Dimensional expansion for the Ising limit of quantum field theory

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A recently proposed technique, called dimensional expansion, uses the space-time dimension D as an expansion parameter to extract nonperturbative results in quantum field theory. Here we apply dimensional-expansion methods to examine the Ising limit of a self-interacting scalar field theory. We compute the first few coefficients in the dimensional expansion of γ_{2n}, the renormalized $2n$-point Green's function at zero momentum, for $n=2$, 3, 4, and 5. Because the exact results for γ_{2n} are known at $D=1$ we can compare the predictions of the dimensional expansion at this value of D. We find typical accuracies of less than 5%. The radius of convergence of the dimensional expansion for γ_{2n} appears to be $2n/(n-1)$. As a function of the space-time dimension D, γ_{2n} appears to rise monotonically with increasing D and we conjecture that it becomes infinite at $D=2n/(n-1)$. We presume that for values of D greater than this critical value γ_{2n} vanishes identically because the corresponding ϕ^{2n} scalar quantum field theory is free for $D>2n/(n-1)$.

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In a recent Letter [1] we proposed a new technique called dimensional expansion, which can be used to obtain nonperturbative results in quantum field theory. The dimensional series uses the space-time dimension D as an expansion parameter. The first term in such an expansion is easy to obtain because quantum field theory can be solved in closed form in zero-dimensional space-time. An advantage of dimensional expansions is that some of the nontrivial aspects of the interaction already appear at $D=0$. (Traditional perturbative methods yield only noninteractive results in leading order.) The obvious question is how one can obtain the coefficients of higher powers of D. A detailed explanation of how to do so is given in a subsequent paper [2].

Here we use the dimensional expansion to compute the first four γ_{2n}, the renormalized $2n$-point Green's functions at zero external momentum, for a self-interacting scalar quantum field theory in the Ising limit. Specifically, we calculate γ_4 to fourth order in powers of D, γ_6 to fifth order in powers of D, γ_8 to sixth order in powers of D, and γ_{10} to seventh order in powers of D:

$$\begin{aligned}\gamma_4&=\tfrac{1}{12}[1+(1.180\pm0.001)D+(0.620\pm0.001)D^2\\&\quad+(0.18\pm0.02)D^3\\&\quad+(0.03\pm0.02)D^4+\cdots],\\ \gamma_6&=\tfrac{1}{30}[1+(2.20\pm0.02)D+(2.30\pm0.03)D^2\\&\quad+(1.50\pm0.03)D^3+(0.55\pm0.04)D^4\\&\quad+(0.12\pm0.04)D^5+\cdots],\\ \gamma_8&=\tfrac{1}{56}[1+(3.0\pm0.1)D+(4.5\pm0.1)D^2\\&\quad+(4.2\pm0.1)D^3+(2.6\pm0.1)D^4\\&\quad+(1.2\pm0.2)D^5+(0.6\pm0.2)D^6+\cdots],\\ \gamma_{10}&=\tfrac{1}{90}[1+(4.11\pm0.02)D+(8.0\pm0.1)D^2\\&\quad+(10.0\pm0.3)D^3+(8.0\pm0.3)D^4\\&\quad+(4.5\pm0.3)D^5+(1.8\pm0.3)D^6\\&\quad+(0.7\pm0.4)D^7\cdots].\end{aligned} \tag{1}$$

To obtain these dimensional expansions we use the graphical methods described in Ref. [2]. These graphical methods rely on lattice strong-coupling techniques that were developed and explained in an earlier series of papers [3–6]. For the Lagrangian

$$\mathcal{L}=\tfrac{1}{2}[\partial\phi(x)]^2+\tfrac{1}{2}m^2\phi(x)^2+\tfrac{1}{4}g\phi(x)^4 \tag{2}$$

the Ising limit [7,8] is defined as the limit in which the unrenormalized coupling constant g tends to infinity while the renormalized mass M is held fixed. The Ising limit is conveniently obtained by choosing $m^2\propto -g$. In the limit $g\to\infty$ the theory asymptotically approaches a two-state system. The Green's functions of this system are universal in the sense that they are independent of the power of ϕ in the self-interaction term in (2); $g\phi^{2k}$ gives the same results as $g\phi^4$ for all $k\geq 2$.

