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Perturbation dynamics for membranes and strings governed by the Dirac-Goto-Nambu action in curved space

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It is shown how to set up a concise and fully covariant formalism, in terms of an appropriate (geodesically defined) displacement vector ξ^μ, in such a way that the corresponding second order perturbation of the Dirac-Goto-Nambu action itself provides the action for a convenient secondary variation principle governing the linearized dynamics of first order perturbations of the relevant membrane and string models in an arbitrarily curved background, as well as determining the corresponding conserved symplectic current associated with any pair of distinct solutions ξ^μ and $\grave{\xi}^\mu$ on the world sheet.

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This paper presents an efficient formalism for treating perturbations in an arbitrary curved n dimensional spacetime background of the simplest kind of relativistic string, membrane, or other "brane" [1] model, as governed by an action $\mathcal{I}$ that is the world sheet integral of a scalar Lagrangian $\tilde{\mathcal{L}}$ that is just a *constant:*

$$\mathcal{I}=\int \tilde{\mathcal{L}}\, d\tilde{\Sigma}, \quad \tilde{\mathcal{L}}=-m^{p+1} , \tag{1}$$

using units with $\hbar=c=1$, where $d\tilde{\Sigma}$ is the surface measure element induced on the timelike world sheet by the background metric, $g_{\mu\nu}$ say, and m is a fixed parameter having the dimensionality of a mass (which would be of order of magnitude of the relevant Higgs boson mass scale for a cosmic monopole, string or membrane obtained from a spontaneously broken vacuum symmetry [2]) while the power index $p+1$ is the dimension of the relevant timelike world sheet, so that $p=0$ for an ordinary point particle, $p=1$ for a Goto-Nambu string, $p=2$ for a Dirac membrane, and so on for higher "p branes" up to $p=n-2$.

It is well known that in a flat background spacetime the dynamic equations for a Goto-Nambu string (and *a fortiori* for a free point particle) are expressible exactly in a linear, and thus fully integrable, form, making it unnecessary to bother about using a linearized perturbation theory. However in a not so highly symmetric background, such as that of a rotating black hole in an asymptotically de Sitter universe [3], one can only hope to obtain an explicit analytic solution for special cases (such as configurations that are invariant with respect to residual symmetries of the background) while for higher dimensional cases, starting with the Dirac membrane, the dynamic equations are nontrivial even in flat space.

In view of such considerations, Larsen and Frolov [4] have recently developed a perturbation formalism that is applicable to Goto-Nambu strings and their higher dimensional Dirac-type generalization, and that is covariant in the weak sense of being expressed in terms of quantities of the bitensorial kind introduced by Eisenhart [5], involving two kinds of indices $A,B=0,\ldots,p$ and $\mu,\nu=0,\ldots,n-1$ respectively, referring to distinct world sheet and background coordinate systems. The purpose of the present work is to show how newer geometrical methods [6] can be used to achieve the same purpose by means of a more powerful formalism that is covariant in the strong sense of being expressed in terms, not of bitensors, but only of ordinary tensors, referring just to external background coordinates x^μ, whose physical significance is more directly interpretable.

Instead of working in terms of internal metric, $h_{AB}=g_{\mu\nu}x^\mu{}_{,A}x^\nu{}_{,B}$ (using a comma for partial differentiation), the present treatment will be carried out in terms of the (first) *fundamental tensor* of the $(p+1)$-dimensional world sheet, i.e., the corresponding (rank $p+1$) spacetime pullback, whose coordinates $\eta^{\mu\nu}=h^{AB}x^\mu{}_{,A}x^\nu{}_{,B}$, like those of the complementary rank $(n-p-1)$ *orthogonal projection* tensor $\perp^\mu{}_\nu=g^\mu{}_\nu-\eta^\mu{}_\nu$, are quite independent of any choice of internal (world sheet) coordinate system. Instead of the many internal coordinate-dependent first differentiated quantities needed for the Eisenhart formalism, the present approach needs just the unique *second fundamental tensor* that is given in terms of the tangential covariant differentiation operator $\tilde{\nabla}_\mu=\eta^\nu{}_\mu\nabla_\nu$ (where ∇_μ denotes Riemannian covariant differentiation with respect to $g_{\mu\nu}$) by

