

**(1 + 1)-dimensional large- $N$  QCD coupled to adjoint fermions**

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We consider (1 + 1)-dimensional QCD coupled to Majorana fermions in the adjoint representation of the gauge group  $SU(N)$ . Pair creation of partons (fermion quanta) is not suppressed in the large- $N$  limit, where the glueball-like bound states become free. In this limit the spectrum is given by a linear light-cone Schrödinger equation, which we study numerically using the discretized light-cone quantization. We find a discrete spectrum of bound states, with the logarithm of the level density growing approximately linearly with the mass. The wave function of a typical excited state is a complicated mixture of components with different parton numbers. A few low-lying states, however, are surprisingly close to being eigenstates of the parton number, and their masses can be accurately calculated by truncated diagonalizations.

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**I. INTRODUCTION**

QCD has become universally accepted as the correct theory of strong interactions, on the basis of a large body of experimental and theoretical evidence. However, there are few reliable nonperturbative calculations that can be carried out starting from first principles. One of the most explored approaches to nonperturbative QCD has been lattice gauge theory [1]. In fact, numerical studies of pure glue theory appear to be close to the continuum limit, and significant progress is being made on various versions of lattice QCD with quarks. It is important, however, to look for other nonperturbative methods, in the hope that they will lead to new qualitative and quantitative insight. One such approach makes use of the light-cone quantization and subsequent numerical diagonalization of the light-cone Hamiltonian [2]. It may provide tools for calculating the hadron spectrum, as well as the wave functions in the infinite momentum frame, the decay amplitudes, and the interaction cross sections. Among its other advantages is the ability to introduce chiral fermions without obvious complications. This approach has been successfully applied to QCD and other model field theories in 1+1 dimensions [3], and is currently being generalized to (3+1)-dimensional theories [4]. In this paper we consider an application of the light-cone approach to a new type of models [5] where, we feel, it is particularly well suited.

As proposed by 't Hooft, QCD simplifies when generalized to a large number of colors,  $N$  [6]. When combined with the light-cone quantization, this simplification be-

comes particularly striking: in the  $N \rightarrow \infty$  limit meson and glueball wave functions are solutions of *linear* light-cone Schrödinger equations [7,8]. This is related to the fact that mesons and glueballs become free in the large- $N$  limit. The linearity of the equations, however, is a special property of the light-cone quantization. Recall, for comparison, that the loop equations remain nonlinear in the large- $N$  limit [9].

In this paper we will study the spectrum of such linear light-cone Schrödinger equation for a particular model, (1 + 1)-dimensional large- $N$  QCD coupled to matter in the adjoint representation of  $SU(N)$  [5]. This model is far more complex than the large- $N$  QCD coupled to quarks in the fundamental representation, where 't Hooft derived and numerically solved the bound-state equation for mesons [7]. The quanta of the adjoint matter resemble gluons in that there are two color flux tubes attached to each quantum. The resulting glueball-like bound states may contain any number of quanta connected into a closed string by the color flux tubes (see Fig. 1). As we will see, the eigenstates are generally complex mixtures of such strings with different numbers of partons. This should be contrasted with the 't Hooft model where, due to the absence of transverse gluons, all the meson bound states have the structure of a quark and an antiquark connected by a color flux tube. Our introduction of adjoint matter has the purpose of imitating some transverse gluon effects. In fact, if we dimensionally reduce (2 + 1)-dimensional gauge theory, the zero mode of the transverse gluon field acts as the adjoint matter field coupled

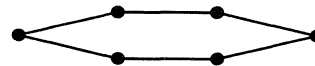


FIG. 1. A glueball-like bound state of six partons.

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to (1+1)-dimensional QCD. Therefore, this model seems to be the simplest setting where one can study some genuine QCD effects, such as the pair creation of partons. We will argue that the light-cone quantization supplemented with a regulator in the form of discretized longitudinal momenta [3,10] allows one to extract significant amount of physical information about the large- $N$  theory.

Consider the pure glue  $SU(N)$  gauge theory in 2+1 dimensions:

$$S = -\frac{1}{4g_3^2} \int d^3x \text{Tr} F_{\mu\nu} F^{\mu\nu}. \quad (1)$$

If one of the spatial dimensions is made compact,  $y \sim y + L$ , then as  $L \rightarrow 0$  we may ignore the dependence of fields on  $y$ , i.e.,  $\partial A^\mu / \partial y = 0$ . The action then reduces to

$$S_{\text{sc}} = \int dx^0 dx^1 \text{Tr} \left[ \frac{1}{2} D_\alpha \phi D^\alpha \phi - \frac{1}{4g^2} F_{\alpha\beta} F^{\alpha\beta} \right], \quad (2)$$

where  $g^2 = g_3^2 / L$  and  $\phi(x^0, x^1) = A_y / g$  is a traceless  $N \times N$  Hermitian matrix field, whose covariant derivative is given by  $D_\alpha \phi = \partial_\alpha \phi + i[A_\alpha, \phi]$ . Therefore,  $\phi$  represents the remnants of the transverse gluon degrees of freedom. If we choose the light-cone gauge  $A_- = 0$  and add a mass term for  $\phi$ , we obtain

$$S_{\text{sc}} = \int dx^+ dx^- \text{Tr} \left[ \partial_+ \phi \partial_- \phi - \frac{1}{2} m^2 \phi^2 + \frac{1}{2g^2} (\partial_- A_+)^2 + A_+ J^+ \right], \quad (3)$$

where  $x^\pm = (x^0 \pm x^1) / \sqrt{2}$ , and the longitudinal momentum current  $J_{ij}^+ = i[\phi, \partial_- \phi]_{ij}$ . The mass term for  $\phi$ , which does not destroy the (1+1)-dimensional gauge invariance, is necessary to absorb the logarithmically divergent mass renormalization. The light-cone quantization and the spectrum of the theory (3) were considered in Ref. [5]. However, in the numerical diagonalization no proper account was taken of the divergent mass renormalization.<sup>1</sup> The bare mass was held fixed, and hence all the bound-state masses were diverging in the continuum limit. A proper numerical diagonalization, where the renormalized mass is held fixed, is in progress, and we hope to report on it in the future.

In this paper we will examine, instead, a simpler model where the adjoint scalar is replaced by an adjoint Majorana fermion [5]. Thus, we obtain a (1+1)-dimensional gauge theory coupled to the zero mode of a transverse gluino. The bound states again are built of any number of partons connected into a closed string by color flux tubes. The new feature is that the bound states are fermions or bosons depending on whether the number of partons is even or odd. This theory has the advantage of being perfectly finite; moreover, it is supersymmetric for a special value of the fermion mass [11]. In Ref. [5], Dalley and one of the authors carried out the light-cone quantization

of this theory, and began a numerical investigation of the low-lying spectrum. Here we continue this program with further analytical and numerical results.