Lattice strong-coupling methods are especially well suited to obtain the dimensional expansion of Green's functions in quantum field theory because the lattice integral for each graph is a *polynomial* in powers of D. This property leads to an efficient organization of the graphs that contribute to each order in the D series. We achieve a high order in the graphical expansion by eliminating all graphs except those that contribute to the coefficients in the dimensional expansion under consideration. We employ an intermediate renormalization scheme to calculate the renormalized mass M and dimensionless renormalized $2n$-point scattering amplitudes γ_{2n} at zero momentum. We perform a mass renormalization of the scattering amplitudes by eliminating the bare mass

m in γ_{2n} in favor of the renormalized mass M. We then use Padé extrapolation methods to derive a sequence of approximants for each coefficient in the dimensional expansions in (1) for each of the scattering amplitudes γ_{2n} in the continuum limit. We believe that each Padé sequence gives an accurate approximation to the true coefficient in the dimensional expansion for γ_{2n} because these dimensional series are in good numerical agreement with known exact results for γ_{2n} [3]. The series in (1) are exact at $D=0$. At $D=1$ the exact results are $\gamma_4=\frac{1}{4}$, $\gamma_6=\frac{1}{4}$, $\gamma_8=\frac{5}{16}$, $\gamma_{10}=\frac{7}{16}$, and the results for (1) are $\gamma_4=0.250\pm0.003$ (1% error), $\gamma_6=0.26\pm0.01$ (4% error), $\gamma_8=0.31\pm0.01$ (4% error), and $\gamma_{10}=0.42\pm0.02$ (5% error).

We obtain the graphical rules for the lattice strong-coupling expansion by observing that in the limit of large g the kinetic term in the Lagrangian (2) can be viewed as a small perturbation. Therefore, the generating function

$$Z[J]=\mathcal{N}\int \mathcal{D}\phi(x)\exp\left[-\int d^Dx\{\tfrac{1}{2}[\partial\phi(x)]^2+\tfrac{1}{2}m^2\phi(x)^2+\tfrac{1}{4}g\phi(x)^4-J(x)\phi(x)\}\right] \tag{3}$$

for the quantum field theory associated with the Lagrangian (2) can be rewritten as

$$Z[J]=\exp\left[\frac{1}{2}\int d^Dx\, d^Dy\frac{\delta}{\delta J(x)}\mathcal{D}^{-1}(x-y)\frac{\delta}{\delta J(y)}\right]\times Z_0[J]\,, \tag{4}$$

where

$$\mathcal{D}^{-1}(x-y)=\partial^2\delta^D(x-y)$$

and

$$Z_0[J]=\mathcal{N}\int\mathcal{D}\phi\exp\left[-\int d^Dx\left[\frac{1}{2}m^2\phi(x)^2+\frac{1}{4}g\phi(x)^4-J(x)\phi(x)\right]\right]. \tag{5}$$

The factorization in (4) of the partition function leads to the strong-coupling lattice expansion. By introducing a D-dimensional hypercubic lattice with lattice spacing a we rewrite (5) as

$$Z_0[J]=\mathcal{N}\prod_i\int_{-\infty}^{\infty}dt\exp\left[-\frac{1}{2}a^Dm^2t^2-\frac{1}{4}a^Dgt^4+a^DJ_it\right]. \tag{6}$$

Next, we expand in powers of J_i and, to obtain the Ising limit, we set

$$m^2=-\alpha g a^{2-D}\,, \tag{7}$$

where α is a dimensionless parameter considered to be small in the strong-coupling limit:

$$Z_0[J]=\mathcal{N}\prod_i\sum_{n=0}^{\infty}\frac{1}{(2n)!}(a^DJ_i)^{2n}\int_0^{\infty}dt\,t^{n-1/2}\exp\left[-\frac{1}{4}a^Dg(t^2-2\alpha a^{2-D}t)\right]. \tag{8}$$

In the limit $g\to\infty$ the integral in (8) is asymptotic to α^n multiplied by a constant independent of n which we absorb into $\mathcal{N}$. Thus, we write $Z_0[J]$ in (8) as

$$Z_0[J]=\mathcal{N}\exp\left[a^D\sum_i\left[\sum_{n=1}^{\infty}\frac{1}{(2n)!}J_i^{2n}V_{2n}\right]\right], \tag{9}$$

where the vertices are $V_2=a^2\alpha$, $V_4=-2a^{4+D}\alpha^2$, $V_6=16a^{6+2D}\alpha^3$, $V_8=-272a^{8+3D}\alpha^4$, $V_{10}=7936a^{10+4D}\alpha^5$, and so on. The propagator on the lattice can be written in vector notation as

$$\mathcal{D}^{-1}=a^{-D-2}[(\underline{1})-2D(\underline{0})]\,.$$

This notation was introduced in Ref. [4] where this discrete form of the propagator was used to evaluate lattice integrals. The lattice strong-coupling expansion is organized by the number of free propagators $\mathcal{D}^{-1}$ (in contrast with weak-coupling expansions where the number of *vertices* and not the number of lines determines the order).

