PHYSICAL REVIEW D VOLUME 48, NUMBER 10 15 NOVEMBER 1993

Semiclassical electrodynamics: A model for semiclassical gravity

Miles Blencowe
Enrico Fermi Institute, University of Chicago, 5640 South Ellis Avenue, Chicago, Illinois 60637
(Received 27 January 1993)

We consider the QED process involving the breakup of a neutral, spherical object into a charged inner core and equal, but oppositely charged expanding outer shell, such that a substantial amount of pair creation occurs. We show, under certain conditions, that semiclassical electrodynamics gives a good description at late times, when pair creation has ceased to good approximation. We also outline a method for testing the semiclassical approximation during the dynamical stage, when pair creation is taking place. The underlying motivation for this investigation is to understand better some of the issues concerning semiclassical gravity.

PACS number(s): 12.20.Ds, 04.60.+n

I. INTRODUCTION

The semiclassical theory of gravity treats the metric classically and matter quantum mechanically:

$$G_{ab}=8\pi G\langle \hat{T}_{ab}\rangle \ . \tag{1}$$

Here, G_{ab} is the Einstein tensor and $\langle \hat{T}_{ab}\rangle$ is the expectation of the energy-momentum tensor operator with respect to some quantum state. This theory may account for a wider range of gravitational phenomena than can be described by either the classical Einstein equations or quantum field theory on a fixed curved spacetime. A possible example of one such phenomenon is black-hole mass loss due to the Hawking effect.

There are several problems which must be addressed, however, before semiclassical gravity can be considered a viable theory [1].

(1) A calculation of $\langle \hat{T}_{ab}\rangle$ requires a regularization and removal of a divergent part. However, demanding that the remaining finite term satisfy all physically reasonable conditions (covariant conservation, etc.) does not uniquely specify the infinite part to be removed: the resulting finite $\langle \hat{T}_{ab}\rangle$ turns out to have a two-parameter ambiguity.

(2) These ambiguous terms involve higher derivatives which give rise to unphysical, "runaway" solutions to the semiclassical equations.

Let us suppose that we are somehow able to overcome these two problems: we have an unambiguous procedure for extracting well-behaved solutions from the semiclassical equations.[1] We should then address the following problem.

(3) Given that the correct theory is quantum gravity, under what conditions do we expect the semiclassical theory to give an accurate description? In other words, what is the domain of validity of semiclassical gravity?

To date, there has been little progress in finding a solution to these problems. Perhaps the main reason for this is our lack of understanding of dynamical gravity/particle production processes, a consequence of the extreme difficulty in calculating $\langle \hat{T}_{ab}\rangle$ for curved spacetimes. Furthermore, to provide a proper answer to (3) would require knowledge of the full quantum theory of gravity.

Recognizing the difficulties in trying to directly address the above problems, a possible line of attack might be to first consider a simpler, better understood quantum field theory whose associated semiclassical theory has similar properties to semiclassical gravity, such as particle production. If, for this analogous theory, we can find concrete examples of a process involving a substantial amount of particle production (so that back reaction effects are important) *which is well described semiclassically*, then we will have a little more confidence that problems (1) and (2) can be overcome and that the resulting semiclassical gravity theory can accurately describe nontrivial processes such as black hole evaporation. Most importantly, such an investigation may give us hints concerning the nature of the resolution of these problems.

As analogues of quantum gravity and semiclassical gravity, we shall consider Minkowski spacetime quantum electrodynamics and the associated semiclassical electrodynamics theory which treats the vector potential classical and Dirac field quantum mechanically.[2] With black-hole evaporation in mind, we shall investigate the following quantum electrodynamics process: at early times we have an electrically neutral spherical object which should be thought of as comprising equal number densities of positive and negative charges. At a later time, the negative charges leave the object as an expanding spherical shell. When the potential energy difference between the outer shell and core center exceeds $2m$, where m is the

[1] Simon [2] has given a possible resolution to problem (2).

[2] The idea of using semiclassical electrodynamics as a model for semiclassical gravity has been considered before [3]. See also Ref. [4] for investigation of another model system: two-dimensional dilaton gravity. An electrodynamic analogue of this model is given in Ref. [5].

electron mass, the charge-zero vacuum becomes unstable and spontaneous pair creation occurs. This should be viewed as an analogue of the Hawking effect. We suppose that the created field in the region of the positively charged spherical core is such that a substantial amount of e^+/e^- pair creation takes place. Over a period of time the produced electrons will screen the charged core, while the positrons will stream radially outwards from the core towards the outer shell. The decrease in the net charge in the core region is analogous to black hole mass loss.

The full QED and semiclassical descriptions of this process are obtained from the following respective equation pairs:[3]

$$\partial_\nu \hat{F}^{\nu\mu} = e\hat{\bar{\psi}}\gamma^\mu\hat{\psi} + J^\mu \,, \quad (i\partial\!\!\!/ - e\hat{A\!\!\!/} - m)\hat{\psi} = 0 \,, \tag{2}$$

and

$$\partial_\nu F^{\nu\mu} = \langle 0|(e\hat{\bar{\psi}}\gamma^\mu\hat{\psi})|0\rangle + J^\mu \,, \quad (i\partial\!\!\!/ - eA\!\!\!/ - m)\hat{\psi} = 0 \,. \tag{3}$$

In these equations, $\hat{j}^\mu := e\hat{\bar{\psi}}\gamma^\mu\hat{\psi}$ is the electron-positron current operator. The semiclassical Maxwell equation (3) is the analogue of the semiclassical Einstein equation (1), with $\langle \hat{j}^\mu \rangle$ and $\langle \hat{T}^{\mu\nu} \rangle$ playing the same roles: namely, as back reaction producing sources. Note, however, that problem (1) does not arise for $\langle \hat{j}^\mu \rangle$ (see, e.g., [6]). J^μ is a classical background charge/current density which we assume can well approximate the bare core and expanding shell charges. At early times, $J^\mu \approx 0$, and the full QED and semiclassically described systems are in their respective vacuum states $|0\rangle$.

Our aim will be to try to find examples of the above-described process for which the full QED expectation value $\langle \hat{F}^{\mu\nu} \rangle$ is well approximated by $F^{\mu\nu}$ determined from the semiclassical equations. The process can be controlled to the extent that J^μ can be varied (but without violating current conservation). Important parameters will be the degree of spatial variation of charge density of the inner core, and the rate at which the outer shell expands.

When we approximate full QED by its semiclassical theory, we neglect certain pair and photon production contributions. Similarly, when we use semiclassical gravity, we neglect certain pair and graviton production contributions. Even if we begin with a solar mass black hole for which the mass-loss rate is extremely low, it is not obvious that neglecting the nonsemiclassical contribution will not produce a significant error over long enough times for substantial mass loss. However, we can be a little more hopeful that this will in fact be the case if for the analogous "slow" QED process we find that the semiclassical description is accurate.