## II. LIGHT-CONE QUANTIZATION

Consider  $N^2 - 1$  Majorana (real) fermions which transform in the adjoint representation of  $SU(N)$ . They can be combined into a traceless Hermitian matrix  $\Psi_{ij}$ . Upon gauging the  $SU(N)$  symmetry we obtain the action

$$S_f = \int d^2x \text{Tr} \left[ i\Psi^T \gamma^0 \gamma^\alpha D_\alpha \Psi - m \Psi^T \gamma^0 \Psi - \frac{1}{4g^2} F_{\alpha\beta} F^{\alpha\beta} \right], \quad (4)$$

where the transposition acts only on the Dirac indices, and the covariant derivative is defined by  $D_\alpha \Psi = \partial_\alpha \Psi + i[A_\alpha, \Psi]$ . The fermion field  $\Psi_{ij} = 2^{-1/4} \begin{pmatrix} \chi_{ij} \\ \psi_{ij} \end{pmatrix}$  is a two-component spinor, where  $\chi$  and  $\psi$  are traceless Hermitian  $N \times N$  matrices of Grassmann variables. Choosing the light-cone gauge  $A_- = 0$ , and the representation  $\gamma^0 = \sigma_2$ ,  $\gamma^1 = i\sigma_1$ , we find the action

$$S_f = \int dx^+ dx^- \text{Tr} \left[ i\psi \partial_+ \psi + i\chi \partial_- \chi - i\sqrt{2} m \chi \psi + \frac{1}{2g^2} (\partial_- A_+)^2 + A_+ J^+ \right], \quad (5)$$

where the longitudinal momentum current is now of the form  $J_{ij}^+ = 2\psi_{ik} \psi_{kj}$ .

In light-cone quantization  $x^+$  is treated as time, and the canonical anticommutation relations are imposed at equal  $x^+$ :

$$\{\psi_{ij}(x^-), \psi_{kl}(y^-)\} = \frac{1}{2} \delta(x-y^-) \left[ \delta_{il} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl} \right]. \quad (6)$$

The action does not contain time derivatives of  $A_+$  and  $\chi$ , and these nondynamical fields can be eliminated by their constraint equations. As a result, the light-cone components of total momentum can be expressed in terms of  $\psi$  only:

$$P^+ = \int dx^- \text{Tr} \left[ i\psi \partial_- \psi \right], \quad (7)$$

$$P^- = \int dx^- \text{Tr} \left[ -\frac{im^2}{2} \psi \frac{1}{\partial_-} \psi - \frac{1}{2} g^2 J^+ + \frac{1}{\partial_-^2} J^+ \right].$$

Our goal is to solve the eigenvalue problem

$$2P^+ P^- |\Phi\rangle = M^2 |\Phi\rangle. \quad (8)$$

Since  $[P^+, P^-] = 0$ ,  $|\Phi\rangle$  is a simultaneous eigenstate of  $P^+$  and  $P^-$ . In practice, it is easy to ensure that  $|\Phi\rangle$  carries a definite  $P^+$ , but the subsequent solution of Eq. (8) is highly nontrivial. All the physical states must also satisfy the zero-charge constraint

$$\int dx^- J^+ |\Phi\rangle = 0, \quad (9)$$

<sup>1</sup>We thank D. Kutasov for helpful discussions on this issue.

arising from integration over the zero mode of  $A_+$ , which acts as a Lagrange multiplier.

In order to make Eq. (8) explicit we introduce the mode expansion

$$\psi_{ij}(x^-) = \frac{1}{2\sqrt{\pi}} \int_0^\infty dk^+ [b_{ij}(k^+) e^{-ik^+ x^-} + b_{ji}^\dagger(k^+) e^{ik^+ x^-}] . \quad (10)$$

$$P^+ = \int_0^\infty dk k b_{ij}^\dagger(k) b_{ij}(k) ,$$

$$P^- = \frac{m^2}{2} \int_0^\infty \frac{dk}{k} b_{ij}^\dagger(k) b_{ij}(k) + \frac{g^2 N}{\pi} \int_0^\infty \frac{dk}{k} C(k) b_{ij}^\dagger(k) b_{ij}(k) + \frac{g^2}{2\pi} \int_0^\infty dk_1 dk_2 dk_3 dk_4 \{ A(k_i) \delta(k_1 + k_2 - k_3 - k_4) b_{k_j}^\dagger(k_3) b_{k_i}^\dagger(k_4) b_{k_l}(k_1) b_{k_l}(k_2) + B(k_i) \delta(k_1 + k_2 + k_3 - k_4) [b_{k_j}^\dagger(k_4) b_{k_l}(k_1) b_{k_l}(k_2) b_{ij}(k_3) - b_{k_j}^\dagger(k_1) b_{ij}^\dagger(k_2) b_{li}^\dagger(k_3) b_{ki}(k_4)] \} , \quad (13)$$

where

$$A(k_i) = \frac{1}{(k_4 - k_2)^2} - \frac{1}{(k_1 + k_2)^2} ,$$

$$B(k_i) = \frac{1}{(k_2 + k_3)^2} - \frac{1}{(k_1 + k_2)^2} , \quad (14)$$

$$C(k) = \int_0^k dp \frac{k}{(p-k)^2} ,$$

and we have dropped the superscript plus on  $k_i$  for brevity. The mass renormalization proportional to  $C$  arises from the normal ordering of the quartic term in  $P^-$ . If one uses 't Hooft's principal value prescription, then  $C(k) = -1$ , so that the mass renormalization is finite. Other prescriptions, however, such as the one we will use, render  $C(k)$  linearly divergent. It is important to keep in

$$|\Phi_b(P^+)\rangle = \sum_{j=1}^\infty \int_0^{P^+} dk_1 \cdots dk_{2j} \delta \left[ \sum_{i=1}^{2j} k_i - P^+ \right] f_{2j}(k_1, k_2, \dots, k_{2j}) N^{-j} \text{Tr}[b^\dagger(k_1) \cdots b^\dagger(k_{2j})] |0\rangle . \quad (16)$$

This state is trivially an eigenstate of  $P^+$ , and the problem is to ensure that it is an eigenstate of  $P^-$ . Similarly, the fermionic states are of the form

$$|\Phi_f(P^+)\rangle = \sum_{j=1}^\infty \int_0^{P^+} dk_1 \cdots dk_{2j+1} \delta \left[ \sum_{i=1}^{2j+1} k_i - P^+ \right] f_{2j+1}(k_1, k_2, \dots, k_{2j+1}) N^{-j-1/2} \text{Tr}[b^\dagger(k_1) \cdots b^\dagger(k_{2j+1})] |0\rangle . \quad (17)$$

