Coalescing binary systems of compact objects to (post)$^{5/2}$-Newtonian order. IV. The gravitational wave tail

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The contribution to gravitational radiation from the gravitational wave "tail" is studied using an expression previously obtained by Blanchet and Damour. The expression for the tail of the radiation contains integrals over the past history of the system, thus exhibiting the backscattering nature of radiation in curved spacetime; however the dependence on the remote past of the system is shown to be weak for relevant astrophysical systems. In the cases of circular orbits and high-speed, low-deflection (bremsstrahlung) encounters, the integrals can be evaluated analytically. For circular orbits the frequency of the tail radiation is twice the orbital frequency, just as for the quadrupole radiation, but is phase shifted from it. Using explicit two-body multipoles of the radiation we have published elsewhere, along with the tail terms developed here, we present gravitational waveforms which are accurate to (post)$^{3/2}$-Newtonian order [i.e., $O((Gm/rc^2)^{3/2}) = O((v/c)^3)$ beyond the usual quadrupole radiation] for a coalescing binary system of compact objects in a nearly circular orbit. For the case of two orbiting neutron stars very near coalescence we estimate the tail contribution to the waveform to be roughly half the amplitude of the usual quadrupole radiation. We also compute the correction to the radiation energy flux produced by the tail radiation. We show that this results in an increased rate of decay for a binary in a circular orbit. We show that the same orbital decay rate can also be obtained directly from the tail-transported part of the near-zone radiation reaction force. In the Appendix we give a heuristic, "physical" explanation of the behavior of the tail radiation.

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I. INTRODUCTION

General relativity is a nonlinear gravitational theory, but because of the inherently weak nature of gravity in such places as the solar system, most experimental tests of gravitational theory do not probe or only mildly probe the nonlinear aspects of the theory. However, this situation is likely to change when laser interferometric gravitational wave detectors that are currently under development are operational [1]. The detectors will ultimately observe gravitational waves emitted from highly nonlinear, dynamical spacetime regions, such as near the coalescence of two compact objects such as black holes or neutron stars. In this paper we explore some theoretical and observational aspects of a particular nonlinear phenomenon in general relativity, the gravitational wave tail, which may have important observational consequences in gravitational wave astronomy.

To date, our best laboratory for strong field gravity and gravitational waves has been the binary pulsar PSR 1913+16 (and other similar systems). Indeed the changing orbital period of this system has given excellent confirmation of the "quadrupole" energy loss formula [2], and thereby given good (albeit indirect) evidence that gravitational waves exist. In spite of the high precision of the observations of this system they do not require the *full* nonlinearity of general relativity for their explanation. The quadrupole energy loss formula, and thus the orbital decay and orbital period decrease, can be computed from the lowest-order multipole contribution to the gravitational radiation [3]. Although nonlinearities are encountered in this low-order calculation, they are relatively benign. For example, the lowest-order contributions (and even the first post-Newtonian correction term) to each of the radiative multipoles are very "electromagnetic" in nature: (1) As in the electromagnetic case, they can be expressed as functions of only the retarded time; (2) they can be expressed as integrals over only the *material* source (just as electromagnetic multipoles can be obtained from integrals over the charge distribution), if the formalism of Blanchet, Damour, and Iyer [4–9] is used. (This is in contrast with the formalism of Epstein and Wagoner [10] which does require integrals over the infinite extent of the external fields even at lowest order.) However, at higher order these two statements no longer hold true. It is the failure of the first statement that we wish to explore here in detail. The failure of the second point is illustrated by the effect recently discussed by Christodoulou [11], where the outgoing gravitational radiation itself serves as source for the gravitational radiation. (See Wiseman and Will [12] and Thorne [13] for discussion.)

The fact that the radiative multipoles ultimately depend not only on the motion of the source at the retarded time, but also on the integrated past history of the source, has a simple physical interpretation [14]. As the radiation propagates outward from the source, it scatters off the background curvature of the spacetime. In other

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words, the radiation interacts with the monopole field of the source. Thus there is a component of the radiation that observers see which did not propagate directly to them; rather it left the source prior to the retarded time.

This "tail" contribution to the gravitational radiation, while formally a higher-order effect, could have significant observational consequences. For example, Fig. 1 shows the contributions to the gravitational radiation from the usual quadrupole radiation and the contribution from the tail of the radiation for a binary system of two $1.4M_\odot$ neutron stars very near coalescence. Although the tail radiation is very weak well prior to the coalescence, it is roughly equal to (half of) the amplitude of the quadrupole radiation at separations of $r \approx 6m\ (9m)$, where m is the total mass (i.e., at separations very near coalescence). This illustrates the *possibility* that a Laser Interferometric Gravitational Wave Observatory-(LIGO-)type detector may be able to detect the tail radiation or, conversely, that tail effects may have to be considered in order to interpret observed waveforms of binaries near coalescence. Also notice that the tail radiation has basically the same frequency as the quadrupole radiation; however, it lags in phase by approximately 61.1°. The tail radiation also carries energy away from the system, and thus it accelerates the orbital decay. This slow, but secular, change in the orbit will have a large cumulative effect on the evolution of the gravitational wave phase over the many orbits of the decay. This effect has been explored by Cutler *et al.* [16].

