to find out under what conditions inflation is possible in multidimensional models and to check if the process of dimensional reduction of the internal space can be achieved by inflationary contraction. In the multidimensional case there are three different types of inflation in physical space.

(1) Power law (or extended) inflation:

$$R \sim t^p \quad \text{with} \quad p > 1.$$

(2) Exponential inflation:

$$R \sim \exp(Ht), \quad \text{with} \quad H > 0.$$

(3) Violent inflation:

$$R \sim |t-t_*|^{-q}, \quad \text{with} \quad q > 0.$$

The violent inflation occurs at the final singularity and it can be realized only in multidimensional cosmological models. From the discussion in Sec. IV it follows that when $H > 0$ the final singularity appears when $s \to s_-$. Close to the singularity $H = a_-/(t-t_*)$, $R \sim |t-t_*|^{a_-}$, and $h = s_- a_-/(t-t_*)$, so $r \sim |t-t_*|^{s_- a_-}$, where $a_- < 0$. The physical space expands while the internal space contracts so the dimensional reduction takes place. The expansion of the physical space is inflationary ($\ddot{R} > 0$), while the multidimensional volume ($V = R^k r^{k'}$) decreases ($\dot{V} < 0$).

This scenario was discussed by several authors [8, 9]. Sahdev [8] was the first to notice this possibility of dimensional reduction but he did not specify a mechanism to stabilize contraction of the internal space at the Planck scale. Kolb *et al.* [9] pointed out that even if one can find some satisfactory mechanism of stabilizing the internal space the entropy produced in this scenario is less than the observed entropy of the universe. Kolb *et al.* [9] suggested that the problem encountered in this model might be solved by allowing the equation of state to change. Finally Maeda [10] showed that when $\dot{V} < 0$ even rapidly changing equation of state cannot stabilize contraction of the internal space. The process of quantum particle creation poses another problem. This process tends to isotropize the initially anisotropic evolution and counteracts the process of compactification [11]. Therefore the violent inflation does not provide an efficient mechanism of dimensional reduction.

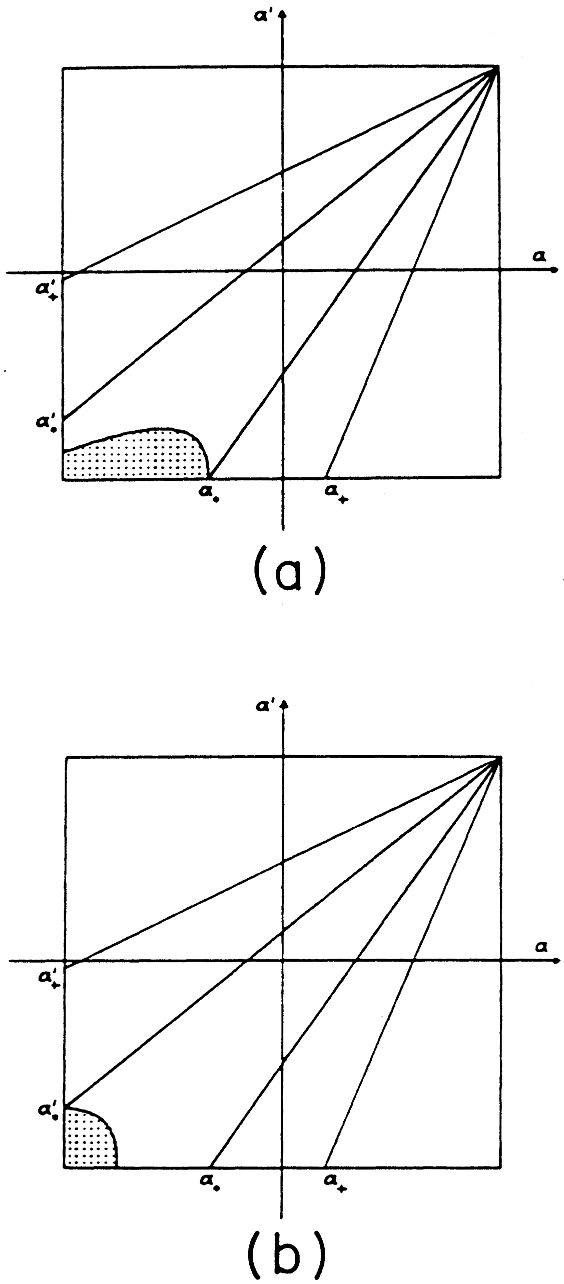

FIG. 6. Dotted region represents equations of state which allow (a) power law inflation in the physical space ($k = 3$, $k' = 7$) $a_0 > 1$; (b) curvature in the internal space is not important when $s \to s_0$ and $s_0 a_0 > 1$.