Because of the fermionic statistics of the oscillators, the wave functions have cyclic symmetry:

$$f_i(k_2, k_3, \dots, k_i, k_1) = (-1)^{i-1} f_i(k_1, k_2, \dots, k_i) . \quad (18)$$

The increased complexity of the coupling to adjoint matter arises mainly from the fact that the eigenstates are mixtures of states with different numbers of partons. This can be traced to the presence of pair production and pair annihilation terms in  $P^-$ . One easily checks that

From Eq. (6) it follows that

$$\{b_{ij}(k^+), b_{lk}^\dagger(\bar{k}^+)\} = \delta(k^+ - \bar{k}^+) \left[ \delta_{il} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl} \right] . \quad (11)$$

In terms of the oscillators, Eq. (7) assumes the form

$$P^+ = \int_0^\infty dk k b_{ij}^\dagger(k) b_{ij}(k) , \quad (12)$$

mind, however, that  $C$  is not a physical quantity because it enters in the mass of a colored object. The physical quantities are the masses of the colorless bound states, and they must be independent of which consistent prescription is used to define  $P^-$ . This was the case for the 't Hooft model [12], and we expect the same to be true here.

An important advantage of the light-cone quantization is that the oscillator vacuum satisfies

$$P^+ |0\rangle = 0, \quad P^- |0\rangle = 0 . \quad (15)$$

Other states in the Fock space are constructed by acting with creation operators  $b_{ij}^\dagger$  on the vacuum. The zero-charge condition (9) requires that all the color indices be contracted. Therefore, we look for bosonic eigenstates of Eq. (8) in the form

these appear in the leading order of  $1/N$  expansion, provided that  $g^2 N$  is kept fixed in the large- $N$  limit. Indeed, for the model with adjoint matter an extra pair of partons can be produced *inside* a color singlet. Furthermore, the terms in  $P^-$  that take one color singlet into two are suppressed by  $1/N$ . Therefore, our bound states are stable in the large- $N$  limit, and their wave functions satisfy *linear* eigenvalue equations [8,5]. These equations are not hard to write down explicitly. Upon introducing longitudinal momentum fractions  $x_i = k_i^+ / P^+$ , we find the following set of coupled integral equations by acting on states (16) and (17) with  $P^-$ ,

$$\begin{aligned}
M^2 f_i(x_1, x_2, \dots, x_i) &= \frac{m^2}{x_1} f_i(x_1, x_2, \dots, x_i) + \frac{g^2 N}{\pi(x_1 + x_2)^2} \int_0^{x_1 + x_2} dy f_i(y, x_1 + x_2 - y, x_3, \dots, x_i) \\
&+ \frac{g^2 N}{\pi} \int_0^{x_1 + x_2} \frac{dy}{(x_1 - y)^2} [f_i(x_1, x_2, x_3, \dots, x_i) - f_i(y, x_1 + x_2 - y, x_3, \dots, x_i)] \\
&+ \frac{g^2 N}{\pi} \int_0^{x_1} dy \int_0^{x_1 - y} dz f_{i+2}(y, z, x_1 - y - z, x_2, \dots, x_i) \left[ \frac{1}{(y + z)^2} - \frac{1}{(x_1 - y)^2} \right] \\
&+ \frac{g^2 N}{\pi} f_{i-2}(x_1 + x_2 + x_3, x_4, \dots, x_i) \left[ \frac{1}{(x_1 + x_2)^2} - \frac{1}{(x_2 + x_3)^2} \right] \\
&\pm \text{cyclic permutations of } (x_1, x_2, \dots, x_i). \tag{19}
\end{aligned}$$

For odd  $i$  all cyclic permutations enter with positive sign, while for even  $i$  they enter with alternating signs. This is related to the cyclic symmetry (18). In the second line of Eq. (19) the Coulomb double pole is partly compensated by the zero of the numerator at  $y = x_1$ . Therefore, the integral is finite in the principal value sense, and the equation contains no ambiguity. The same holds true for the 't Hooft equation (33)

Equation (19) possesses a  $\mathbb{Z}_2$  symmetry  $T$  [11]. For any fermionic (odd  $i$ ) eigenstate,

$$f_i(x_1, x_2, \dots, x_i) = T(-1)^{(i-1)/2} f_i(x_i, \dots, x_2, x_1), \tag{20}$$

while for any bosonic (even  $i$ ) eigenstate,

$$f_i(x_1, x_2, \dots, x_i) = T(-1)^{i/2} f_i(x_i, \dots, x_2, x_1). \tag{21}$$

The  $\mathbb{Z}_2$  quantum number  $T$  has two possible values: 1 and  $-1$ . In terms of the original field,  $T: \psi_{ij} \rightarrow \psi_{ji}$ , which obviously leaves  $P^\pm$  invariant [11]. Physically, every bound state can be thought of as a superposition of oriented closed strings, and the quantum number  $T$  describes the transformation property under a reversal of orientation.

### III. THE DISCRETIZED APPROXIMATION

The system of equations (19) involves an infinite number of multivariable functions. The complexity of the adjoint matter model is evidently much greater than that of the 't Hooft model, where each bound state is specified by a single function of one variable. Even there, however, one needs to resort to numerical methods to find the eigenvalues of the linear light-cone Schrödinger equation. Here we follow a similar strategy and replace the continuum equations (19) by a sequence of discretized approximations, such that the eigenvalues of the discretized problems eventually converge to the eigenvalues of (19). In the light-cone quantization, a simple discretized approximation is obtained by replacing the continuous momentum fractions  $x$  by a discrete set  $n/K$ , where  $n$  are odd positive integers, and the positive integer  $K$  is sent to infinity as the cutoff is removed [10,3]. Thus, the functions  $f_i(x_1, x_2, \dots, x_i)$  are replaced by finite collections

of numbers which specify their values at the discrete set of  $x$ , and

$$\int_0^1 dx \rightarrow \frac{2}{K} \sum_{\text{odd } n > 0}^K.$$

Moreover, the constraint  $\sum_{j=1}^i x_j = 1$  eliminates all states with over  $K$  partons, so that the discretized eigenvalue problem becomes finite dimensional. Any given eigenstate of the continuous problem should be well approximated by the discretizations with large enough  $K$ , although in practice the convergence may be very slow for highly excited states.