$$K_{\mu\nu}{}^\rho=\eta^\lambda{}_\nu\tilde{\nabla}_\mu\eta^\rho{}_\lambda , \tag{2}$$

which is characterized by the nontrivial symmetry property [6] $K_{\mu\nu}{}^\rho=K_{\nu\mu}{}^\rho$ (interpretable as the Weingarten integrability condition) and the obvious tangentiality and orthogonality conditions $\perp^\lambda{}_\mu K_{\lambda\nu}{}^\rho=0, K_{\mu\nu}{}^\lambda\eta^\rho{}_\lambda=0$.

The *trace* of the second fundamental tensor gives the world sheet orthogonal *curvature vector*

$$K^\nu=K^\mu{}_\mu{}^\nu=\tilde{\nabla}_\mu\eta^{\mu\nu}, \quad \eta^\mu{}_\nu K^\nu=0 , \tag{3}$$

which plays a particularly important role in the present context because the special "harmonicity" [7] condition that K^ν should vanish expresses the context of the well-known Dirac-Goto-Nambu dynamic equations that are gotten, as shown below, from the action (1). The neat, obviously vectorial expression (3) is to be contrasted with what is obtained in the weakly covariant Eisenhart-type notation scheme of the traditional approach [4] based on the explicit use of the world sheet metric h_{AB}, which in terms of the corresponding internal Dambertian operator $\Box$, say on the world sheet, and of the components $\Gamma_\lambda{}^\mu{}_\nu$ of the Riemannian background connection, gives $K^\mu = \Box x^\mu + h^{AB}\Gamma_\lambda{}^\mu{}_\nu x^\lambda{}_{,A} x^\nu{}_{,B}$.

We shall need to consider not only first but also *second* order perturbations of the action, since according to a very general principle it is the latter that will govern the first order perturbations of the equations of motion with which we are ultimately concerned. As with solid or fluid media [8], in dealing with strings and membranes it is often most convenient in actual calculations to work with perturbations that are *Lagrangian* in the sense of being defined with respect to a reference system that is comoving with the relevant displacement. The relevant finite Lagrangian difference operator Δ will be expansible in terms of the corresponding infinitesimal Lagrangian differential, δ_L, say in the form $\Delta = \delta_L + \delta^2{}_L/2! + \cdots$.

"Lagrangian" variations (as opposed to the "Eulerian" and "parallel" kinds described below) are particularly convenient for dealing with an action integral of the general form (1), whose first order variation specifies the corresponding surface stress momentum energy density tensor $\tilde{T}^{\mu\nu}$ via the formula [9] $\delta\mathcal{I} = \frac{1}{2}\int \tilde{T}^{\mu\nu}(\delta_L g_{\mu\nu})d\tilde{\Sigma}$, where

$$\delta\mathcal{I} = \int [(\delta_L \tilde{L})d\tilde{\Sigma} + \tilde{L}\delta_L(d\tilde{\Sigma})] , \tag{4}$$

in which the induced variation of the surface measure element is given simply by

$$\delta_L(d\tilde{\Sigma}) = \tfrac{1}{2}\eta^{\mu\nu}(\delta_L g_{\mu\nu})d\tilde{\Sigma} . \tag{5}$$

In the present case the Lagrangian scalar in (1) is fixed, so that its variation will be given trivially by $\delta_L \tilde{L} = 0$. For the corresponding Dirac-Goto-Nambu surface stress momentum energy density tensor one thus obtains the simply (internally Lorentz invariant) result

$$\tilde{T}^{\mu\nu} = -m^{p+1}\eta^{\mu\nu} . \tag{6}$$

For a field, such as the metric, whose support is not confined to a particular brane world sheet, there will be a corresponding *fixed point* or "Eulerian" differential, δ_E say, given by $\delta_E = \delta_L - \vec{\xi}\mathcal{L}$, where $\vec{\xi}\mathcal{L}$ denotes Lie differentiation with respect to the relevant infinitesimal *displacement vector* field $\vec{\xi}$, whose components can be identified with the corresponding first order change in the scalar coordinate fields x^μ of the background, i.e.,