Figure 2 shows the quadrupole and tail contributions to the waveform during a high-speed, low-deflection encounter confined to the xy plane. This is the gravitational analogue of bremsstrahlung. Note that although the $h^{xy}_{\rm quad}$ component of the quadrupole radiation exhibits linear "memory" (i.e., it has different and persistent values before and after the encounter) the tail correction to radiation has no memory. The absence of a tail contribution to the gravitational wave memory is briefly discussed in the Appendix.

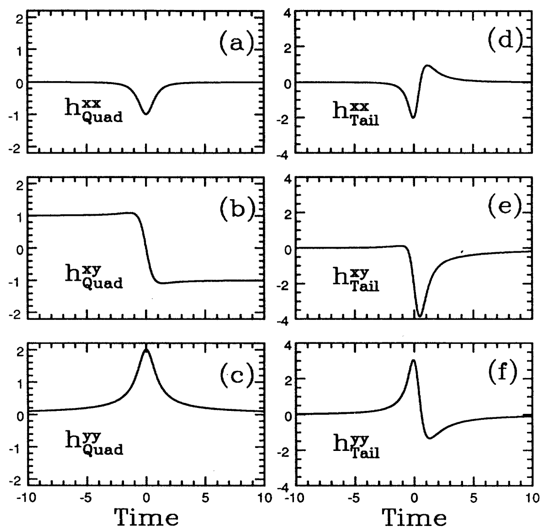

FIG. 2. Gravitational waveform emitted during a high-speed, low-deflection encounter plotted against $(v_\infty/b)T$. (a)–(c) The quadrupole contribution. Plotted is $\frac{R}{2\mu}(b/m)h^{ij}_{\rm quad}$. (d)–(f) The tail contribution. Plotted is $\frac{R}{2\mu}\left[\frac{4v_\infty^3(m/b)}{e}\right]^{-1} h^{ij}_{\rm tail}$.

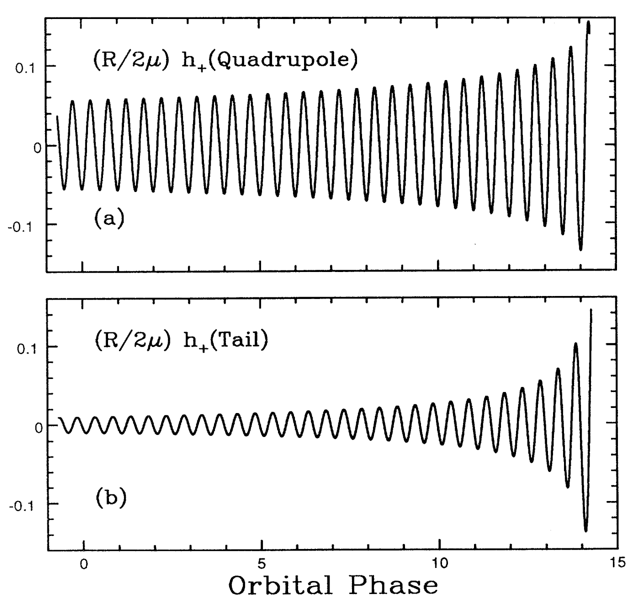

FIG. 1. Gravitational waveform plotted against orbital phase. Plotted is $(R/2\mu)h_+$ with $m_1 = m_2$. The portion of the waveform shown is for the orbital decay from $r \approx 18m$ on the left to $r \approx 6m$ on the right. The position of the observer is in the equatorial plane.

In the remainder of this paper we show the details of the calculations. In large part this paper examines a number of very general results of Blanchet and Damour [6] and shows how they apply to very specific, potentially observable, astrophysical events. In Sec. II we discuss a general formula for the gravitational wave tail, which has previously been given by Blanchet and Damour [6]. In Sec. III we examine the tail portion of the radiation for the case of a high-speed, low-deflection encounter of two compact stars. In Sec. IV we examine the tail of the radiation for the case of binary stars in nearly circular orbits. We also show how the energy carried off by the tail of the radiation (the tail contribution to the luminosity) affects the orbital decay rate. In Sec. V we show that the orbital decay rate obtained in Sec. IV can also be obtained from the near-zone, tail-transported, radiation reaction force. This is done without appealing to far-zone energy balance arguments. In the Appendix we discuss several qualitative features of the tail radiation.