An equivalent way to describe our cutoff is in terms of the discretized light-cone quantization [3]. There one makes  $x^-$  compact and imposes antiperiodic boundary conditions:<sup>2</sup>

$$\psi_{ij}(x^-) = -\psi_{ij}(x^- + 2\pi L).$$

Therefore,  $k^+$  is restricted to discrete values  $n/(2L)$  where  $n$  are odd positive integers. The total light-cone momentum is  $P^+ = K/(2L)$ , where  $K$  is odd for the fermionic bound states whose wave functions are antiperiodic in  $x^-$ , and  $K$  is even for the bosonic bound states which are periodic in  $x^-$ . The mode expansion can now be written as

$$\begin{aligned}
\psi_{ij}(x^-) &= \frac{1}{\sqrt{4\pi}} \sum_{\text{odd } n > 0} [B_{ij}(n) e^{-iP^+ nx^-/K} \\
&+ B_{ji}^\dagger(n) e^{iP^+ nx^-/K}], \tag{22}
\end{aligned}$$

with the oscillator algebra

$$\{B_{ij}(n), B_{lk}^\dagger(n')\} = \delta_{nn'} \left[ \delta_{il} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl} \right]. \tag{23}$$

<sup>2</sup>In Ref. [5] periodic boundary conditions were used instead. Although for either choice of the boundary conditions the theory eventually converges to the limit of continuous  $k^+$ , we find that the convergence is appreciably faster for the antiperiodic boundary conditions.

The matrix that has to be diagonalized can be constructed in terms of the oscillators:

$$2P^+P^- = \frac{g^2N}{\pi}K(xV+T), \quad (24)$$

$$\begin{aligned} T = & 4 \sum_n B_{ij}^\dagger(n)B_{ij}(n) \sum_m^{n-2} \frac{1}{(n-m)^2} \\ & + \frac{2}{N} \sum_{n_i} \left[ \delta_{n_1+n_2, n_3+n_4} \left[ \frac{1}{(n_4-n_2)^2} - \frac{1}{(n_1+n_2)^2} \right] B_{kj}^\dagger(n_3)B_{ji}^\dagger(n_4)B_{kl}(n_1)B_{li}(n_2) \right. \\ & \left. + \delta_{n_1+n_2+n_3, n_4} \left[ \frac{1}{(n_3+n_2)^2} - \frac{1}{(n_1+n_2)^2} \right] \right. \\ & \left. \times [B_{kj}^\dagger(n_4)B_{kl}(n_1)B_{li}(n_2)B_{ij}(n_3) - B_{kj}^\dagger(n_1)B_{ji}^\dagger(n_2)B_{li}^\dagger(n_3)B_{ki}(n_4)] \right]. \end{aligned} \quad (26)$$

All the summations above are restricted to positive odd integers.

In order to perform the diagonalizations, we may consider a basis of states normalized to 1 in the large- $N$  limit:

$$\frac{1}{N^{i/2}\sqrt{s}} \text{Tr}[B^\dagger(n_1) \cdots B^\dagger(n_i)]|0\rangle, \quad \sum_{j=1}^i n_j = K. \quad (27)$$

The states are defined by ordered partitions of  $K$  into  $i$  positive odd integers, modulo cyclic permutations. If  $(n_1, n_2, \dots, n_i)$  is taken into itself by  $s$  out of  $i$  possible cyclic permutations, then the corresponding state receives a normalization factor  $1/\sqrt{s}$ . In the absence of special symmetries,  $s=1$ . For even  $i$ , however, some such states vanish due to the fermionic statistics of the oscillators: all partitions of  $K$  where  $i/s$  is odd do not give rise to states.

In actual calculations it is advantageous to consider separately the even and odd sectors under  $T$ :  $B_{ij}^\dagger(n) \rightarrow B_{ji}^\dagger(n)$ . The states that carry a definite quantum number  $T$  are in general linear combinations of the states

where  $x = \pi m^2/g^2N$  is the dimensionless parameter. The mass term is

$$V = \sum_n \frac{1}{n} B_{ij}^\dagger(n)B_{ij}(n), \quad (25)$$

while the term generated by the gauge interaction is

(27). Construction and proper normalization of such states is a combinatorial problem that is easily solved with a computer program. We will show the solution for a simple example, setting  $K=10$ .

In the  $T=1$  sector the normalized states are

$$\begin{aligned} |1\rangle &= \frac{1}{N^3} \text{Tr}[B^\dagger(5)B^\dagger(1)B^\dagger(1)B^\dagger(1)B^\dagger(1)B^\dagger(1)]|0\rangle, \\ |2\rangle &= \frac{1}{N^3} \text{Tr}[B^\dagger(3)B^\dagger(1)B^\dagger(3)B^\dagger(1)B^\dagger(1)B^\dagger(1)]|0\rangle, \\ |3\rangle &= \frac{1}{N^2\sqrt{2}} \{ \text{Tr}[B^\dagger(5)B^\dagger(3)B^\dagger(1)B^\dagger(1)] \\ & \quad + \text{Tr}[B^\dagger(1)B^\dagger(1)B^\dagger(3)B^\dagger(5)] \} |0\rangle, \quad (28) \\ |4\rangle &= \frac{1}{N} \text{Tr}[B^\dagger(9)B^\dagger(1)]|0\rangle, \\ |5\rangle &= \frac{1}{N} \text{Tr}[B^\dagger(7)B^\dagger(3)]|0\rangle. \end{aligned}$$

Here the matrix to be diagonalized is

$$xV+T = \begin{pmatrix} \frac{26x}{5} + \frac{7}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{14x}{3} + \frac{7}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{38x}{15} + \frac{359}{144} & -\frac{17}{72\sqrt{2}} & \frac{4\sqrt{2}}{9} \\ 0 & 0 & -\frac{17}{72\sqrt{2}} & \frac{10x}{9} + \frac{107}{72} & -\frac{8}{9} \\ 0 & 0 & \frac{4\sqrt{2}}{9} & -\frac{8}{9} & \frac{10x}{21} + \frac{47}{18} \end{pmatrix}. \quad (29)$$