$$\xi^\mu = \delta_L x^\mu . \tag{7}$$

In a more general analysis designed for evaluating the effect of incoming gravitational waves, one might wish to allow for a nonzero Eulerian metric variation, but the present analysis will be restricted to the case in which *the background metric is fixed,* i.e., $\delta_E g_{\mu\nu} = 0$. This implies that the corresponding first order Lagrangian variation of the metric will simply be given by an expression of the familiar form

$$\delta_L g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\mu \xi_\nu , \tag{8}$$

whose substitution in (4) gives

$$\delta\mathcal{I} = \int \tilde{\nabla}_\mu(\tilde{T}^\mu{}_\nu \xi^\nu)d\tilde{\Sigma} - \int \xi^\nu \tilde{\nabla}_\mu \tilde{T}^\mu{}_\nu d\tilde{\Sigma} , \tag{9}$$

in which the first integrand is a pure surface divergence. The condition that $\delta\mathcal{I}$ should vanish for any local displacement ξ^μ thus gives a dynamic equation expressible as the vanishing of the relevant surface stress momentum energy tensor divergence, which is proportional, in the present case (6), just to the curvature vector: $\tilde{\nabla}_\nu \tilde{T}^{\mu\nu} = -m^{p+1}K^\mu$.

Having thus recapitulated the standard results obtained from the first order action variation, we are now ready to move on to new ground by considering the corresponding second order action variation which is obtainable from (4) as

$$\delta^2\mathcal{I} = \tfrac{1}{2}\int [\tilde{T}^{\mu\nu}(\delta_L^2 g_{\mu\nu})d\tilde{\Sigma} + (\delta_L \tilde{T}^{\mu\nu})(\delta_L g_{\mu\nu})d\tilde{\Sigma} + \tilde{T}^{\mu\nu}(\delta_L g_{\mu\nu})\delta_L(d\tilde{\Sigma})] . \tag{10}$$

The last two terms in (10) can be evaluated using the formulas given above together with the formula for the first order Lagrangian variation of the fundamental tensor $\eta^{\mu\nu}$ itself which is given simply by

$$\delta_L \eta^{\mu\nu} = -\eta^{\mu\rho}\eta^{\nu\sigma}\delta_L g_{\mu\nu} . \tag{11}$$

However the evaluation of the first term in (10) poses a problem of a new kind, because it involves the second variation of the metric, and so depends on $\delta_L^2 x^\mu$, which has not yet been specified.

As can be seen from the example of the Bunting identity [10], a particularly convenient way of using a vector $\vec{\xi}$ to provide a covariant specification of a spacetime displacement Δx^μ, not just to first infinitesimal order but also to higher order as required here, is to prescribe that the displacement be given by the end point of the corresponding geodesic, so that, in particular, the second order analogue of the first order coordinate variation formula (7) will be given by $\delta_L^2 x^\mu = -\Gamma^\mu_{\lambda\nu}\xi^\lambda\xi^\nu$.

When the displacement is defined in this way, it can be seen to follow [11] (subject, as before, to the understanding that the Eulerian variation is zero) that the second order analogue of the first order contribution (8) to the Lagrangian variation induced in the metric will be given by

$$\delta^2{}_L g_{\mu\nu} = 2(\nabla_\mu \xi^\rho)\nabla_\nu \xi_\rho - 2\mathcal{R}_{\mu\rho\nu\sigma}\xi^\rho\xi^\sigma , \tag{12}$$

where $\mathcal{R}_{\mu\rho\nu\sigma}$ denotes the Riemann curvature tensor of the background.