II. GENERAL TAIL FORMULAS

We start with the standard symmetric trace-free (STF) multipolar decomposition of the gravitational radiation field [17]:

$$h^{ij} = \frac{2}{R}\Bigg[\overset{(2)}{I^{ij}} + \frac{1}{3} \overset{(3)}{I^{ijk}}\; n^k + \frac{1}{12} \overset{(4)}{I^{ijkl}}\; n^k n^l$$

$$+ \frac{1}{60} \overset{(5)}{I^{ijklm}}\; n^k n^l n^m \cdots$$

$$+ \epsilon^{kl(i}\Bigg(+ \frac{4}{3} \overset{(2)}{J^{j)k}}\; n^l + \frac{1}{2} \overset{(3)}{J^{j)km}}\; n^l n^m$$

$$+ \frac{2}{15} \overset{(4)}{J^{j)kmn}}\; n^l n^m n^n \cdots \Bigg)\Bigg]_{\rm TT} \; . \tag{2.1}$$

Here R is the distance from the source to the observer, the $I^{ij\cdots}$ represent the radiative mass multipoles of the radiation field, the $J^{ij\cdots}$ represent the radiative current multipoles, the number over each multipole represents the order of time differentiation, and the subscript TT denotes that only the transverse traceless projection is to be taken. See Refs. [17–19] for other conventions and notation. Each of the successive sets of terms in Eq. (2.1) is $O(v/c)$ smaller than the previous set of terms; i.e., the mass octopole $\overset{(3)}{I^{ijk}}$ and the current quadrupole $\overset{(2)}{J^{ij}}$ are $O(v/c) = O((Gm/rc^2)^{1/2})$ smaller than the mass quadrupole term, the mass hexadecapole $\overset{(3)}{I^{ijkl}}$ and the current octopole $\overset{(3)}{J^{ijk}}$ are $O((v/c)^2)$ smaller, and so on. Terms grouped sequentially are of the same "order" as each other. In general the leading order contribution as well as the first post-Newtonian correction to each of the mass and current multipoles can be computed using the formalism of Blanchet, Damour, and Iyer [4–9]. (Unfortunately there is no compact formula for the first post-Newtonian corrections to the current multipoles of octopole and higher degree.) Explicit *two-body* formulas for the multipoles through $O((v/c)^3)$ beyond the lowest quadrupole order, comprising all the terms displayed in Eq. (2.1) (modulo the tail contribution, which we are about to treat), have been extensively studied in Ref. [20].

The contribution to Eq. (2.1) that we will study in this paper is the tail correction to the radiative quadrupole moment in Eq. (2.1). In other words, we will study a relativistic correction to I^{ij} that enters the waveform at the same order as the radiation from the multipoles I^{ijklm} and J^{ijkl} [i.e., $O((v/c)^3)$ beyond the lowest-order quadrupole radiation]. Thus, the inclusion of this tail term with the multipoles of the radiation we have examined in Ref. [20] gives a complete two-body description of the radiation through $O((v/c)^3)$ beyond the lowest-order quadrupole radiation.

Taking Eq. (3.4a) from Blanchet and Damour [6] we have the contribution to the radiation from the mass quadrupole term in Eq. (2.1):

$$h^{ij}_{I^{ij}}(T) = \frac{2}{R}\Bigg[\frac{d^2}{dT_{\rm ret}^2}\left(\int x^i x^j \sigma(\mathbf{x}, T_{\rm ret}) d^3x\right)$$

$$+ \frac{1}{14}\frac{d^4}{dT_{\rm ret}^4}\int x^i x^j \mid \mathbf{x}^2 \mid \sigma(\mathbf{x}, T_{\rm ret}) d^3x$$

$$- \frac{20}{21}\frac{d^3}{d^3T_{\rm ret}}\left(\int x^i x^j x^k \sigma_k(\mathbf{x}, T) d^3x\right)\Bigg]_{\rm TT}$$

$$+ h^{ij}_{\rm tail} \, , \tag{2.2a}$$

where

$$h^{ij}_{\rm tail}(T) = \frac{4m}{R}\int_0^\infty \overset{(4)}{Q^{ij}}\, (T_{\rm ret} - u)\left(\ln(u/2s) + \frac{11}{12}\right) du_{\rm TT} \tag{2.2b}$$

and the $T_{\rm ret} = T - R$. The fundamental feature of the tail of radiation is immediately apparent from Eq. (2.2b); it depends directly on the integrated past history of the system. Also note the explicit appearance of the system's total mass m indicating the interaction with the monopole gravitational field of the source. The scale parameter s and the definition of the radiative coordinate time T in Eq. (2.2b) are the subject of discussion below. Expressions for the sources σ and σ_k that enter Eq. (2.2a) are given in Refs. [4–9]. The lowest-order contribution to the first term in Eq. (2.2a) gives the usual quadrupole contribution to the radiation,

$$h^{ij}_{\rm quad} = \frac{2}{R}\frac{d^2}{dT^2} Q^{ij}_{\rm TT}, \tag{2.3a}$$

where

$$Q^{ij} = \int \rho y^i y^j d^3y = \mu x^i x^j \; . \tag{2.3b}$$

μ is the reduced mass of the binary system, and x^i represents components of the relative position of body 1 relative to body 2. (We do not take the traceless part of Q^{ij}; rather, it is to be understood that in the end we take the transverse traceless projection of h^{ij} to get the waveform.)

We now turn to several technical issues relating to Eq. (2.2b): (1) Does the integral in Eq. (2.2b) converge? (2) How sensitive is Eq. (2.2b) to the remote past of the system? (3) What is the meaning of the "arbitrary" scale factor s? Is the waveform independent of s? Although our answers to these questions are somewhat general, the emphasis will be on answers that pertain to the two astrophysical situations that we are considering: gravitational bremsstrahlung and coalescing binaries in decaying circularized orbits.