From the dynamical equations (2.7) it follows that the exponential inflation is possible only when $\alpha = \alpha' = -1$. The exponential inflation of the physical space implies inflation of the internal space. Therefore dimensional reduction is not compatible with exponential inflation.

The power law inflation is possible in the region of parameter space shown in Fig. 6(a). The power law inflation is not compatible with dimensional reduction which can occur only in regions IV and V of the parameter space.

When α and α' are confined to regions IV and V of the parameter space and initially the physical and the internal spaces are expanding then after some time the internal space begins to contract. The contraction to appropriate small scale takes too much time to be compatible with subsequent three-dimensional evolution of the Universe [12].

Finally let us discuss the effects of curvature in the physical or the internal space on the behavior of our model. In order to take into account the curvature it is necessary to modify the dynamical equations by adding to (2.7a) a term proportional to $1/R^2$ or to (2.7b) a term proportional to $1/r^2$ [5]. If $|RH|$ ($|rh|$) increases then the influence of curvature of the physical (internal) space decreases and asymptotically the curvature is not important. This condition is always satisfied close to a singularity. Therefore all singularities present in spatially flat multidimensional models appear also in general models with curvature. When we consider nonsingular situation $t \to \pm\infty$, the condition $|RH| \to \infty$ is satisfied in the same region of the parameter space where the power law inflation is possible [see Fig. 6(b)].

When the internal space is more than one dimensional all singularities are vacuumlike (cigarlike). It is interesting to note that in the one-dimensional internal space the behavior of spacetime near singularity which occurs when $s \to \infty$ does depend on the equation of state. Therefore in this case there are three types of singularities: cigarlike, balloonlike, and pancakelike. This conclusion is also relevant in the case of standard three-dimensional Bianchi type I cosmology ($k = 2, k' = 1$) contrary to the previous claims [7].

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APPENDIX

The nonzero components of the Riemann tensor for the metric (2.5) are

$$R^a_{00a} = -R^a_{0a0} \sim (s-s_+)^{2p_+}(s-s_-)^{2p_-}(s-s_0)^{2p_0}[1+(k+k's)-A(s-s_+)(s-s_-)], \tag{A1}$$

$$R^0_{a0a} = -R^0_{aa0} \sim (s-s_+)^{2(q_++p_+)}(s-s_-)^{2(q_-+p_-)}(s-s_0)^{2(q_0+p_0)} \times[1-(k+k's)-A(s-s_+)(s-s_-)], \tag{A2}$$

$$R^a_{bab} = -R^a_{bba} \sim (s-s_+)^{2(q_++p_+)}(s-s_-)^{2(q_-+p_-)}(s-s_0)^{2(q_0+p_0)}, \tag{A3}$$

$$R^i_{aia} = -R^i_{aai} = sR^a_{bab}, \tag{A4}$$

where $a,b=1,2,\ldots,k$; $i,j=k+1,\ldots,k+k'=n$ and

$$q_+ + p_+ = \frac{a_+ - 1}{Aa_+(s_+-s_0)(s_+-s_0)}, \tag{A5}$$

$$q_- + p_- = \frac{a_- - 1}{Aa_-(s_--s_0)(s_--s_+)}, \tag{A6}$$

$$q_0 + p_0 = \frac{a_0 - 1}{Aa_0(s_0-s_-)(s_0-s_+)}. \tag{A7}$$

All other nonzero components of the Riemann tensor can be obtained from (A1)–(A4) by substituting $a \leftrightarrow i$, $k \leftrightarrow k'$, $s_\pm \leftrightarrow 1/s_\mp$, and $s_0 \leftrightarrow 1/s_0$ everywhere including the exponents $p_0, p_\pm, q_0, q_\pm$ and the constant A.

Using the relations $p_0+p_++p_- = -1$ and $q_0+q_++q_- = 0$ we see that the components of the Riemann tensor can become infinite only when s approaches one of the three separatrixes s_0, $s_\pm$.

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