To compute γ_{2n} it is necessary to calculate the one-particle-irreducible $2n$-point functions Λ_{2n} for $n=1, 2, 3, 4$, and 5, in the strong-coupling expansion and to find their Fourier transforms $\tilde{\Lambda}_{2n}$ in momentum space at zero external momentum. We must also compute

$$\left.\frac{\partial}{\partial(p^2)}\tilde{\Lambda}_2(p^2)\right|_{p^2=0}$$

to obtain the wave-function renormalization constant defined by

$$Z^{-1}\equiv 1+\left.\frac{\partial}{\partial(p^2)}\tilde{\Lambda}_2^{-1}\right|_{p^2=0}.$$

We define the scattering amplitudes γ_{2n} as the *dimensionless* renormalized one-particle-irreducible vertices at zero external momentum:

$$\gamma_{2n} \equiv \tilde{\Gamma}_{2n}^{R}(0,0,\ldots,0)M^{D(n-1)-2n}, \tag{10}$$

where M is the renormalized mass defined as $M^2 \equiv \tilde{\Gamma}_2^R(0,0)$. There are simple rules giving Γ_{2n} in terms of Λ_{2m}, $m \le n$, which are explained in Ref. [4]:

$$\Gamma_2 = \Lambda_2^{-1},$$

$$\Gamma_4 = -\Lambda_4\Lambda_2^{-4},$$

$$\Gamma_6 = -\Lambda_6\Lambda_2^{-6} + \frac{6!}{2\times 3!3!}\Lambda_4^2\Lambda_2^{-7},$$

$$\Gamma_8 = -\Lambda_8\Lambda_2^{-8} + \frac{8!}{3!5!}\Lambda_4\Lambda_6\Lambda_2^{-9} - \frac{8!}{2\times 2!3!3!}\Lambda_4^3\Lambda_2^{-10}, \tag{11}$$

$$\Gamma_{10} = -\Lambda_{10}\Lambda_2^{-10} + \frac{10!}{3!7!}\Lambda_8\Lambda_4\Lambda_2^{-11} + \frac{10!}{2(5!)^2}\Lambda_6^2\Lambda_2^{-11} - \frac{10!}{2\times 3!5!}\Lambda_6\Lambda_4^2\Lambda_2^{-12} - \frac{10!}{2(3!)^2 4!}\Lambda_6\Lambda_4^2\Lambda_2^{-12} + \frac{10!}{2(2!)^2(3!)^2}\Lambda_4^4\Lambda_2^{-13} + \frac{10!}{(3!)^4}\Lambda_4^4\Lambda_2^{-13}.$$

We use the wave-function renormalization constant Z to renormalize the one-particle-irreducible vertices in an intermediate renormalization scheme according to

$$\tilde{\Gamma}_{2n}^{R}(0,\ldots,0) = Z^n\tilde{\Gamma}_{2n}(0,\ldots,0).$$

In order to mass renormalize the scattering amplitudes γ_{2n}, we eliminate the bare mass m, which is related to α through (7), in favor of the renormalized mass M. To that end, we simply invert the relation obtained for the renormalized mass,

$$M^2a^2 = \alpha^{-1} - 2D + (2D - \tfrac{2}{3})\alpha + (4D^2 - \tfrac{26}{3}D + \tfrac{194}{45})\alpha^3 + (32D^3 - 132D^2 + \tfrac{2584}{15}D - \tfrac{68\,164}{945})\alpha^5 + (-2048D^5 - 4096D^4 - 1024D^3 - 800D^2 + 480D - 64)\alpha^6 + \cdots, \tag{12}$$

to expand α in terms of $y \equiv a^{-2}M^{-2}$:

$$\alpha = y - 2Dy^2 + (4D^2 + 2D - \tfrac{2}{3})y^3 + (-8D^3 - 12D^2 + 4D)y^4 + (16D^4 + 48D^3 - 4D^2 - 14D + \tfrac{26}{5})y^5 + (-32D^5 - 160D^4 - \tfrac{200}{3}D^3 + 140D^2 - 52D)y^6 + (64D^6 + 480D^5 + 560D^4 - 720D^3 + 20D^2 + 272D - \tfrac{636}{7})y^7 + \cdots. \tag{13}$$