After carrying out the investigation outlined above, we will have a better idea of the domain of validity of semiclassical electrodynamics. The specification of the domain will take the form of a certain set of conditions on the space-time variability of $\langle \hat{F}^{\mu\nu} \rangle$. Considering other theories as well, such as scalar electrodynamics, and comparing the various obtained conditions may point out common and hence more universal physical criteria for validity. With such knowledge, we can then perhaps make a more sensible guess of the domain of validity of semiclassical gravity. An attractive possibility is that problems (1)–(3) are not independent: the problems of ambiguous and unphysical behavior can only be overcome provided one is within the range of validity of the semiclassical approximation. Clearly, then, to be able to make an educated guess concerning problem (3) would be helpful.

A cautionary remark: we should be careful not to draw too close an analogy between electrodynamics and gravity. Experience has taught us that gravity is rather unique. As a model, we expect semiclassical electrodynamics can only tell us so much concerning the problems of semiclassical gravity.

In the present paper, we will mostly be concerned with the latetime stage of the above-described electrodynamics process, when pair creation and other dynamical effects have ceased to good approximation. We will show that the semiclassical theory can give an accurate description at late times. We shall discuss the dynamical stage in a future paper. By itself, the late time stage of the electrodynamics process has little relevance for black-hole evaporation and so we will not have much further to say about semiclassical gravity in this paper. Despite this, we believe it is a good strategy to concentrate on understanding the late time stage before tackling the more difficult dynamical part: first, we have been able to learn about certain methods of analysis and techniques to estimate the semiclassical part and QED correction which may usefully generalize to the dynamical problem. Second, we have gained some knowledge about spatial conditions for validity of the semiclassical approximation.

In the next section the late time results are presented, together with a discussion of them. In Sec. III, we derive the results and in the concluding section we map out, in brief, a plan of attack for the dynamical problem.

II. THE RESULTS

Subsequent to the shell expansion, we generally expect the behavior of $\langle \hat{F}^{\mu\nu} \rangle$ to be very complicated. However, as long as $\langle \hat{F}_{\mu\nu} \rangle$ is changing in time, the region of the core will lose energy through photon and e^+/e^- emission. Semiclassically, the core region can only lose energy through e^+/e^- emission. This situation is atypical, however, since an arbitrarily small departure from spherical symmetry allows energy loss through Maxwell radiation. Thus, after a sufficiently long time, we expect there to be no dynamical effects to good approximation.

We shall address the question of the validity of the semiclassical approximation at late times in the region of the inner core. The sole function of the expanding outer shell is to produce an electric field which is such as to

[3] We work with units $\hbar = c = 1$; $\alpha := e^2/4\pi = \frac{1}{137}$; $e = -|e|$.

give rise to a substantial amount of pair creation. We suppose that at late times the shell is at rest and with sufficiently large radius that the measured potential difference between the inner and outer surfaces of the shell is negligibly small compared to the potential difference between the core center $r=0$ and the outer surface of the shell. The shell can therefore be ignored.

Since at late times there are no dynamical effects to good approximation, we can choose a gauge such that only the time component of the vector potential is nonzero as $t \to +\infty$. Denote

$$V(r) := e\langle \hat{A}^0(r,t)\rangle|_{t\to+\infty} ,$$

where $\langle \hat{A}^\mu \rangle$ is the full QED expectation of the vector potential. Fix the remaining gauge freedom by demanding that $V(r) \to 0$ as $r \to +\infty$.

Now, suppose V satisfies the following conditions in the region of the core: (i) $|V(r)| \gg m$, (ii) $\lambda^2(r)|V''(r)|$, $\lambda(r)|V'(r)| \lesssim |V(r)|$; $\lambda(r) = |(e/\pi)V(r)|^{-1}$, i.e., $V(r)$ is slowly varying on the scale $\lambda(r)$, and (iii) $\alpha/3\pi \ln|V'(r)/m^2| \ll 1$; then V is obtained from

$$\nabla^2 V = -e^2(n_e - n_b)$$

$$[\text{boundary conditions } V'(r)=0 \text{ at } r=0 \text{ and } V(r)\to 0 \text{ as } r\to+\infty] , \quad (4)$$

with the produced electron number density n_e expressed as a function of V through the key relation

$$n_e(r) \approx -\frac{1}{3\pi^2}\left\{ V(r) + n_e^{1/3}(r)\left[\frac{1}{2}\left[\frac{3}{\pi}\right]^{1/3}\alpha - \frac{1}{2\pi}\left[\frac{3}{\pi}\right]^{1/3}\alpha^2\ln\alpha + (\cdot)\alpha^2 + \cdots \right]\right\}^3 . \quad (5)$$

In (4), n_b is the background number density. The V term on the right-hand side of relation (5) is the semiclassical contribution, while the $n_e^{1/3}$ term is the full QED correction. The semiclassical result was obtained in Ref. [7].[4] Since the second term of the series in α is smaller than the first by a factor of $\alpha \ln\alpha$, it is reasonable to assume that keeping only the first term in the series gives a good approximation to the QED correction. From (5), we therefore have

$$n_e(r) \approx -\frac{1}{3\pi^2}\left[1 - \frac{3}{2\pi}\alpha\right]\left[V(r)\right]^3 , \quad (6)$$

and substituting this into (4), we obtain

$$\nabla^2 V \approx -e^2\left[-\frac{1}{3\pi^2}\left[1-\frac{3}{2\pi}\alpha\right]V^3 - n_b\right] . \quad (7)$$

Gauss' law gives for the measured electric field

$$|E(r)| \approx \frac{1}{r^2}\left| \int_0^r dr' r'^2 e\left[-\frac{1}{3\pi^2}\left[1-\frac{3}{2\pi}\alpha\right]V^3 - n_b\right]\right| . \quad (8)$$

From (8), we see that the semiclassical approximation is generally a good one. However, the approximation may break down if the difference between n_b and the semiclassically determined n_e is of order αn_e (or less) throughout the region $0 \le r' \le r$.

We now explain the meaning of conditions (i)–(iii) and also demonstrate that they are attainable.

Recall that we want to find examples of processes with a non-negligible back-reaction effect and which are well described semiclassically. Now the background bare core will have a total charge much larger than $|e|$ (we are assuming that it can be well approximated classically). Since a background potential difference of $2m$ between $r=0$ and ∞ must be introduced in order to spontaneously create just two e^+/e^- pairs (because of Pauli's principle), the background charge density should be such that the background potential satisfies $|V_b| \gg m$ in the region of the core in order to spontaneously create a large number of e^+/e^- pairs and thus have the possibility of a substantial back reaction. Using Eqs. (3), it is then straightforward to show that condition (i) holds.[5] Note that even for $|V_b| \lesssim m$, it may be possible to have a large number of electrons in the core region if the field had previously been time varying sufficiently rapidly so as to stimulate the production of large numbers of e^+/e^- pairs. However, in the case of non-negligible back reaction, it is not clear whether the final measured field would be well approximated semiclassically. With such time varying fields, the final number of produced electrons in the core region determined semiclassically may differ substantially from the actual full QED determined number. Also, the full QED final state will always be a degenerate ground state, whereas semiclassically we will typically expect an excited state since spontaneous photon emission (which makes the excited state unstable and hence decay to a ground state) is not taken into account. With condition (i), the final semiclassical and full QED-determined field strengths do not depend to a good approximation on the way in which the final background V_b is achieved. We will show this in the next section. It is this fact which enables us to investigate the semiclassical approximation at

[4]A nonrelativistic estimate of the correction to the semiclassical result is also given in Ref. [7]. Since we are in the ultrarelativistic domain with the above conditions on V, such an estimate is not well justified *a priori*. See also Ref. [8] where the semiclassical result is given without derivation.