In the  $T=-1$  sector the normalized states are

$$\begin{aligned}
|1\rangle &= \frac{1}{N^4} \text{Tr}[B^\dagger(3)B^\dagger(1)B^\dagger(1)B^\dagger(1)B^\dagger(1)B^\dagger(1)B^\dagger(1)B^\dagger(1)]|0\rangle, \\
|2\rangle &= \frac{1}{N^3} \text{Tr}[B^\dagger(3)B^\dagger(3)B^\dagger(1)B^\dagger(1)B^\dagger(1)B^\dagger(1)]|0\rangle, \\
|3\rangle &= \frac{1}{N^2} \text{Tr}[B^\dagger(7)B^\dagger(1)B^\dagger(1)B^\dagger(1)]|0\rangle, \\
|4\rangle &= \frac{1}{N^2\sqrt{2}} \{ \text{Tr}[B^\dagger(5)B^\dagger(3)B^\dagger(1)B^\dagger(1)] - \text{Tr}[B^\dagger(1)B^\dagger(1)B^\dagger(3)B^\dagger(5)] \} |0\rangle, \\
|5\rangle &= \frac{1}{N^2} \text{Tr}[B^\dagger(5)B^\dagger(1)B^\dagger(3)B^\dagger(1)]|0\rangle, \\
|6\rangle &= \frac{1}{N^2} \text{Tr}[B^\dagger(3)B^\dagger(3)B^\dagger(3)B^\dagger(1)]|0\rangle,
\end{aligned} \tag{30}$$

and the matrix to be diagonalized is

$$xV + T = \begin{pmatrix} \frac{22x}{3} + 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{14x}{3} + \frac{137}{36} & -\frac{5}{36} & \frac{3}{4\sqrt{3}} & 0 & 0 \\ 0 & -\frac{5}{36} & \frac{22x}{7} + \frac{89}{36} & -\frac{3}{4\sqrt{2}} & 0 & 0 \\ 0 & \frac{3}{4\sqrt{2}} & -\frac{3}{4\sqrt{2}} & \frac{38x}{15} + \frac{247}{72} & -\frac{11}{18\sqrt{2}} & \frac{4\sqrt{2}}{9} \\ 0 & 0 & 0 & -\frac{11}{18\sqrt{2}} & \frac{38x}{15} + \frac{47}{18} & -\frac{8}{9} \\ 0 & 0 & 0 & \frac{4\sqrt{2}}{9} & -\frac{8}{9} & 2x + \frac{37}{9} \end{pmatrix}. \tag{31}$$

The calculations above were repeated for higher values of  $K$  with the help of a computer program. The number of states increases rapidly with  $K$ . Our biggest diagonalization in the fermionic sector was carried out for  $K=25$ , where there are 3312 states in the  $\mathbb{Z}_2$  odd sector and 3400 states in the  $\mathbb{Z}_2$  even sector. In the bosonic sector we reached  $K=24$  where there are 2197  $\mathbb{Z}_2$  even states and 2141  $\mathbb{Z}_2$  odd states.

#### IV. THE NUMERICAL RESULTS

A good numerical procedure is to calculate the spectrum for a fixed  $x$  and a range of values of  $K$ , and then to extrapolate the results to infinite  $K$ , the continuum limit. We will also assume that some bulk properties of the spectrum can be estimated from the results at a fixed large  $K$ . We will be most interested in two special values of  $x$ ,  $x=0$ , which corresponds to the limit of massless quanta, and  $x=1$  ( $m^2=g^2N/\pi$ ), where the theory is supersymmetric [11].

In Fig. 2(a) we show the spectrum of fermionic states for  $x=0$  and  $K=25$ , with the mass plotted vs the expectation value of the number of partons,  $n$ . It is immediately obvious that the density of states increases rapidly with the mass, and that almost all the states lie within a band bounded by two  $\langle n \rangle \sim M$  lines. Below we will try to quantify these effects.

One interesting feature of our results, already noted in Ref. [5] for smaller  $K$ , is that for a few low-lying eigen-

states the wave functions are strongly peaked on states with a definite number of partons. For example, for  $K=25$  the ground state has probability 0.999 93 to consist of three partons, and the first excited state has probability 0.994 43 to consist of five partons. As the excitation number increases, however, the wave functions typically become quantum superpositions of states with different parton numbers. It is physically plausible that a typical excited state contains some number of virtual pairs, and our data support this expectation. In order to quantify this effect, we will call a state pure if it has probability  $>0.9$  to be in one of the number sectors. Table I shows the total number of states and the number of pure states in each mass interval of Fig. 2(a). We also show the expectation value of the number of partons averaged over all states in each mass interval. Evidently, a few low-lying states are pure, while there are no pure states among the high excitations.

As the mass of the quantum increases we expect the pair creation to become somewhat suppressed. In order

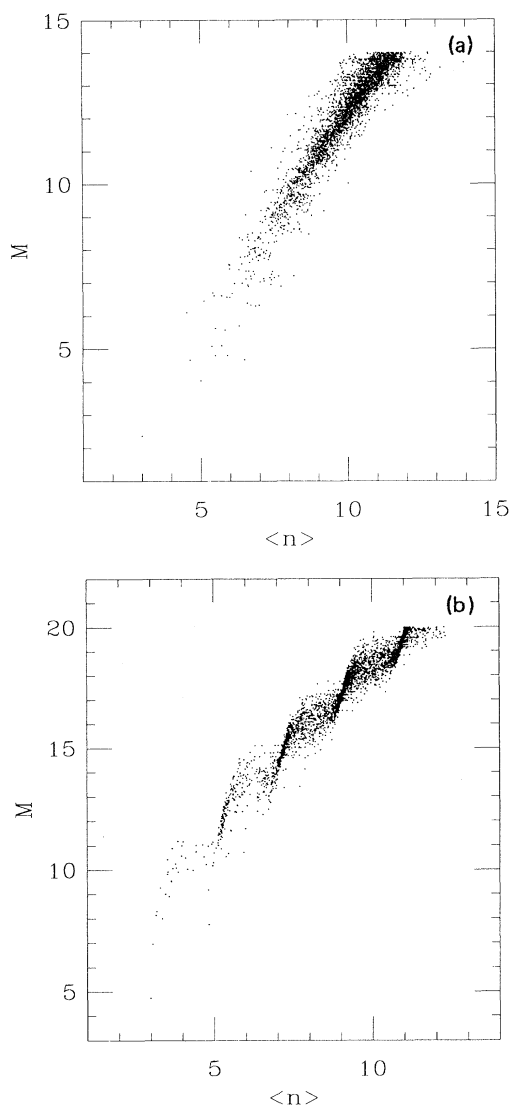


FIG. 2. The spectrum of fermionic states for  $K=25$ : (a)  $x=0$ ,  $M < 14$ ; (b)  $b=1$ ,  $M < 20$ ; mass  $M$  is measured in units of  $\sqrt{g^2 N}/\pi$  and plotted vs the expectation value of the parton number.

to study this effect, we plot in Fig. 2(b) the spectrum of fermionic states for  $x=1$  and  $K=25$ , and in Table II we quantify their purity. We find that, indeed, there are more pure states for low excitation numbers, but for highly excited states the pair creation again becomes important. In Figs. 3(a) and 3(b) we show the results for  $K=24$  and for  $x=0$  and 1, respectively, in order to

demonstrate that for the bosonic bound states the qualitative picture is the same. Table III shows the numerical data for  $x=0$ . In Ref. [11] the spectrum of highly excited states was found in the approximation where the number changing processes were ignored, i.e., all such states were assumed to be pure. The observed tendency of the excited states to be quantum superpositions of many number sectors, as well as the distributions of states in Figs. 2 and 3, do not seem to support this approximation. In principle, it is possible that we have not reached a high enough value of  $K$  for our discretized approximations to detect these pure highly excited states. We are inclined to believe, however, that a typical highly excited state does contain a number of virtual pairs of partons. More discussion of this issue will follow in Sec. V.