We then substitute (13) for α in every Γ_{2n}^R to obtain

$$\gamma_4 = \tfrac{1}{12}[1 + 4Dy + (4D^2 - 10D)y^2 + 16Dy^3 + (-80D^2 + 30D)y^4 + (256D^3 + 104D^2 - 192D)y^5 + (-704D^4 - 1736D^3 + 2508D^2 - 656D)y^6 + (1792D^5 + 10\,432D^4 - 11\,232D^3 - 3872D^2 + 4992D)y^7 + \cdots], \tag{14}$$

$$\gamma_6 = \tfrac{1}{30}[1 + 6Dy + (12D^2 - 6D)y^2 + (8D^3 - 12D^2 - 20D)y^3 + (48D^2 + 48D)y^4 + (-96D^3 - 816D^2 + 528D)y^5 + (192D^4 + 4640D^3 - 2736D^2 - 560D)y^6 + (-384D^5 - 18\,432D^4 - 10\,800D^3 + 46\,512D^2 - 23\,040D)y^7 + \cdots], \tag{15}$$

$$\gamma_8 = \tfrac{1}{56}[1 + 8Dy + (24D^2 - 8D)y^2 + (32D^3 - 32D^2)y^3 + (16D^4 - 32D^3 - 36D^2 - 18D)y^4 + (896D^2 - 448D)y^5 + (-4192D^3 - 920D^2 + 2816D)y^6 + (13\,184D^4 + 52\,064D^3 - 92\,800D^2 + 38\,400D)y^7 + \cdots], \tag{16}$$

$$\gamma_{10} = \tfrac{1}{90}[1 + 10Dy + (40D^2 - 10D)y^2 + (80D^3 - 60D^2)y^3 + (80D^4 - 120D^3 + 30D)y^4 + (32D^5 - 80D^4 - 300D^2 + 108D)y^5 + (1280D^3 + 5040D^2 - 4240D)y^6 + (-2560D^4 - 64\,080D^3 + 76\,880D^2 - 23\,040D)y^7 + \cdots]. \tag{17}$$

The strong-coupling expansions in (14)–(17) were obtained by treating the dimensionless parameter $\alpha = -a^{D-2}m^2/g$ as small in the limit where the bare coupling g tends to infinity. The relation in (12) explicitly carries the assumption of smallness over to the parameter y. This justifies the reversion of (12) into (13) and the subsequent reexpansion of the scattering amplitudes γ_{2n} in powers of y. Up to this point we have taken the lattice spacing a to be held fixed. We expect that in the continuum limit $a \to 0$ our expressions for γ_{2n} become the corre-

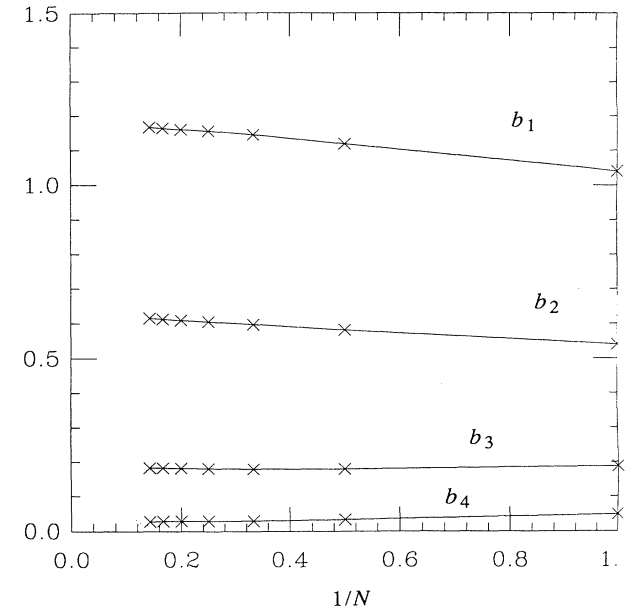

FIG. 1. Padé extrapolation for the first four coefficients, $b_i(i=1,\ldots,4)$, in the dimensional expansion of $\gamma_4=\frac{1}{12}(1+b_1D+b_2D^2+b_3D^3+b_4D^4+\cdots)$. For each coefficient b_i we plot the value of the $(0,N)$-Padé (shown as cross) as a function of $1/N$ for $N=1,\ldots,7$. The continuum value of each b_i is the extrapolation of the sequence to $N=\infty$. In Eq. (1) we list the results of this procedure for γ_4, γ_6, γ_8, and γ_{10}.