[5]If, on the contrary, we had $|V(r)| \lesssim m$ in the region of the core, then from Dirac Eq. (3), we would obtain $n_e \ll n_b$ in the region of the core. But then Maxwell Eq. (3) becomes $\nabla^2 V \approx e^2 n_b \Longrightarrow |V| \gg m$, a contradiction.

late times. From now on, we suppose V_b satisfies $|V_b| \gg m$ in the region of the core.

With condition (ii), the derivation of the approximate dependence of n_e on V is considerably simplified. This condition justifies several approximations having similar form but different origin. For example, when deriving the semiclassical contribution we neglect a spin part since it contains terms such as $(V')^2/V^4$ and V''/V^3. Also, the QED correction is approximated as a local term. Actually, we can only make such an approximation if we have the stronger condition with "$\lesssim$" replaced by "$\ll$" in (ii). In general, the QED correction is nonlocal, probing the functional dependence of n_e in a volume having dimensions $\lambda(r)$ about a given position $\mathbf{r}$. Nevertheless, we shall assume that the correction term in (5) is still a reasonable order of magnitude estimate when the weaker condition (ii) holds. Equation (7) prevents us from imposing the stronger condition. We now argue that solutions V obtained from (7) can satisfy condition (ii), given suitable choices for n_b.

First, we need to show that V satisfies

$$V_b(r) < V(r) \le 0 \; . \tag{9}$$

Integrating (7) we have, neglecting the QED correction,

$$V'(r) = \frac{1}{r^2}\int_0^r ds\, s^2 \left[\frac{e^2}{3\pi^2}V^3(s)\right] + V_b'(r) \tag{10}$$

and

$$V(r) = -\int_r^\infty \frac{dt}{t^2}\int_0^t ds\, s^2 \left[\frac{e^2}{3\pi^2}V^3(s)\right] + V_b(r) \; . \tag{11}$$

Now, suppose we had $V(0) \ge 0$. Because $V_b'(r) \ge 0$ (and is strictly positive for some r), using (10) to generate the solution $V(r)$ we would find $V(r) \to +\infty$ as $r \to +\infty$. Thus, we must have $V(0) < 0$. In fact, we have $V(r) \le 0$ for all r, because if $V(r)$ were to become positive then there would have to be a turning point at which $V'' < 0$ [so that we might have $V(r) \to 0$ as $r \to +\infty$], but from (7) $V'' > 0$ at any turning point where $V > 0$. Finally, with $V(r) \le 0$ (and strictly negative for some r), it follows from (11) that also $V(r) > V_b(r)$ for all r.

Choose n_b such that in the core region V_b satisfies

$$\nabla^2 V_b (= e^2 n_b) \approx \frac{e^2}{3\pi^2}|V_b|^3 \; , \lambda_b |V_b'| \lesssim |V_b| \; , \qquad \lambda_b = \left|\frac{e}{\pi}V_b\right|^{-1} . \tag{12}$$

From (7) and (9) we then have

$$0 < \nabla^2 V \approx \frac{e^2}{3\pi^2}(V^3 + |V_b|^3) < \frac{e^2}{\pi^2}(V^3 + |V_b|^3) \approx \frac{e^2}{\pi^2}|V|^3 \; , \tag{13}$$

where we have used the fact that $|V_b|^3$, $|V|^3$, and $|V_b|^3 - |V|^3$ must be of the same order of magnitude, for otherwise we would get a contradiction solving (7). Since $\nabla^2 V > 0$, from (10) and (12) we have

$$0 < V'(r) < V_b'(r) \lesssim \frac{e}{\pi}V_b^2 \approx \frac{e}{\pi}V^2 \tag{14}$$

[with $V'(r) = V_b'(r) = 0$ at $r = 0$]. Condition (ii) on V immediately follows from (13) and (14).

From the preceding analysis we see that, for V_b satisfying (12), the resulting V describe solutions where $n_e(r) < n_b(r)$, $n_e(r)$, $n_b(r)$, and $n_b(r) - n_e(r)$ are comparable in magnitude, and both $n_e(r)$ and $n_b(r)$ are monotonically decreasing with increasing r. Condition (12) is just a special case of condition (ii) for V_b. It would be of interest to study other types of solutions to (7) and also determine whether, given *any* V_b satisfying (ii) (with $\lambda_b = |(e/\pi)V_b|^{-1}$), it follows that the solutions V satisfy (ii) [with $\lambda = |(e/\pi)V|^{-1}$].

Condition (iii), together with condition (ii), allows us to neglect the "virtual" vacuum polarization contribution to n_e, as is evident from the following estimate of this contribution [7,9]:

$$n_{e\,\mathrm{vac}} \approx \left[\frac{V''}{12\pi^2} + \frac{\alpha}{3\pi}\ln\left|\frac{V'}{m^2}\right|(n_e - n_b)\right] . \tag{15}$$

Condition (iii) places an upper limit on the allowed electric-field strength.

Note that in the derivation of Eq. (5) giving n_e as a function of V, the various terms which are neglected can be comparable to the full QED correction. In light of this, we make the following crucial remark concerning our various approximations and assumptions, which should be kept in mind while reading this paper: since we are concerned solely with investigating the validity of the semiclassical approximation, we do not need a precise calculation of the semiclassical part and QED correction. Order of magnitude estimates are sufficient. Such an approach does make life easier. Some work is required, however, since confidence in the validity of the semiclassical approximation can only be achieved if some justification is given for the estimates.

When conditions (ii) and (iii) are violated, all our various approximations break down and we know longer expect relation (5) to give reasonable estimates for the semiclassical part and QED correction. We might speculate that, given V satisfying condition (i), conditions (ii) and (iii) are not only sufficient, but also *necessary* for the semiclassical approximation to be a good one. We are unable to provide evidence to support this, though; very little is known about QED for such rapidly varying and intense fields.

III. THE DERIVATION

A. Preliminaries

We now derive the results of the previous section. Several of the required calculations have been given elsewhere, so we shall try to be concise, referring the reader to the relevant papers. We assume throughout this section that conditions (i), (ii), and (iii) on V hold.