The logarithm is integrable as $u \to 0$ since $\overset{(4)}{Q^{ij}}$ is well behaved at $u = 0$; however, there is a more serious convergence problem as $u \to \infty$. In the case of gravitational bremsstrahlung, the term $\overset{(4)}{Q^{ij}}$ goes to zero as $1/u^n$, where n = 2, 3, or 5 depending on which component of the quadrupole we are considering. [See Eq. (3.3) below.] This is sufficient to guarantee that the integrand vanishes as $u \to \infty$ sufficiently rapidly that the integral converges. For the case of bound orbits, Walker and Will [21] have shown that all such orbits were unbound hyperbolic orbits (or parabolic) in the infinite past. The integral will have the same behavior as discussed above for the bremsstrahlung case, and thus there is no convergence problem. (The special case of asymptotically parabolic orbits converge slightly more slowly.)

Now that we have shown that the tail integral is mathematically well defined for the astrophysical situations we are considering, we now address whether or not the integral is unreasonably sensitive to the remote past (ancient history) of the system (i.e., as $u \to \infty$). "Physical intuition" leads us to believe that it is not; however, it is unclear how to reconcile the physical intuition with the diverging logarithm in Eq. (2.2b). If the tail integral is thought of as describing the radiation that took an indirect path to the observer (i.e., scattered off the background curvature before arriving), it is hard to see how this could sensitively depend on the ancient history of the system. Any radiation that left the source long before the direct radiation and yet arrived at the distant observer at the same time as the direct radiation would have had to propagate well away from the source before being scattered back to the observer. However, the monopole field of the source (the curvature) falls off rapidly ($\approx 1/(\text{distance})^3$) as we move away from the source. Therefore, in order for Eq. (2.2b) to have a strong sensitivity to the ancient history of the system there would have to be substantial scattering even in the regions of very weak curvature. This seems unlikely.

To show that the physical intuition about the sensitivity to the ancient history of the system matches our mathematical expression Eq. (2.2b) we invoke the argument of [6]. We split up the region of integration by $\int_0^\infty = \int_0^\tau + \int_\tau^\infty$, and then integrate Eq. (2.2b) by parts twice to obtain

$$\frac{R}{2\mu} h^{ij}_{\text{tail}}(T) = 2\frac{m}{\mu}\Bigg[[11/12 - \ln(2s/m)] \overset{(3)}{Q^{ij}} (T_{\text{ret}}) + \ln(\tau/m) \overset{(3)}{Q^{ij}} (T_{\text{ret}} - \tau) + (1/\tau) \overset{(2)}{Q^{ij}} (T_{\text{ret}} - \tau) + \int_0^\tau \overset{(4)}{Q^{ij}} (T_{\text{ret}} - u) \ln(u/m) du - \int_\tau^\infty \overset{(2)}{Q^{ij}} (T_{\text{ret}} - u)(1/u^2) du \Bigg]_{\text{TT}} . \qquad (2.4)$$

Note that this formula is valid for any value of τ, including the limit as $\tau \to 0$. [In the limit all the divergent quantities in Eq. (2.4) identically cancel.] Also note that the remote past history of the system enters Eq. (2.4) only through the last integral. Thus, we often use τ as the parameter which divides the "recent history" of the system from the "ancient history" of the system.

If we assume that the system has never been appreciably more radiative in the past than it is now (i.e., $| \overset{(2)}{Q^{ij}} |$ is bounded; a criterion is certainly satisfied by the asymptotic arguments of Walker and Will [21] stated above) the last integral in Eq. (2.4) falls of as $1/\tau$. When we treat the very specific case of circular orbits in Sec. IV the oscillatory behavior of $\overset{(2)}{Q^{ij}}$ rapidly destroys the contribution from the last integral in Eq. (2.4). The contribution from the remote past falls off like $(1/\tau^2)$.

In spite of the appearance of Eq. (2.2b) we will show that the total waveform is nearly independent of the scale factor s that appears in Eqs. (2.2) and (2.4). To do this it is necessary to realize that s also enters the definition of the radiative coordinate retarded time T_{ret}:

$$T_{\text{ret}} = T - R = t - R - 2m \ln(R/s) , \qquad (2.5)$$

where t is the harmonic coordinate time. (Note that the scale parameter s is equivalent to P^{rad} in Ref. [6]; here we reserve P to denote the true period of the orbital system.) The scale parameter enters Eq. (2.5) in the following way. In the near zone the radiation generation problem has been solved in harmonic coordinates with flat and rigid light cones $t - R$ =const, where t is the harmonic coordinate time. However, we also know that in the far zone the signal propagates to the observer along logarithmically corrected light cones, $t - R - 2m\ln(R)$ =const. The logarithmic term corresponds simply to the Shapiro time delay of a signal propagating radially in the monopole field of the source. (For a discussion of the "Shapiro time delay" in the context of solar system tests of general relativity, see Ref. [3].) We then match these two light cones in the intermediate zone, i.e., a distance s from the source, thus fixing the constant to be $2m\ln(s)$, and resulting in the radiative retarded time of Eq. (2.5). There is also a portion of the tail radiation in Eq. (2.2b) which depends on the location of this matching point, leading to the s dependence in Eq. (2.2b). The two effects precisely offset each other. [In the language of Kovacs and Thorne [22] this s-dependent portion of the radiation in Eq. (2.2b) is called the "transition" radiation, which has its origin in the time-changing Shapiro time delay near the source (i.e., within a distance s from the source).]