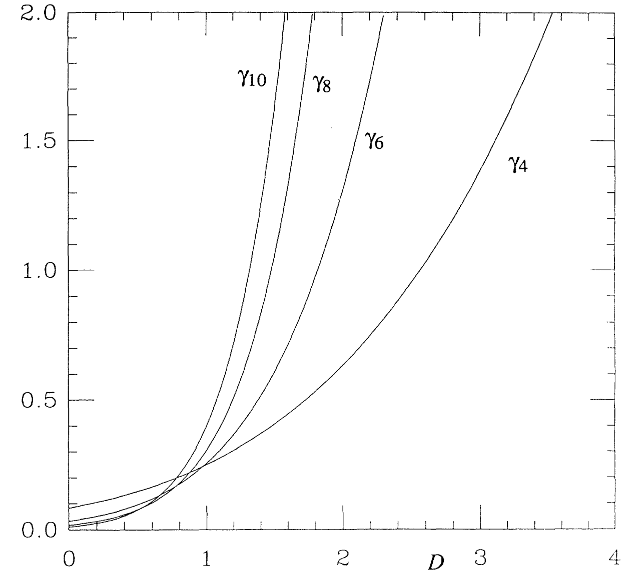

FIG. 2. Plot of γ_4, γ_6, γ_8, and γ_{10} in Eq. (1) as functions of D. For each n, γ_{2n} rises monotonically and γ_{2n+2} is growing faster than γ_{2n} for increasing D. We believe that the radius of convergence of the D series for each γ_{2n} in Eq. (1) is $D=2n/(n-1)$.

sponding quantities of the continuum theory. The continuum limit is subtle because as $a\to 0$ the parameter y that we have taken to be small actually becomes infinite. Hence, subsequent terms in this expansion for the scattering amplitudes γ_{2n} in (14)–(17) are increasingly singular as a series in y in the limit where $a\to 0$.

We use Padé extrapolation techniques to extract information from perturbation series like those in (14)–(17), where the perturbative parameter tends to infinity. The Padé extrapolation method employed here uses as input a perturbation series of the form

$$f(y)=y^r(c_0+c_1y+c_2y^2+\cdots)\quad(r\neq 0)\,, \tag{18}$$

where we assume that $f(\infty)$ is finite. We first take the rth root of both sides of (18) and divide by y to obtain

$$\frac{f(y)^{1/r}}{y}=(\text{new power series in } y)\,. \tag{19}$$

We then take the Nth power of the right side of (19) for $N=1,2,3,\ldots$, reexpand and form the $(0,N)$-Padé approximant. By extracting the coefficient of y^N in the denominator of the $(0,N)$-Padé approximant and raising it to the power $-r/N$, we create a sequence of Padé extrapolants for $f(\infty)$ [9]. We apply this method to the scattering amplitudes γ_{2n} in (14)–(17) and expand each $(0,N)$-Padé approximant for all γ_{2n} in powers of D. For each n, we obtain a sequence in N of $(0,N)$-Padé approximants for each coefficient in the D series of γ_{2n}. In Fig. 1 we plot the $(0,N)$-Padé extrapolants for the first four coefficients in the dimensional expansion of γ_4 as functions of $1/N$. We indicate the errors in the Padé determination of the coefficients in (1). We truncate the dimensional expansions for each γ_{2n} after that coefficient for which the estimated error of the following coefficient becomes significant compared to its absolute size.

Observe that the series in (1) all have positive coefficients and therefore each γ_{2n} is a monotonically rising function of D. Each of these functions is plotted in Fig. 2. For each n, γ_{2n+2} is growing faster than γ_{2n} for increasing D. We believe that the radius of convergence of the D series for γ_{2n} is likely to be $D=2n/(n-1)$, the space-time dimension for which the coupling constant g of a $g\phi^{2n}$ theory becomes dimensionless and the theory becomes renormalizable. Since we expect that for values of $D>2n/(n-1)$, γ_{2n} vanishes [10–15], we assume that there is a singularity (possibly a natural boundary) in the complex-D plane at $D=2n/(n-1)$.

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