We note a difference in the distribution of states for  $x=0$  and 1. For  $x=0$ , Figs. 2(a) and 3(a), states are distributed within a band almost uniformly. For  $x=1$ , however, we see an increase in the number of fermionic states with an average parton number near 5,7,9,11, . . . , Fig. 2(b), and a similar increase in the number of bosonic states with an average parton number near 4,6,8,10, . . . , Fig. 3(b). We believe this effect to be related to the turning on of the mass, and that it gets stronger as the mass increases. It would be interesting to investigate this further.

A striking property of Figs. 2 and 3 is the rapid growth of the density of states with increasing mass. In Fig. 4 we plot the logarithm of the number of states vs the mass for the data in Table I. For a certain range of masses the graph is approximately linear. The deviation from linearity for large enough mass is clearly due to the effects of the cutoff. Our results indicate that the density of states grows roughly exponentially with the mass, exhibiting the Hagedorn behavior

$$\rho(m) \sim m^\alpha e^{\beta m}, \quad (32)$$

as suggested in Ref. [11]. Thus, although the mass spectrum is discrete, it rapidly becomes virtually indistinguishable from a continuum. From our data we estimate that the inverse Hagedorn temperature is  $\beta \approx (0.7-0.75)\sqrt{\pi/(g^2 N)}$ .

Another physical effect that is pronounced in our results (Tables I–III) is that the mass increases roughly linearly with the average number of partons. In Fig. 5 we plot these results for  $x=0$  and  $K=24$  (Table III). We will attempt to give a simple heuristic explanation of this effect. Suppose that the light-cone Hamiltonian of a glueball-like state containing on the average  $n$  partons is replaced by that of  $n$  nonrelativistic particles connected into a closed string by harmonic springs. It is not hard to see that the ground state energy of such a system, to be

TABLE I. Numerical data for  $K=25$ ,  $x=0$ , shown in Fig. 2(a); the 3.0 bin includes all states whose masses are 1.5–3.0, etc.

$M$	3.0	4.5	6.0	7.5	9.0	10.5	12.0	13.5	15.0
Number of states	1	1	9	37	104	362	897	1668	2040
Number of pure states	1	1	0	0	0	0	0	0	0
Average length	3.00	5.00	5.70	6.63	7.31	8.27	9.38	10.50	11.73

TABLE II. Numerical data for  $K = 25$ ,  $x = 1$  plotted in Fig. 2(b).

$M$	6.0	7.5	9.0	10.5	12.0	13.5	15.0	16.5	18.0	19.5
Number of states	1	1	5	17	56	131	296	580	942	1230
Number of pure states	1	1	3	1	0	0	0	0	0	0
Average length	3.00	3.06	3.63	3.99	5.05	5.84	6.87	7.88	9.07	10.36

identified with  $M^2$ , indeed behaves as  $\sim n^2$  for sufficiently large  $n$  [13]. Perhaps such a heuristic picture can indeed help one in a qualitative description of a typical bound state.

Now we need to address the question of convergence towards the continuum limit. In Fig. 6(a) we show the fermionic and bosonic ground states for  $x=0$ , as well as their extrapolation towards infinite  $K$ ; in Fig. 6(b) we repeat the plot for  $x=1$ . We have used the Bulirsch-Stoer algorithm which has proved to be particularly efficient for extrapolating short series [14]. The supersymmetry of the spectrum for  $x=1$  guarantees that the continuum values of the fermionic and bosonic ground states are equal, and our extrapolations indeed agree very well. The numerical values are shown in Tables IV and V.

Since the low-lying states are very pure, they can be well approximated by truncating the diagonalization to a single parton number sector. For instance, for  $x=0$  and  $K=24$  the ground state has probability 0.973 66 to consist of two partons. We can, therefore, obtain a good upper bound on its energy by truncating the eigenvalue equations (19) to the two-parton sector. The resulting eigenvalue problem is

$$M^2 \phi(x) = m^2 \phi(x) \left[ \frac{1}{x} + \frac{1}{1-x} \right] + \frac{2g^2 N}{\pi} \int_0^1 dy \frac{\phi(x) - \phi(y)}{(y-x)^2}, \quad (33)$$

where  $\phi(x) = f_2(x, 1-x)$ . Equation (33) is the 't Hooft equation with  $g^2 \rightarrow 2g^2$ . The doubling of the strength of the interaction term is due to the presence of two color flux tubes connecting a pair of partons (in a meson there is only one). Another important new effect is that, because of the fermionic statistics,  $\phi(x) = -\phi(1-x)$ . This forbids half of the eigenstates of the 't Hooft problem, including the ground state. In particular, for  $m=0$  the  $M=0$  solution  $\phi(x)=1$  is excluded. This provides a heuristic argument for the absence of massless bound states, even as the parton mass  $m$  is taken to zero. A more precise argument will be given in Sec. V. An approximation to the bosonic ground state of the adjoint fermion model is provided by the lowest antisymmetric wave function, whose eigenvalue is  $M^2 \approx 11.76g^2 N/\pi$ . This upper bound is quite close to the extrapolated value from Fig. 6(a), which is  $M^2 \approx 10.7g^2 N/\pi$ . The lowest antisymmetric eigenstate of Eq. (3) for  $m^2 = g^2 N/\pi$  ( $x=1$ ) has  $M^2 \approx 26.56g^2 N/\pi$ , which is a good upper bound on the extrapolated value from Fig. 6(b),  $M^2 \approx 25.9g^2 N/\pi$ .