At late times, the produced positrons are either concentrated in the region of the outer shell or escaping to $r = \infty$ and therefore their effect on the measured electric field throughout the region of the inner core is negligible. Thus, to a good approximation we can take as our system the static (i.e., having existed for all time) charged core only, in a certain bound state. Of course, the choice of bound state depends on what happens during the dynamical stage: we expect a rapidly expanding shell to cause more pairs to be produced, and hence more bound electrons in the core region, than a slowly expanding shell. We shall show at the end of the section, however, that for the given conditions on V the error involved in working with an incorrect bound state is small.

Recall one of the main reasons for studying the late time stage first is to learn about possible fruitful methods for addressing the validity of the semiclassical approximation during the more difficult dynamical stage; clearly a method of approximation which is inadequate for investigating the late time stage has little chance of usefully generalizing when there are dynamical effects as well.

From Eq. (4), the potential V and hence the electric field can be readily determined once we know the electron charge density. The latter quantity can in turn be obtained from some (connected) electron two-point Green's function which we denote as $G(x,y)$ $[=G(\mathbf{x},\mathbf{y};t_x-t_y)]$. How are we to determine $G(x,y)$? In the usual method, one develops a series approximation in α, the background charge density and the free electron and photon Green's functions. This method is inadequate, since it amounts to perturbing about the usual zero-field vacuum which is unstable for the conditions being considered: we expect the series to diverge badly. A less approximate method [10] involves developing a series approximation in α, the free photon Green's function and the background field electron Green's function. We do not expect that much of an improvement, however, since the actual net charge density will differ substantially from the background charge density. (Recall there is a nonnegligible back-reaction effect.) We expect the series to converge only slowly, if at all. What is required is a method of approximation whereby the semiclassical contribution and QED correction can to some extent be analyzed separately: the semiclassical part is handled nonperturbatively, while the QED correction is treated with a combination of perturbative and/or nonperturbative techniques.

The coupled Dyson-Schwinger (DS) equations [10 (Sec. 2.3)], [11 (Sec. 5.34)], [12 (Sec. 10.1)] for the various electrodynamics Green's functions meet these requirements. Note, however, that, because of problems to do with renormalization [12 (Sec. 10.1.2)], there is at present no effective method for solving the DS equations nonperturbatively. Nevertheless, given that we are only concerned with obtaining order of magnitude estimates (see remark previous section), we shall assume that our investigation of these equations is meaningful.

Assume $G(x,y)$ can be decomposed as follows:

$$G(x,y)=i\left[\theta(t_x-t_y)\sum_{\substack{p\\(\epsilon_p>\epsilon_F)}}\phi_p(\mathbf{x})\bar{\phi}_p(\mathbf{y})\exp[-i\epsilon_p(t_x-t_y)]-\theta(t_y-t_x)\sum_{\substack{p\\(\epsilon_p<\epsilon_F)}}\phi_p(\mathbf{x})\bar{\phi}_p(\mathbf{y})\exp[-i\epsilon_p(t_x-t_y)]\right]. \quad (16)$$

Such a decomposition results if the Dirac field operates in a Fock representation. This is allowed semiclassically. However, in full QED we expect this is okay only if the QED correction turns out to be small and hence can be treated perturbatively. Substituting this decomposition into the DS equation for $G(x,y)$, we obtain a (rather nonlinear) differential equation for the functions $\phi_p(\mathbf{x})$ [13]:

$$[\epsilon_p+i\gamma_0\boldsymbol{\gamma}\cdot\nabla_{\mathbf{x}}-\gamma_0 m-V(|\mathbf{x}|)]\phi_p(\mathbf{x}) - \left[\gamma_0\int d\mathbf{z}\,M(\mathbf{x},\mathbf{z};\epsilon_p)\phi_p(\mathbf{z})\right]=0 , \quad (17)$$

where

$$\nabla^2 V=-e^2\{i\,\mathrm{tr}[\gamma^0 G(x,x)]-n_b\} , \quad (18)$$

and

$$M(\mathbf{x},\mathbf{z};\epsilon_p)=\int d(t_x-t_z)M(x,z)e^{i\epsilon_p(t_x-t_z)} ,$$

with

$$M(x,z):=-ie^2\gamma^\mu\int dy\,dy'\,G(x,y)\Gamma^\nu(y,z,y')D_{\nu\mu}(y',x) . \quad (19)$$

Let us now say a few things about these equations, beginning with (16).

In (16), the sums are carried out over a complete set of "single-particle" wave functions. We choose to work with these wave functions rather than with the whole of $G(x,y)$ since our various estimates require only a determination of the ϕ_p's with ϵ_p in a finite interval about ϵ_F. The "Fermi level" ϵ_F, which determines the particular bound state being considered, can take any value between $-m$ and $+m$ (see, e.g., Ref. [14] for more details concerning the present discussion). If $\epsilon_F=-m$, then we are in what might be termed the "vacuum" state, the state with the smallest possible number of bound electrons. Note that the vacuum state is still highly negatively charged. It is not possible to reduce the negative charge further, since removal of an electron produces an unstable state: spontaneous pair production occurs and a positron is ejected from the core, allowing a return to the vacuum state. If instead, $\epsilon_F=0$, then we are in the state with lowest measured total energy, which we call the "ground" state. Observe that the vacuum and ground states do not coincide: to get from the vacuum to ground state requires a filling of all the (discrete) levels in the interval $[-m,0]$ and the energy expended in creating an electron at $r=\infty$ (which is just m) is more than recovered by putting it into one of these orbits around the core. Charge conservation prevents states with $-m\leq\epsilon_F<0$ from decaying into the lower-energy ground state: electrons cannot actually be created singly, but only through spontaneous pair production which is energetically impossible unless $\epsilon_F < -m$. As has already been promised, we later show that the precise choice of ϵ_F is not important. For definiteness, however, we shall take $\epsilon_F = 0$.

Equation (17) will be the key equation in our investigation of the semiclassical contribution and QED correction. If we neglect the self-mass term, that is, the term involving $M(\mathbf{x},\mathbf{z};\epsilon_p)$, then from (18) we see that (17) reduces to just the semiclassical approximation. With the self-mass term included, the full QED correction is taken into account. In (19), $D_{\mu\nu}(x,y)$ is the (connected) two-point photon propagator, while $\Gamma^\mu(x,y,z)$ is the (irreducible) three-point vertex function. There are additional, coupled DS equations for these Green's functions. We shall give these equations when we come to estimating the contribution of the self-mass term.

B. Semiclassical part

We first neglect the self-mass term in (17) and estimate the semiclassical contribution. This part of the problem has been analyzed in Ref. [7] and so we shall be brief, presenting only the essential steps.