Using (12) to complete the evaluation of (10) leads to the principal result of this work, which is the second variation formula

$$\delta^2\mathcal{I} = \int \tilde{T}^\mu{}_\nu[\perp_{\rho\sigma}(\tilde{\nabla}_\mu\xi^\rho)\tilde{\nabla}^\nu\xi^\sigma + 2(\tilde{\nabla}_{[\rho}\xi^\rho)\tilde{\nabla}_{\mu]}\xi^\nu - \mathcal{R}_{\mu\rho}{}^\nu{}_\sigma\xi^\rho\xi^\sigma]d\Sigma , \tag{13}$$

(where square brackets denote index antisymmetrization) in which it is important to notice that (as indicated by the tilde on the symbol $\tilde{\nabla}$) the only derivations involved are tangentially projected. This makes it manifest that [unlike the variations (8) and (12) of the metric, which depend on the way the displacement is extrapolated over the background] the final result (13) depends only on the value of the vector field $\vec{\xi}$ on the world sheet itself, which is all that is physically relevant.

According to the general principles of second order variation theory [8], one expects the second order action variation $\delta^2\mathcal{I}$, to be utilizable as an action in its own right for the corresponding dynamic equations governing the vectorial perturbation components ξ^μ on the world sheet. This can be confirmed explicitly by considering the change $\delta^2 \mapsto \delta^2 + 2\check{\delta}\delta + \check{\delta}^2$ in (13) due to an alteration

$$\xi^\mu \mapsto \xi^\mu + \grave{\xi}^\mu \ . \tag{14}$$

By an integration by parts of the usual kind, the bilinear contribution is convertible to the form

$$\check{\delta}\delta\mathcal{I} = \int \nabla_\mu [\tilde{T}^\mu{}_\nu (\perp_{\rho\sigma} \grave{\xi}^\rho \tilde{\nabla}^\nu \xi^\sigma + 2\grave{\xi}^{[\nu} \tilde{\nabla}_\rho \xi^{\rho]})] d\tilde{\Sigma} + m^{p+1} \int \grave{\xi}^\mu (\delta_P K_\mu + K_\mu \tilde{\nabla}_\nu \xi^\nu) d\tilde{\Sigma} \ , \tag{15}$$

using the notation δ_P to denote the *parallel* differential, as defined with respect to a frame that is parallel propagated along the geodesic specified by the displacement vector $\vec{\xi}$.

The operator δ_P is a sort of compromise between the comparatively "subjective" Lagrangian differential δ_L and the more "objective" Eulerian differential δ_E. For a field (such as the background metric) for which the latter is well defined, the parallel differential will be given by $\delta_P = \delta_E + \xi^\mu \nabla_\mu$ (so that in the case of the background metric its parallel derivative will be the same as its Eulerian one, which is postulated to be zero in the present work). It can, however, be seen that the derivative contributions to the equivalent expression $\delta_P = \delta_L + \xi^\mu \nabla_\mu - \xi\vec{\mathcal{L}}$ will cancel out so that the total will remain well defined, giving the correct result, even for a quantity that is well defined only on the world sheet (so that its Eulerian differential does not exist) as is the case for the curvature covector K_μ, whose parallel differential will be given in terms of its less objective but more computationally convenient Lagrangian differential, by $\delta_P K_\mu = \delta_L K_\mu - K_\nu \nabla_\mu \xi^\nu$.

Using (11) to work out the Lagrangian variation of the formula (3) for the curvature vector, the corresponding parallel variation is explicitly obtainable in the form

$$\delta_P K_\mu = \perp_{\mu\lambda} \tilde{\nabla}_\nu \tilde{\nabla}^\nu \xi^\lambda - 2K_\lambda{}^\nu{}_\mu \tilde{\nabla}_\nu \xi^\lambda - K_\lambda \tilde{\nabla}_\mu \xi^\lambda + \perp_{\mu\nu} \eta^{\rho\sigma} \mathcal{R}_\rho{}^\nu{}_{\sigma\lambda} \xi^\lambda \ , \tag{16}$$

which again (unlike $\delta_L K_\mu$) has the property of depending only on the values of ξ^μ *on the world sheet.* It can be verified (using the generalized Codazzi identity satisfied by the third fundamental tensor [6]) that the effect of a gauge adjustment whereby ξ^μ undergoes a modification of the form (14) for a displacement $\grave{\xi}^\mu$ that is *tangential* to the world sheet is given simply by the corresponding tangential covariant derivative, i.e.,

$$\perp^\mu{}_\nu \grave{\xi}^\nu = 0 \Longrightarrow \delta_P K_\mu \mapsto \delta_P K_\mu + \grave{\xi}^\nu \tilde{\nabla}_\nu K^\nu \ . \tag{17}$$