Since the bosonic ground state is not perfectly pure, the upper bounds can be improved by including in the

truncated diagonalization all the 2-, 4-, and 6-bit states. For  $x=0$  and  $K=24$  the ground state has probability 0.999 98 to be in this sector. Extrapolating to infinite  $K$ , we find 0.999 95, and therefore this truncation is highly reliable. In Table IV we compare the full and truncated calculations. We have performed the truncated diagonalizations up to  $K=34$ , and extrapolating these results to

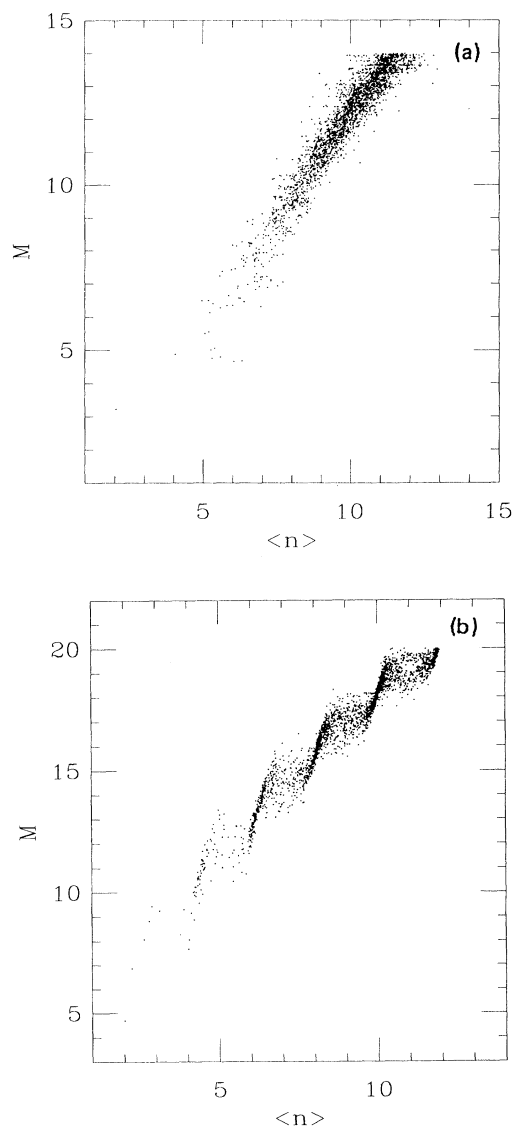


FIG. 3. The spectrum of bosonic states for  $K=24$ : (a)  $x=0$ ,  $M < 14$ ; (b)  $x=1$ ,  $M < 20$ .



TABLE III. Numerical data for  $K=24$ ,  $x=0$ , shown in Fig. 3(a).

$M$	4.5	6.0	7.5	9.0	10.5	12.0	13.5	15.0
Number of states	1	9	35	108	315	767	1229	1257
Number of pure states	1	1	0	0	0	0	0	0
Average length	2.05	5.36	6.43	7.21	8.23	9.32	10.48	11.74

infinite  $K$ , we find the upper bounds  $M^2 \approx 10.75g^2N/\pi$  for  $x=0$  and  $M^2 \approx 25.90g^2N/\pi$  for  $x=1$ . These are extremely close to the extrapolations from Figs. 6(a) and 6(b). This shows that, by judiciously truncating the space of states, certain eigenvalues can be determined to a good accuracy with relatively small diagonalizations.

Similar approximations work even better for the fermionic ground state because it is purer than the bosonic ground state: for  $x=0$  and  $K=25$  this state has probability 0.999 932 to consist of three partons, and probability 0.999 997 to consist of three or five partons. Extrapolating these probabilities to infinite  $K$ , we find 0.999 83 and 0.999 993, respectively. With the three-parton truncation and  $K$  up to 75, we find that the fermionic ground state has the extrapolated eigenvalue  $M^2 \approx 5.72g^2N/\pi$  for  $x=0$ , and  $M^2 \approx 26.05g^2N/\pi$  for  $x=1$ . These values provide good upper bounds on the extrapolations from Figs. 6(a) and 6(b). Furthermore, the truncation that involves three- and five-parton states is almost exact for the lowest fermionic eigenvalue, for all accessible values of  $K$ . The advantage of this truncation is that we can access higher value of  $K$  (up to 49) than in the full diagonalization and extrapolate more reliably. We find that the lowest fermionic eigenvalue extrapolated in this fashion is  $M^2 \approx 5.70g^2N/\pi$  for  $x=0$ , and  $M^2 \approx 25.94g^2N/\pi$  for  $x=1$ . Good agreement of these values with Figs. 6(a) and 6(b) gives us some confidence that our methods are consistent.

## V. DISCUSSION

One interesting property of our model is that, even in the limit  $m \rightarrow 0$ , there are no massless bound states. This result was found numerically in Ref. [5] and can easily be

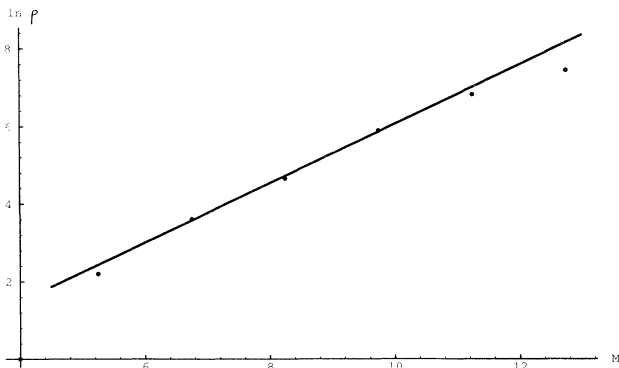


FIG. 4. Logarithm of the density of states and a linear fit for  $K=25$  and  $x=0$ .

explained analytically [15]. For  $m=0$  all the bound state masses are measured in units of  $g$ . If we are interested only in the massless states, we can send  $g \rightarrow \infty$  so that the relevant action is

$$S = \int d^2x \text{Tr}(i\Psi^T \gamma^0 \gamma^\alpha \partial_\alpha \Psi + A_\alpha J^\alpha). \quad (34)$$

The left-moving currents  $J_{ij}^+$  and the right-moving currents  $J_{ij}^-$  generate two independent level- $N$  Kac-Moody algebras. The gauge fields act as Lagrange multipliers that enforce the zero-current conditions. Calculation of the central charge of the Virasoro algebra in such theories is well known (see, for instance, Ref. [16]). For  $SU(N)$  the result is

$$c = c_m - (N^2 - 1) \frac{K_m}{K_m + N},$$

where  $c_m$  is the central charge before gauging, and  $K_m$  is the level of the current algebra. In the theory with an adjoint Majorana fermion  $c_m = (N^2 - 1)/2$ , and we find  $c=0$ . This establishes the absence of massless bound states. Similarly, in a gauge theory coupled to a fundamental Dirac fermion, we have  $c_m = N$  and  $K_m = 1$ , so that  $c=1$ . This proves the existence of one massless meson. In the large- $N$  limit this phenomenon can be attributed to the breaking of the  $U(1)$  chiral symmetry [17]. For comparison, note that the  $m=0$  version of the theory (4) has no chiral symmetry because we are considering Majorana fermions. This provides another physical reason for the absence of massless bound states.