Without the self-mass term, we have the time-independent Dirac equation with semiclassical, spherically symmetric potential $V(r)$. Angular momentum is a good quantum number and we can choose, as basis for the wave functions [12 (Sec. 2.3.2)],

$$\phi_{n\chi m}(\mathbf{r}) := \begin{bmatrix} i\dfrac{G_{n\chi}(r)}{r}\psi_{\chi m}(\theta,\phi) \\ \dfrac{F_{n\chi}(r)}{r}\psi_{-\chi m}(\theta,\phi) \end{bmatrix} , \tag{20}$$

where n denotes the (yet to be determined) discrete and/or continuous radial quantum label, $\chi = \pm(j+\frac{1}{2})$ with $j = \frac{1}{2}, \frac{3}{2}, \ldots$ the total angular momentum quantum number, m is the J_z eigenvalue and $\psi_{\chi m}$ denotes a spinor harmonic. For this basis, the Dirac equation reduces to two coupled first-order differential equations in the radial functions F and G. Making the substitution $G = (m+\epsilon-V)^{1/2}A$, these two equations can be replaced by a single second-order Schrödinger-like equation for A:

$$A'' + 2(E-U)A = 0 , \tag{21}$$

where the effective energy E is given by

$$E := \tfrac{1}{2}(\epsilon^2 - m^2) , \tag{22}$$

and the effective potential U is

$$U := \left[\epsilon V - \frac{1}{2}V^2 + \frac{1}{2r^2}\chi(\chi+1)\right] + \left[-\frac{\chi}{2r}(m+\epsilon-V)^{-1}V' + \frac{3}{8}(m+\epsilon-V)^{-2}(V')^2 + \frac{1}{4}(m+\epsilon-V)^{-1}V''\right] . \tag{23}$$

We approximate U as

$$U \approx \left[\epsilon V - \frac{1}{2}V^2 + \frac{1}{2r^2}\chi(\chi+1)\right] . \tag{24}$$

Equation (21) then becomes just the radial equation for a scalar field and so we are neglecting the electron spin contribution. This approximation will be justified shortly.

Suppose the single-particle energy ϵ is in the range $-m < \epsilon < +m$. Then the effective energy E is negative and from condition (i) on V, the effective potential U is also negative for a large part of the region of the core, provided $|\chi|$ is not too large. Since $U(r) \to 0$ as $r \to \infty$, we therefore have the possibility of confined wave functions with ϵ taking discrete values. For single-particle energies in the range $|\epsilon| > m$, E is positive and therefore the wave functions are unconfined with ϵ continuous. Note, however, that since $V<0$, for $\epsilon < -m$ the effective potential has a barrier. Therefore, if for given ϵ and χ the barrier is wide enough (a precise criterion is given below), then the associated wave function can be approximated as being confined.

In terms of the single-particle wave functions, the electron number density is

$$n_e(r) = \frac{1}{2}\left[\sum_{\substack{n\chi m \\ (\epsilon_{n\chi}<0)}} \phi^\dagger_{n\chi m}(\mathbf{r})\phi_{n\chi m}(\mathbf{r}) - \sum_{\substack{n\chi m \\ (\epsilon_{n\chi}>0)}} \phi^\dagger_{n\chi m}(\mathbf{r})\phi_{n\chi m}(\mathbf{r})\right] , \tag{25}$$

where we have used (16) and the first term on the right-hand side of (18). Recall we have chosen $\epsilon_F = 0$. It is assumed that (25) is well approximated by

$$n_e(r) \approx \sum_{\substack{n\chi m \\ (\text{well confined})}} \phi^\dagger_{n\chi m}(\mathbf{r})\phi_{n\chi m}(\mathbf{r}) , \tag{26}$$

where the sum is only over wave functions which are confined to a good approximation and the wave functions are normalized to 1 in the region of approximate confinement. We are therefore neglecting the "virtual" vacuum polarization contribution to n_e (see previous section).

Let us now estimate the sum (26). Solving (21) using the WKB approximation (which we justify shortly), we obtain, for the radial functions,

$$G(r) = C\left[\frac{\epsilon - V(r) + m}{p(r)}\right]^{1/2}\sin\theta(r)$$
$$F(r) = C\left[\frac{\epsilon - V(r) - m}{p(r)}\right]^{1/2}\sin[\theta(r)+\eta(r)] , \tag{27}$$

where

$$p(r) = \{[\epsilon - V(r)]^2 - m^2 - \chi^2/r^2\}^{1/2} , \tag{28}$$

$$\theta(r) = \int_{r_1}^{r} p(r')dr' + \frac{\pi}{4} , \tag{29}$$

with r_1 the innermost turning point and

$$\eta(r)=\arcsin\{p(r)[(\epsilon-V(r))^2-m^2]^{-1/2}\} . \quad (30)$$

The transmission coefficient of the effective potential barrier for a given wave function is

$$\gamma(\epsilon,\chi)\approx\exp\left[-2\int_{r_2}^{r_3}|p(r)|dr\right] , \quad (31)$$

where r_2 and r_3 are the inner and outer turning points, respectively, of the barrier. If we have

$$\gamma(\epsilon,\chi)\ll 1 , \quad (32)$$

then the wave function can be approximated as being confined, and we have the following quantization and normalization conditions:

$$\int_{r_1}^{r_2} p_{n\chi}(r)dr=(n+\tfrac{1}{2})\pi , \quad (33)$$

$$C_{n\chi}=\left[\frac{1}{\pi}\frac{\partial\epsilon_{n\chi}}{\partial n}\right]^{1/2} . \quad (34)$$

Summing over m and replacing the oscillating functions $\sin^2\theta$ and $\sin^2(\theta+\eta)$ by their averages $(=\frac{1}{2})$, (26) becomes

$$n_e(r)\approx\sum_{n\chi}\frac{(2j+1)C_{n\chi}^2}{4\pi r^2 p_{n\chi}(r)}[\epsilon_{n\chi}-V(r)] . \quad (35)$$

If we fix χ and integrate ϵ [note from (34), we have $\sum_n C_{n\chi}^2\cdots\approx 1/\pi\int d\epsilon\cdots$] over the range $\epsilon_+(r)\le\epsilon\le\epsilon_F(=0)$, where $\epsilon_+(r)$ is the energy at which r coincides with the turning point r_2, we obtain

$$n_e(r)\approx\sum_\chi\frac{|\chi|}{2\pi^2r^2}\left[V(r)^2-\frac{\chi^2}{r^2}\right]^{1/2} \quad (36)$$

[where, because of condition (i), we have neglected the m^2 term compared with V^2]. Summing χ over the range $0\le|\chi|\le rV(r)$, not forgetting the factor 2 degeneracy $(\chi=\pm|\chi|)$, we finally obtain

$$n_e(r)\approx-\frac{1}{3\pi^2}[V(r)]^3 , \quad (37)$$

which is just result (5) with the full QED correction neglected.

Not all the wave functions satisfy (32) in the sums over χ and ϵ carried out above: for given χ, the wave functions throughout the range $\epsilon_+(r)\le\epsilon\le 0$ (with the exception of $\epsilon\approx 0$) become less confined the smaller $|\chi|$ is. However, for the given conditions on V, one typically finds that the contribution to n_e from wave functions violating condition (32) is small.