The explanation given by Blanchet and Damour [8] for this logarithmic term in Eq. (2.5) is to transform to a "coordinate system in which the metric admits an expansion at infinity in inverse powers of the radial distance (without logarithms)." This seems to play the same role as the "truncation" process described by Thorne and Kovacs [15] (also Kovacs and Thorne [22] and Crowley and Thorne [23]). Recognizing that the last term in Eq. (2.5) is the Shapiro time delay [3] experienced by a signal traveling from a coordinate distance s from the monopole mass m to the distant observer at R, we see that Eq. (2.5) simply redefines the retarded time by subtracting off the bulk of the (logarithmically divergent) Shapiro time delay. Crowley and Thorne (Ref. [23], Sec. II) have noted that the failure to perform this truncation process leads to solutions which are ill defined at large coordinate distances from the source. (See particularly their comparison of the Thorne-Kovacs Green's function with the Green's function of DeWitt and DeWitt.)

The prescription given by Kovacs and Thorne [22] for actually choosing a location for the "truncation" is slightly different than the prescription given by Blanchet and Damour [8] for choosing a value for the scale s. However, we show that in practice they yield essentially the same choice. Blanchet and Damour identify s as a "characteristic period of the system." In the case of a gravitational bremsstrahlung with impact parameter b and speed v_∞ this would correspond to $s = b/v_\infty$. Kovacs and Thorne [22] suggest keeping only the portion of the delay which occurs well outside the source—the contribution from outside a radius of $\approx 10b$; thus they choose $s = 10b$. If we are considering a scattering event with a velocity of $0.1c$ (i.e., slow enough that our post-Newtonian approximation is certainly valid) these two definitions are essentially the same. We also add that the 11/12 term in Eq. (2.4) seems to have no counterpart in the Green's function construction of Thorne and Kovacs.

To demonstrate that the computed waveform is actually independent of the parameter s, we explicitly show the s dependence in the radiative coordinate time in the waveform quadrupole term Eq. (2.3a) and the s-dependent tail term in Eq. (2.4):

$$h^{ij}[T(s)] = \frac{2}{R}\left[\overset{(2)}{Q^{ij}} [t - R - 2m\ln(R/s)] - 2m\ln(2s/m) \overset{(3)}{Q^{ij}} [t - R - 2m\ln(R/s)]\right]_{\rm TT} + (\text{other terms}) . \tag{2.6}$$

If we now consider a change in s, $s \to s + \Delta s$ (not necessarily a small change in s, say, $s \to 2s$) and expand each of the terms and discard higher-order terms (e.g., $\overset{(4)}{Q}$ terms) we obtain $h^{ij}[T(s+\Delta s)] = h^{ij}[T(s)]$. Thus, h^{ij} is actually independent of s for even fairly large changes in s. In order to justify the use of the Taylor expansion in the steps above we must show that the time increment $\Delta t = 2m\ln(1 + \Delta s/s)$ is small compared to the time scales of the system (i.e., the period). Specifically, for the extreme case of a coalescing binary in circularized orbit very near coalescence $r \approx 9m$ and $s \to 2s$, we have

$$\frac{2m\ln(1 + \Delta s/s)}{\text{period}} \approx 0.01 . \tag{2.7}$$

A similar argument can be made for the bremsstrahlung case if we take the "period" to be b/v_∞.

The final point we make concerning the scale parameter s (and equivalently the 11/12 term) in Eq. (2.2b) is that it has essentially no effect on the energy radiated from a nearly periodic source. It enters the flux at lowest order as a cross term with the lowest-order quadrupole radiation, schematically,

$$\frac{dE}{dt} \sim [11/12 - \ln(2s/m)] \overset{(3)}{Q}\overset{(4)}{Q} . \tag{2.8}$$

This is a perfect derivative and will average to zero over one period of the system (i.e., one orbit). Thus for the very important case of a decaying circular orbit the subtleties in the definition of s will have no effect on the decay rate orbit (at least at the order we are considering).