One of the motivations for coupling  $(1+1)$ -dimensional QCD to the adjoint matter is that the parton pair creation is not suppressed by a power of  $N$ . This situation resembles large- $N$  QCD in higher dimensions, where

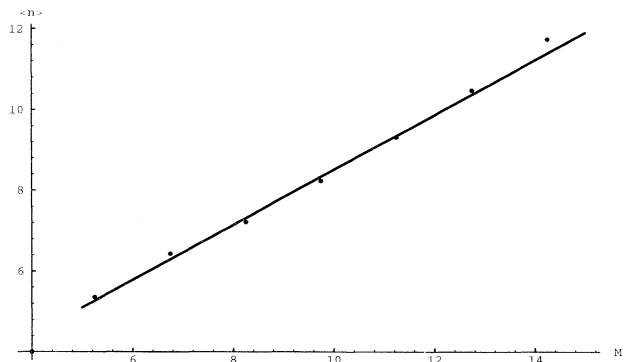


FIG. 5. Average number of partons as a function of mass for  $K=24$  and  $x=0$ .

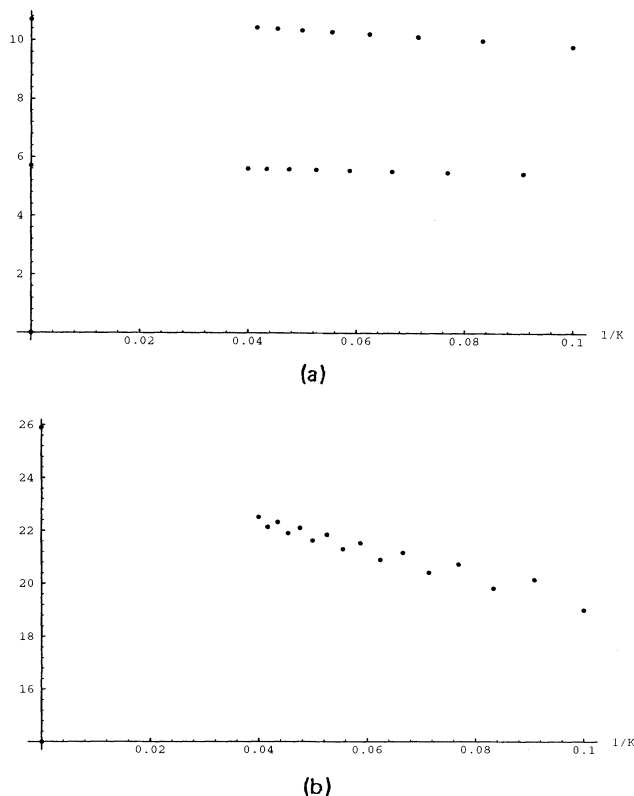


FIG. 6. Fermionic and bosonic ground states and their extrapolation toward infinite  $K$ : (a)  $x=0$ , (b)  $x=1$ .

the quark creation is suppressed, but the gluon creation is not. We find, however, that in the low-lying states the pair creation is suppressed for dynamical reasons. Roughly speaking, in the low-lying states the color flux tubes are not highly stretched, and it is not energetically favorable for a flux to divide into three flux tubes by creating a pair of partons in the middle. Indeed, even as  $m$  is taken to zero, it costs some energy to create a pair of quanta together with the associated flux tubes. This is why the first fermionic excited state, which contains five partons, is considerably heavier than the fermionic ground state, which contains three partons. However, as we pump vibrational energy into a bound state, the pair

creation should become more favored. This is confirmed by our computations. It is remarkable, though, that the lowest states are so pure that completely ignoring pair creation is an excellent approximation for them. It is tempting to speculate that this is somehow connected with the success of the “valence approximation” in (3+1)-dimensional QCD [18].

Another important point concerns the rich structure of the excited states found in (1+1)-dimensional QCD with adjoint matter. This may be the simplest class of models to exhibit a spectrum of Hagedorn type, with an exponentially growing density of states. Since we have taken  $N$  to infinity, all these glueball-like states are stable and can be found as solutions of the linear equation (19). For comparison, the ’t Hooft model has only one state per unit mass squared. These features are related to the presence of a deconfining phase transition in the adjoint matter model, and its absence in the ’t Hooft model [11].

Recently (1+1)-dimensional large- $N$  QCD was connected with closed string theory in a very precise fashion [19]. This was accomplished for the pure glue theory, which is almost topological. It is an interesting question, whether a continuum closed string description exists for the more complicated adjoint matter model, with its rich structure of physical states. The appearance of the exponentially growing density of levels suggests that such a theory should have a hidden transverse dimension. In our light-cone description this dimension manifests itself in the fluctuations of the longitudinal momenta and of the number of partons.

Clearly, a lot remains to be understood about the (1+1)-dimensional large- $N$  QCD coupled to adjoint matter. We hope that this class of models bears some physical similarity with higher-dimensional gauge theories. It may also serve as a good test of the light-cone quantization methods that are promising to become a useful tool for studying the nonperturbative structure of the strong interactions.

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TABLE IV. Numerical values for the bosonic ground state for  $x=0$  and  $x=1$ , as shown in Figs. 6(a) and 6(b).

$K$	$x=0$		$x=1$	
	$M^2$ (full)	$M^2$ (2+4+6-bit)	$M^2$ (full)	$M^2$ (2+4+6-bit)
12	9.9710	9.9711	19.7985	19.7985
14	10.1034	10.1036	20.4120	20.4120
16	10.2004	10.2008	20.8972	20.8972
18	10.2742	10.2747	21.2923	21.2923
20	10.3320	10.3326	21.6214	21.6214
22	10.3783	10.3791	21.9004	21.9004
24	10.4162	10.4171	22.1406	22.1406
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\infty$	10.7	10.75	25.9	25.90

TABLE V. Numerical values for the fermionic ground state for  $x=0$  and  $x=1$ , as shown in Figs. 6(a) and 6(b).

$K$	$x=0$			$x=1$		
	$M^2$ (full)	$M^2$ (3 bit)	$M^2$ (3+5 bit)	$M^2$ (full)	$M^2$ (3 bit)	$M^2$ (3+5 bit)
15	5.5111	5.5119	5.5112	21.1658	21.1722	21.1659
17	5.5388	5.5399	5.5389	21.5335	21.5427	21.5337
19	5.5602	5.5617	5.5603	21.8397	21.8517	21.8400
21	5.5771	5.5790	5.5772	22.0996	22.1143	22.0999
23	5.5908	5.5930	5.5910	22.3234	22.3408	22.3238
25	5.6021	5.6046	5.6022	22.5187	22.5385	22.5189
⋮	⋮	⋮	⋮	⋮	⋮	⋮
$\infty$	5.7	5.72	5.70	25.9	26.05	25.94

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