It was already apparent from (9) that, since the divergence term gives no contribution for a variation with compact support, the unperturbed dynamical equations of motion of the Dirac-Goto-Nambu system characterized by (1) are expressible as the simple harmonicity condition

$$K_\mu = 0 \ . \tag{18}$$

It can now also be seen in the same way [from the requirement that the surviving second term on the right of (15) should be independent of $\grave{\xi}^\mu$] that the linearized system characterized by the action (13) will, as promised, give rise to dynamical equations of motion that are obtainable, using the zero order condition (18), in the form

$$\delta_P K_\mu = 0 \ , \tag{19}$$

so as to provide a correct description of first order perturbations of the original system (18). It is evident, in view of (18), that the linearized system is invariant with respect to gauge transformations of the form (17), i.e., the equations of motion (19) are automatically satisfied in the trivial case of a displacement $\vec{\xi}$ that is tangential to the world sheet. This gauge invariance can of course be removed so as to obtain a dynamical perturbation vector that is physically well defined by imposing the orthogonality condition $\eta^\mu{}_\nu \xi^\nu = 0$. Whether or not one chooses to work in this orthogonal gauge, it is to be noted that, by further use of (18), the perturbation formula (16) can be further simplified so as to allow the linearized dynamical equations for the perturbation components ξ^μ to be finally expressed in the compact form

$$\perp^\mu{}_\lambda \tilde{\nabla}_\nu \tilde{\nabla}^\nu \xi^\lambda = 2K_\lambda{}^{\nu\mu} \tilde{\nabla}_\nu \xi^\lambda - \perp^\mu{}_\nu \eta^{\rho\sigma} \mathcal{R}_\rho{}^\nu{}_{\sigma\lambda} \xi^\lambda \ , \tag{20}$$

in which the terms on the right-hand side allow for the respective effects of the curvature of the unperturbed solution and of the background, in terms of the corresponding second fundamental tensor and Riemann tensor, in a fully covariant manner whose geometric meaning is more patent than that of the series of only weakly covariant terms needed for the equivalent formula in the Eisenhart formalism [4].

It is to be noted in conclusion that, since the bilinear variation (15) must be invariant under interchange of ξ^μ and $\grave{\xi}^\mu$, the difference between the term within the divergence on the right and its analogue under this interchange will evidently give a symplectic Noetherian surface current

$$\mathcal{C}^\mu = (2\tilde{T}^\mu{}_{[\sigma} \eta^\nu{}_{\rho]} - \tilde{T}^{\mu\nu} \perp_{\sigma\rho})(\grave{\xi}^\sigma \tilde{\nabla}_\nu \xi^\rho - \xi^\sigma \tilde{\nabla}_\nu \grave{\xi}^\rho) \tag{21}$$

that will automatically satisfy the surface current conservation law $\tilde{\nabla}_\mu \mathcal{C}^\mu = 0$ whenever both ξ^μ and $\grave{\xi}^\mu$ are solutions of the dynamical perturbation equations (20). As a particular application, this formula can be used to obtain the conserved currents associated with any (continuous) symmetries of the background, using the new solution generated from a previously given solution by the action of the relevant Killing vector k^μ. The expression (21) can be considerably simplified by the use of the orthogonal gauge, which reduces it just to $\mathcal{C}^\mu = \tilde{T}^{\mu\nu}(\xi_\rho \tilde{\nabla}_\nu \grave{\xi}^\rho - \grave{\xi}_\rho \tilde{\nabla}_\nu \xi^\rho)$.

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