III. BREMSSTRAHLUNG TAIL RADIATION

In this section we derive the tail portion of the waveform produced during a high-speed, low-deflection encounter of two compact objects (i.e., gravitational bremsstrahlung). Although it is true that from an observational point of view in-spiraling binary systems offer the best opportunity for actual detection of gravitational radiation (LIGO-type detectors are in effect "tuned" for coalescing binaries), from a theoretical point of view, high-speed, low-deflection encounters of stars offer a much "cleaner" system for examining effects such as gravitational wave memory. At present there is no complete solution to the coalescing binary problem starting from the point where the stars are well separated and emitting very little radiation, through the hydrodynamic coalescence, to the point where the resulting object is again nearly quiescent. Thus, it is impossible to compare the value of the waveform h^{ij} well before the coalescence with h^{ij} well after the coalescence. In contrast, in the case of gravitational bremsstrahlung it is a relatively simple matter to evolve the hyperbolic orbit from a time when the stars are well separated and there is essentially no radiation being emitted, through the "encounter," and back to large separation. Hence, a meaningful comparison of the waveform before and after the encounter can be made. Therefore, in order to see "memory" effects (or the absence of memory effects) in the tail of the radiation we examine the case of gravitational bremsstrahlung.

In this discussion we consider stars on a hyperbolic trajectory with large eccentricity (i.e., we keep only terms to first order in $1/e$). The trajectory in the xy plane is easily obtained, from, e.g., Wagoner and Will [25],

$$\mathbf{x}(T) = \left[-\frac{v_\infty}{e}(t_b^2 + T^2)^{1/2} + b\right]\hat{\mathbf{x}} + \left[v_\infty T + \frac{b}{e}\text{arcsinh}(T/t_b)\right]\hat{\mathbf{y}}, \tag{3.1}$$

where b is the impact parameter, e is the eccentricity, and t_b is the time scale of the encounter, $t_b = b/v_\infty$. The various parameters are related by

$$\frac{v_\infty^2}{e} = \frac{m}{b}, \tag{3.2}$$

where it is assumed that $1/e \ll 1$.

The lowest-order quadrupole radiation can be constructed from Eq. (2.3):

$$\frac{R}{2\mu} h^{xx}_{\rm quad} = -2\frac{m}{b}\left[\frac{1}{[1 + (T/t_b)^2]^{3/2}}\right] \to -2\frac{m}{b}\left(\frac{t_b}{T}\right)^3, \tag{3.3a}$$

$$\frac{R}{2\mu}h^{xy}_{\rm quad} = 2\frac{m}{b}\left[\frac{-2(T/t_b)-(T/t_b)^3}{[1+(T/t_b)^2]^{3/2}}\right] \to \mp 2\frac{m}{b}\,, \tag{3.3b}$$

$$\frac{R}{2\mu}h^{yy}_{\rm quad} = 2\frac{m}{b}\left[\frac{2+(T/t_b)^2}{[1+(T/t_b)^2]^{3/2}}\right] \to 2\frac{m}{b}\left(\frac{t_b}{T}\right)\,. \tag{3.3c}$$

On the right we show the asymptotic time dependence of $h^{ij}_{\rm quad}$ as $T \to \pm\infty$. In Eq. (3.3c) we have omitted a large constant, nonradiative, unmeasurable contribution to $h^{yy}_{\rm quad}$. This constant term makes no contribution to the tail integral Eq. (2.2b).

The values of $h^{xx}_{\rm quad}$, $h^{yy}_{\rm quad}$, and $h^{xy}_{\rm quad}$ are plotted in Fig. 2. Figure 2(b) clearly shows the linear contribution to the "memory" of the burst (i.e., h^{xy} does not return to its original value after the encounter).

The tail portion of the radiation can be constructed by substituting Eq. (3.1) into Eq. (2.3b) and then into Eq. (2.4). Integrating and taking limits as $\tau \to 0$ we obtain

$$\frac{R}{2\mu}h^{xx}_{\rm tail} = \frac{4v^3_\infty(m/b)}{e}\frac{[L(T_b)(3T_b)-(T_b^2-2)(1+T_b^2)^{1/2}+T_b^3-4T_b]}{(1+T_b^2)^{5/2}}, \tag{3.4a}$$

$$\frac{R}{2\mu}h^{xy}_{\rm tail} = \frac{4v^3_\infty(m/b)}{e}\frac{[L(T_b)(T_b^2-2)-(T_b^3+4T_b)(1+T_b^2)^{1/2}-(T_b^4-5T_b^2-1)]}{(1+T_b^2)^{5/2}}, \tag{3.4b}$$

$$\frac{R}{2\mu}h^{yy}_{\rm tail} = \frac{4v^3_\infty(m/b)}{e}\frac{[L(T_b)(-T_b^3-4T_b)+3(1+T_b^2)^{1/2}+5T_b]}{(1+T_b^2)^{5/2}}, \tag{3.4c}$$

where

$$L(T_b) \equiv \tfrac{11}{12} + \ln(t_b/s) - \ln[(1+T_b^2)^{1/2}-T_b] + \ln(T_b^2+1)\,, \tag{3.4d}$$

and T_b is the time scaled by the encounter time t_b (i.e., $T_b \equiv T/t_b$). The appearance of v^3_∞ in the leading coefficient of Eq. (3.4) clearly shows that the tail contribution to the waveform is $O((v/c)^3)$ smaller than the quadrupole radiation Eqs. (3.3). These contributions are plotted in Figs. 2(d)–2(f). In contrast with the quadrupole radiation [Figs. 2(a)–2(c)], the tail radiation is neither time symmetric nor time antisymmetric.