The condition for the WKB approximation to hold is

$$\frac{|(\epsilon-V)V'-\chi^2/r^3|}{|(\epsilon-V)^2-m^2-\chi^2/r^2|^{3/2}}\ll 1 . \quad (38)$$

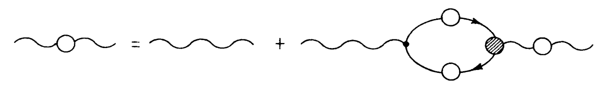
For the range of ϵ and χ giving the main contribution to n_e (see above), this condition becomes $|V'|/V^2\ll 1$ and therefore, from conditions (i) and (ii) on V, the WKB approximation gives a reasonable estimate.

Similarly, conditions (i) and (ii) ensure that the electron spin part is small compared to the first part of (23) for the range of ϵ and χ giving the main contribution to n_e.

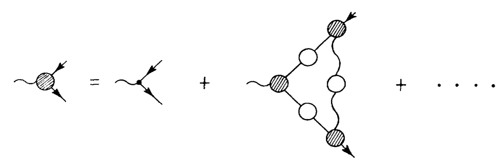
C. QED correction

We now estimate the QED correction. The stronger version of condition (ii) [i.e., $V(r)$ is approximately constant on the scale $\lambda(r)=|(e/\pi)V(r)|^{-1}$] is assumed throughout this subsection.

The Dyson-Schwinger equations for the photon and vertex Green's functions are given diagrammatically as [10,11]

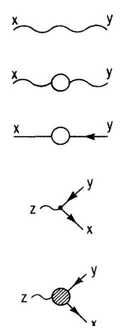


(39)



(40)

In (40), the sum is over irreducible vertex diagrams. The coordinate space Feynman notation is being employed here:



$\leftrightarrow\ iD^0_{\mu\nu}(x-y)$,

$\leftrightarrow\ iD_{\mu\nu}(x,y)$,

$\leftrightarrow\ -iG(x,y)$,

$\leftrightarrow\ -ie\gamma^\mu\delta(x-y)\delta(x-z)$,

$\leftrightarrow\ -ie\Gamma^\mu(x,y,z)$, (41)

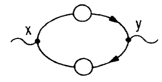
where

$$D^0_{\mu\nu}(x):=-\eta_{\mu\nu}\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot x}}{k^2+i\alpha}$$

is the free photon propagator.

From (39) and (40), we can obtain a series expansion of $D_{\mu\nu}$ and Γ^μ in terms of the free photon propagator $D^0_{\mu\nu}$, the bare vertex $e\gamma^\mu$ and the electron Green's function $G(x,y)$. Substituting these expansions into (19) gives us a corresponding expansion for the self-mass operator $M(x,z)$. Now, this series contains divergent diagrams (see below). We do assume, however, that the diagrams in the expansion can be grouped into classes and that each such class of diagrams can be formally summed to give a single finite contribution. Since we only require an estimate of the QED correction, we do not need to substitute into the self-mass term in (17) the sum of contributions from all the classes; the sum can be truncated after the first few terms.

To motivate the correct grouping of diagrams for the given conditions on V, consider the following polarization tensor diagram:



(42)

From (41) and (16), this is

$$\Delta^{\mu\nu}(\mathbf{x},\mathbf{y};l_0) = -e^2 \int d(t_x - t_y) \mathrm{tr}[\gamma^\mu G(x,y)\gamma^\nu G(y,x)] e^{il_0(t_x - t_y)}$$
$$= -i \left[\sum_{\substack{p,q \\ (\epsilon_p > 0, \epsilon_q < 0)}} [l_0 - (\epsilon_p - \epsilon_q) + i\alpha]^{-1} - \sum_{\substack{p,q \\ (\epsilon_p < 0, \epsilon_q > 0)}} [l_0 - (\epsilon_p - \epsilon_q) - i\alpha]^{-1} \right]$$
$$\times \{ \mathrm{tr}[\gamma^\mu \phi_p(\mathbf{x}) \bar{\phi}_p(\mathbf{y}) \gamma^\nu \phi_q(\mathbf{y}) \bar{\phi}_q(\mathbf{x})] \} \ . \tag{43}$$

For our purposes, it will be sufficient to determine the $\Delta^{00}(\mathbf{x},\mathbf{y};l_0)$ component using the semiclassical wave-function solutions ϕ_p, with $|\mathbf{x}-\mathbf{y}| = O(\lambda(s))$ and $l_0 \le O(\lambda^{-1}(s))$, where $\lambda(s) = [(e/\pi)V(s)]^{-1}$, $\mathbf{s} = \frac{1}{2}(\mathbf{x}+\mathbf{y})$. After some calculation, we find

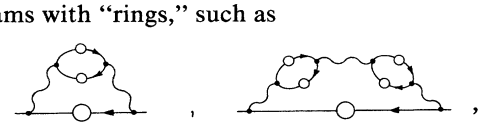
$$\Delta^{00}(\mathbf{x},\mathbf{y};l_0) \approx i\lambda^{-2}(s) \int \frac{d^3l}{(2\pi)^3} \left[1 + \frac{l_0}{2|\mathbf{l}|} \ln \left[\frac{l_0 - |\mathbf{l}|(1 - m^2/2V^2) + i\alpha}{l_0 + |\mathbf{l}|(1 - m^2/2V^2) - i\alpha} \frac{l_0 - i\alpha}{l_0 + i\alpha} \right] \right] e^{i\mathbf{l}\cdot(\mathbf{x}-\mathbf{y})} \ . \tag{44}$$

Very briefly, the derivation of (44) from (43) proceeds as follows: we assume that the main contribution to (43) comes from wave functions for which

$$|\epsilon_p - \epsilon_q| \le O(\lambda^{-1}(s)) \ .$$

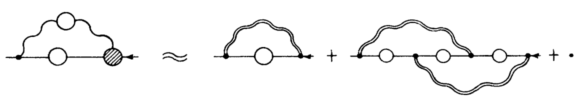
This, together with the fact that ϵ_p and ϵ_q are separated by the Fermi energy ($=0$) in the sums and also (strong) condition (ii) on V, imply condition (38) and we can therefore use the WKB approximation obtained above for the wave functions. The resulting sums over the radial and angular momentum quantum numbers n_p, n_q, χ_p, χ_q, m_p, and m_q can be simplified (by approximation) using extensively the fact that the main contribution to the sums comes from $|\chi_p|, |\chi_q| \gg 1$ with their difference $||\chi_p| - |\chi_q||$ at most an order of magnitude smaller. We can also show that (43) depends only on $V(r)$ in a neighborhood of $\mathbf{x}$ and $\mathbf{y}$ with dimensions $\lambda(s)$. From (strong) condition (ii) on V, (43) therefore depends to good approximation only on the difference $\mathbf{x}-\mathbf{y}$. We can change the above-mentioned simplified sums over the radial and angular momentum quantum numbers to integrals over linear momenta. One of the integrals can be readily carried out and (neglecting various terms small compared to $|V|$) we obtained (44).