Figure 2 also shows that there is no "memory" in the tail radiation (i.e., $h^{ij}_{\rm tail} \to 0$ as $T_b \to \pm\infty$). However, it can also be seen from Fig. 2(e), or equivalently from Eq. (3.4b), that $h^{xy}_{\rm tail}$ goes to zero more slowly than the other components. This is directly attributable to the fact that $h^{xy}_{\rm quad}$, or equivalently $\overset{(2)}{Q}{}^{xy}$, does not go to zero as $T_b \to \pm\infty$.

IV. CIRCULAR ORBITS

We now examine the tail contribution to the radiation emitted from an in-spiraling binary system. Of course this problem could be treated in a general way by choosing some initial conditions and numerically evolving the orbit (e.g., Lincoln and Will [26]), and then, at each value of the retarded time, numerically integrating back in time to compute the tail of the radiation from Eq. (2.4). However, in order to illustrate some of the important features of the tail radiation, we perform the integration analytically by assuming that the in-spiraling binaries are in a quasicircular orbit. Astrophysically, this is a reasonable assumption; Lincoln and Will [26] have shown that virtually all captured binaries will have sufficient time to "circularize" their orbits before plunging to coalescence. (See particularly Fig. 6 in Ref. [26].) A reasonable objection to using this assumption when computing the tail is that the secular decay of the orbit radius may have a cumulative effect in the last integral in Eq. (2.4). However, the influence of the remote past of the system is so severely suppressed that it is reasonable to include only the previous few orbits when evaluating the tail integral Eq. (2.4). In particular, we will show that including only the influence of the previous two orbits results in errors of less than 1% in Eq. (2.4). In other words, if τ (the parameter which is chosen to separate the ancient history of the system from the recent history) is chosen to be two orbital periods of the system, and all influence on the tail from the ancient history is omitted, there is virtually no error.

We now examine the $h^{yy}_{\rm tail}$ component of the radiation by using the quadrupole moment for a binary system in a circular orbit:

$$Q^{yy} = \mu y^2 = \tfrac{1}{2}\mu r^2(1-\cos 2\phi), \tag{4.1}$$

where ϕ is the orbital phase angle. We also select the arbitrary parameter τ, which separates the recent history of the system from the ancient history in Eq. (2.4), by the relationship

$$\tau = nP \quad (n,\ \text{a positive integer})\,, \tag{4.2}$$

where P is the true orbital period. In the end the result must be independent of the parameter τ (or equivalently of n), but we will also explicitly show that the contribution to the tail integral for times prior to nP is insignificant for $n \gtrsim 2$. If Eq. (4.1) is substituted into Eq. (2.4) the result can be written

$$\frac{R}{2\mu}h^{yy}_{\rm tail} = 8(m/r)^{5/2}\Big[(R_s+A_s)\sin 2\phi + (R_c+A_c)\cos 2\phi\Big], \tag{4.3a}$$

where

$$R_s = -\tfrac{11}{12} - \ln(P/2s) - \ln n - 4\pi n \int_0^1 \sin(4\pi n x) \ln x dx \ , \qquad (4.3b)$$

$$R_c = \frac{1}{4\pi n} - 4\pi n \int_0^1 \cos(4\pi n x) \ln x dx \qquad (4.3c)$$

are the contributions to the tail from the recent history of the system (i.e., $T_{\rm ret} - nP < T < T_{\rm ret}$), and

$$A_s = \frac{-1}{4\pi} \int_0^\infty \frac{\sin(4\pi x)}{(n+x)^2} dx, \qquad (4.3d)$$

$$A_c = \frac{-1}{4\pi} \int_0^\infty \frac{\cos(4\pi x)}{(n+x)^2} dx \qquad (4.3e)$$

are the contributions to the tail integral from the ancient history (i.e., $T < T_{\rm ret} - nP$). These integrals can be done analytically using sine- and cosine-integral functions. The results are

$$R_s = -\tfrac{11}{12} - \ln(P/2s) + \gamma + \ln(4\pi) - \mathrm{ci}(4\pi n) \ , \qquad (4.4a)$$

$$R_c = \frac{\pi}{2} + \frac{1}{4\pi n} + \mathrm{si}(4\pi n) \ , \qquad (4.4b)$$

$$A_s = \mathrm{ci}(4\pi n) \approx \frac{-1}{(4\pi n)^2} + O(1/(4\pi n)^4) \ , \qquad (4.4c)$$

$$A_c = -\frac{1}{4\pi n} - \mathrm{si}(4\pi n) \approx \frac{2}{(4\pi n)^2} + O(1/(4\pi n)^5) \ . \qquad (4.4d)$$