The important property to note is that, in contrast with the corresponding *vacuum* polarization diagram, the integrand of (44) is nonzero for $l^2 (= l_0^2 - \mathbf{l}^2) \to 0$. Since $D^0_{\mu\nu}(l)$ behaves as $1/l^2$, we therefore expect self-mass diagrams with "rings," such as



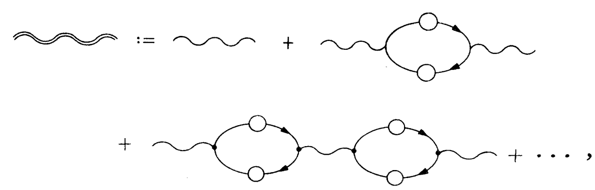
(45)

to be singular. Now, (44) coincides with the corresponding contribution to the polarization tensor for a static, uniform density, zero-temperature electron gas with Fermi energy $\epsilon_F = |V(s)|$, which is stabilized by an equal density of positive background charges [15]. One of the objectives of these references is to calculate the ground-state energy of an electron gas. This involves summing all "bubble" diagrams. It is found that a finite result is obtained by formally summing a class of bubble diagrams formed by taking a bubble with a single free photon propagator and replacing the free photon propagator with propagators involving any number of rings. We therefore assume the self-mass operator in (17) can be approximated as follows:



(46)

where



(47)

with each term in (46) denoting the finite contribution obtained by formally summing the class of diagrams determined by (47). [For example, the first class contains diagrams such as (45).]

Since we are in the ground state, the Hohenberg-Kohn theorems [16] imply that the self-mass operator $M(\mathbf{x},\mathbf{z};\epsilon_p)$ is a unique functional of the electron number density $n_e(r)$. Now, given the form of (44) for $l_0 = 0$, we expect the above formal sums to give rise to a "large-distance" cutoff: we suppose *$M(\mathbf{x},\mathbf{z};\epsilon_p)$ is short range and in fact approximately depends on $n_e(r)$ only in a neighborhood of $\mathbf{x}$ having dimensions $\lambda(|\mathbf{x}|)$*. (A partial demonstration of this can be obtained by straightforwardly generalizing to the relativistic case an argument given in [17] for the corresponding property of a nonrelativistic electron gas.) We shall approximate the self-mass term in (17) by replacing $M(\mathbf{x},\mathbf{z};\epsilon_p)$ with $M(\mathbf{x},\mathbf{z};\epsilon_F = 0)$ for all p. Applying (strong) condition (ii) on V and dimensional analysis, we have (neglecting dependence on m)

$$\gamma_0 \int d^3z \, M(\mathbf{x},\mathbf{z};\epsilon_p)\phi_p(\mathbf{z}) \approx n_e^{1/3}(|\mathbf{x}|) C(\alpha) \phi_p(\mathbf{x}) \ , \tag{48}$$

where $C(\alpha)$ is a dimensionless matrix. Assuming $C(\alpha)$ acts approximately diagonally, (17) is approximately the time-dependent Dirac equation with potential

$$\tilde{V}(r) := V(r) + n_e^{1/3}(r) C(\alpha) \ .$$

We can therefore use the same analysis as for the semiclassical part to obtain relation (5).

Let us now estimate $C(\alpha)$. From the above discussion, the wave functions can be approximated by solutions to the time-independent Dirac equation with *constant* potential

$$\tilde{V} := -[3\pi^2 n_e(|\mathbf{x}|)]^{1/3} .$$

Working with linear momentum, we have

$$\phi_p(\mathbf{z},t_z)=\frac{1}{(2\pi)^{3/2}}[2(\epsilon_p-\tilde{V})(m+\epsilon_p-\tilde{V})]^{-1/2}\exp[-i(\mathbf{p}\cdot\mathbf{z}-\epsilon_p t_z)](\not p+m)\begin{bmatrix}u\\0\\0\end{bmatrix}, \tag{49}$$

where $\epsilon_p=\pm(m^2+\mathbf{p}^2)^{1/2}+\tilde{V}$,

$p_0:=\epsilon_p-\tilde{V}$,

and $u:=\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}$.

Substituting (49) into (16) (with $\epsilon_F=0$), we obtain, for the electron Green's function,

$$G(z)=-\int\frac{d^4p}{(2\pi)^4}\frac{\not p+m}{p^2-m^2+i\alpha}e^{-ip\cdot z}e^{-i\tilde{V}t_z}-i\int_{|\mathbf{p}|<(\tilde{V}^2-m^2)^{1/2}}\frac{d^3p}{(2\pi)^3}\frac{\not p+m}{2E_p}e^{-ip\cdot z}e^{-i\tilde{V}t_z}, \tag{50}$$

where, in the second integral, $p_0\equiv E_p:=(\mathbf{p}^2+m^2)^{1/2}$. Keeping only the first diagram in (46), we have, for the self-mass term (at the Fermi level),

$$\gamma_0\int d^3z\, M(\mathbf{x},\mathbf{z};\epsilon_F=0)\phi_F(\mathbf{z})\approx\gamma_0\int d^3z\int d(t_x-t_z)[-ie^2\gamma^\mu G(x-z)\gamma^\nu D^{\text{ring}}_{\mu\nu}(z-x)]\phi_{\mathbf{p}_F}(\mathbf{z}), \tag{51}$$

where G is given by (50), $D^{\text{ring}}_{\mu\nu}$ by (47) and $\phi_{\mathbf{p}_F}(\mathbf{z})$ by (49) with $\epsilon_p=\epsilon_F=0$ and

$$|\mathbf{p}_F|=(\tilde{V}^2-m^2)^{1/2} .$$

The part of (51) involving the first integral in (50) diverges. We treat this by subtracting off an identical part with $D^{\text{ring}}_{\mu\nu}$ replaced by $D^0_{\mu\nu}$. This ensures that (51) vanishes when $|\tilde{V}|\to m$ and hence $n_e\to 0$. Estimating the order α term, we replace $D^{\text{ring}}_{\mu\nu}$ by $D^0_{\mu\nu}$ in (51) and therefore the renormalized first part does not contribute, while the part of (51) involving the second integral in (50) gives approximately (after extensive use of the condition $|\tilde{V}|\gg m$)

$$-\left[\frac{\alpha\tilde{V}}{2\pi}\right]\phi_{\mathbf{p}_F}(\mathbf{x})=\left[\frac{1}{2}\left[\frac{3}{\pi}\right]^{1/3}\alpha n_e^{1/3}(|\mathbf{x}|)\right]\phi_{\mathbf{p}_F}(\mathbf{x}) . \tag{52}$$

Thus, to order α,

$$C(\alpha)\approx\frac{1}{2}\left[\frac{3}{\pi}\right]^{1/3}\alpha . \tag{53}$$

This disagrees with the result obtained by Migdal, Popov, and Voskresenskii [7] by a factor of $-\frac{1}{2}$. Their estimate can be obtained by repeating the above calculation with $0<|\tilde{V}|-m\ll m$. This is clearly incorrect, given the actual condition on $\tilde{V}$. Migdal, Popov, and Voskresenskii arrived at their estimate by using the (less direct) variational method [16] to obtain the self-energy from the nonrelativistic ground-state energy. If, however, the ultrarelativistic expression for the ground-state energy is used, then the variational method gives the same result as ours (cf., e.g., MacDonald and Vosko [16]). Applying the variational method to obtain the next order correction to (53), we find

$$C(\alpha)\approx\frac{1}{2}\left[\frac{3}{\pi}\right]^{1/3}\alpha-\frac{1}{2\pi}\left[\frac{3}{\pi}\right]^{1/3}\alpha^2\ln\alpha , \tag{54}$$

where the order $\alpha^2\ln\alpha$ ground-state energy correction can be found in, e.g., Ref. [15] [Akhiezer and Peletminskii Eq. (32)]. An estimate of the order α^2 correction to (54) would require consideration of both the first and second diagrams in (46).

Finally, how are the above obtained results modified if we choose some other bound state $\epsilon_F\neq 0$? The key observation is that we must have $|\epsilon_F|\leq m$ (see earlier this section). This, together with condition (i) on V means that $|\epsilon_F|\ll|V|$ and therefore, in the derivation of the semiclassical contribution, we still obtain (37) to good approximation. For the same reasons, the QED correction in relation (5) remains a good approximation.

IV. CONCLUSION

We considered the process involving the breakup of a neutral spherical object into a positively charged inner core and negatively charged expanding outer shell, resulting in a substantial amount of pair creation. We addressed the question of validity of the semiclassical approximation in the region of the core at late times, when pair creation has ceased to good approximation, and argued that the semiclassical description is a good one when certain conditions on the vector potential hold [see conditions (i)–(iii), Sec. II].

The next problem which must be addressed is the validity of the semiclassical approximation during the dynamical stage. This will, in particular, involve determining how rapidly the outer shell can expand. If the expansion is slow enough such that to good approximation the system evolves through a succession of vacuum states and, in addition, conditions (ii) and (iii) hold when (i) does, then the present results guarantee that the semiclassical approximation will be a good one throughout the whole process. [At early times, when (i) does not hold, we can neglect the produced electron contribution: this part of the process is well approximated classically.] Thus, we have an example of a dynamical process which is well-described semiclassically and which can have a non-negligible backreaction. [Note that with condition (i), the vacuum states are highly negatively charged and hence there is a backreaction.] However, we are really interested in the validity of the semiclassical approximation in less trivial situations. A process more analogous to black-hole evaporation will be to have the shell expand just fast enough so that most of the pair production occurs subsequent to the expansion and at a low rate over a long period of time. Addressing the validity of the semiclassical approximation throughout this process will only require that we obtain rough estimates for the semiclassical and full QED descriptions. The simple form of the late time estimates [see Eq. (5)] gives us some hope that, at least for some range of evolution rates, the problem will be tractable.

If the expansion of the shell is so rapid as to cause large amounts of pair production immediately following expansion, then we might expect the semiclassical approximation to hold only for a short time. We should also try to obtain estimates for these time scales.

The Feynman Green's function Dyson-Schwinger equations proved a useful framework within which to investigate the full QED and semiclassical descriptions of the late time (equivalently static) stage. When there are dynamical effects such as pair production, we must work with the expectation value Green's functions. There are also a corresponding set of Dyson-Schwinger equations for these Green's functions [10 (Sec. 3.3)]. The expectation value Dyson-Schwinger equations are therefore an obvious starting point in an investigation of the validity of the semiclassical approximation during the dynamical stage.

ACKNOWLEDGMENTS

I thank Bob Geroch and Bob Wald for many helpful conversations. Helpful discussions with Jonathan Simon are also acknowledged. I am especially grateful to Bob Wald for introducing the problem to me and for providing constant support.

[1] R. M. Wald, in *Gravitation and Quantizations,* Proceedings of the Les Houches Summer School, Les Houches, France, 1992, edited by B. Julia and J. Zinn-Justin, Les Houches Summer School Proceedings Vol. LVII (North-Holland, Amsterdam, 1993).

[2] J. Z. Simon, Phys. Rev. D **41**, 3720 (1990).

[3] S. Randjbar-Daemi, B. S. Kay, and T. W. B. Kibble, Phys. Lett. **91B**, 417 (1980); F. Cooper and E. Mottola, Phys. Rev. D **40**, 456 (1989).

[4] C. G. Callan, S. B. Giddings, J. A. Harvey, and A. Strominger, Phys. Rev. D **45**, 1005 (1992).

[5] A. Peet, L. Susskind, and L. Thorlacius, Phys. Rev. D **46**, 3435 (1992).

[6] J. G. Valatin, Proc. R. Soc. London **A222**, 93 (1954).

[7] A. B. Migdal, V. S. Popov, and D. N. Voskresenskii, Zh. Eksp. Teor. Phys. **72**, 834 (1977) [Sov. Phys. JETP **45**, 436 (1977)].

[8] B. Müller and J. Rafelski, Phys. Rev. Lett. **34**, 349 (1975).

[9] A. B. Migdal, Nucl. Phys. **B52**, 483 (1973).

[10] E. S. Fradkin, D. M. Gitman, and S. M. Shvartsman, *Quantum Electrodynamics with Unstable Vacuum* (Springer, Berlin, 1991).

[11] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Wiley, New York, 1959).

[12] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1985).

[13] P.-G. Reinhard, W. Greiner, and H. Arenhövel, Nucl. Phys. **A166**, 173 (1971); L. Gomberoff and V. Tolmachev, Phys. Rev. D **3**, 1796 (1971).

[14] W. Greiner, B. Müller, and J. Rafelski, *Quantum Electrodynamics of Strong Fields* (Springer, Berlin, 1985).

[15] B. A. Freedman and L. D. McLerran, Phys. Rev. D **16**, 1147 (1977); S. A. Chin, Ann. Phys. (N.Y.) **108**, 301 (1977); B. Jancovici, Nuovo Cimento **25**, 428 (1962); I. A. Akhiezer and S. V. Peletminskii, Zh. Eksp. Teor. Phys. **38**, 1829 (1960) [Sov. Phys. JETP **11**, 1316 (1960)].

[16] P. Hohenberg and W. Kohn, Phys. Rev. **136**, 864 (1964); A. H. MacDonald and S. H. Vosko, J. Phys. C **12**, 2977 (1979).

[17] L. J. Sham and W. Kohn, Phys. Rev. **145**, 561 (1966).