In Eq. (4.4a) γ is Euler's number ($\gamma = 0.577...$). The first term of the asymptotic expansion is shown for A_s and A_c. If the exact results in Eq. (4.4) are substituted into Eq. (4.3a) all dependence on n cancels identically (as it must). However, it is also useful to note that the contribution to the tail integral from the ancient history [i.e., A_s and A_c in Eqs. (4.4c) and (4.4d)] rapidly goes to zero for increasing n. In fact, with $n = 2$ (i.e., two orbits), we would make less than a 1% error by omitting all together the contribution from A_s and A_c. In any case, using the exact formulas, we are left with the following compact results for the tail of the radiation for coalescing binaries:

$$\frac{R}{2\mu} h^{xx}_{\rm tail} = -2(m/r)^{5/2} B \cos(2\phi - \delta) \ , \qquad (4.5a)$$

$$\frac{R}{2\mu} h^{yy}_{\rm tail} = 2(m/r)^{5/2} B \cos(2\phi - \delta) \ , \qquad (4.5b)$$

$$\frac{R}{2\mu} h^{xy}_{\rm tail} = 2(m/r)^{5/2} B \sin(2\phi - \delta) \ , \qquad (4.5c)$$

where

$$B = \left\{ 16[-11/12 + \gamma + \ln(8\pi) - \ln(P/s)]^2 + 4\pi^2 \right\}^{1/2} \qquad (4.6a)$$

$$\approx 13.1 + \text{small logarithmic dependence} \ , \qquad (4.6b)$$

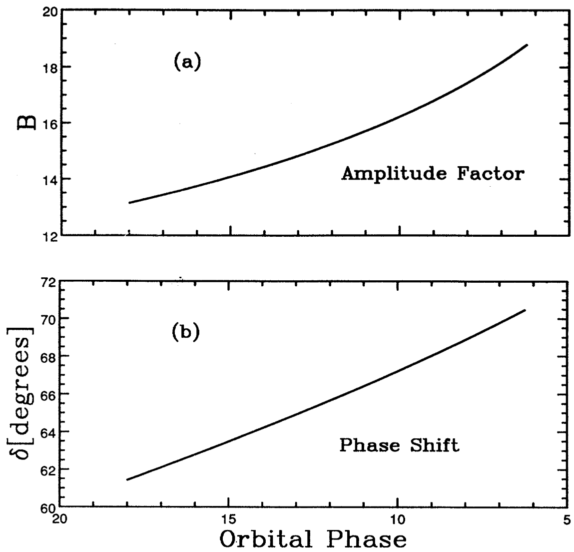

FIG. 3. (a) Amplitude factor B [Eq. (4.6a)] and (b) phase shift δ [Eq. (4.6c)] plotted against orbital separation during the inspiral of an equal-mass binary.

$$\delta = \arctan\left[\frac{2[-11/12 + \gamma + \ln(8\pi) - \ln(P/s)]}{\pi}\right] \qquad (4.6c)$$

$$\approx 61.1^\circ + \text{small logarithmic dependence} \ . \qquad (4.6d)$$

The "small logarithmic dependence" depends on our choice of the scale parameter s. However, as we have shown before, the total waveform (i.e., after we add the quadrupole radiation) is independent of s. A reasonable choice (the choice suggested by Blanchet and Damour [6]) for s is s = *the period of the system*; however, the period of the system is changing as the orbit decays, and thus $\ln(P/s)$ is not constant no matter how we choose s. For inspiraling binaries we choose s = the instantaneous period of the system as the orbit decays through $r = 18m$. The resulting change in the amplitude factor B and the phase shift δ as the binary system spirals from $r = 18m$ to $r = 6m$ is shown in Figs. 3(a) and 3(b).

In Fig. 1 we show the tail radiation computed during the in-spiral. The usual quadrupole radiation is also shown. Notice that the tail correction to the waveform is of comparable size to the leading order quadrupole term as the binary nears coalescence (e.g., $r \sim 10m$). Also notice the tail radiation is shifted in phase from the quadrupole radiation.

For completeness, as well as comparison with other work [27], we present total waveforms which are accurate through (post)$^{3/2}$-Newtonian order [i.e., $O((Gm/rc^2)^{3/2}) = O((v/c)^3) = O(m\omega)$ beyond the usual quadrupole radiation]. These are constructed from the two-body, instantaneous multipoles in Wiseman [20] plus the tail contribution presented above. Using notation very similar to Poisson [27] we can write for the plus polarization

$$h_+(T) = -(\mu/R)(m\omega)^{2/3}[\zeta_+^{(0)} + \zeta_+^{(1)}(m\omega)^{1/3} + (\zeta_+^{(2)} + \eta\zeta_{+(\eta)}^{(2)})(m\omega)^{2/3} + (\zeta_+^{(3)} + \eta\zeta_{+(\eta)}^{(3)} + \zeta_{+(\mathrm{tail})}^{(3)})(m\omega)] \ , \tag{4.7}$$

where

$$\zeta_+^{(0)} = 2(1+c^2)\cos 2\phi \ , \tag{4.8a}$$

$$\zeta_+^{(1)} = \tfrac{1}{4}(\delta m/m)\sin i[(5+c^2)\sin\phi + 9(1+c^2)\sin 3\phi] \ , \tag{4.8b}$$

$$\zeta_+^{(2)} = -\tfrac{1}{3}[(19+9c^2-2c^4)\cos 2\phi + 8(1-c^4)\cos 4\phi] \ , \tag{4.8c}$$

$$\zeta_+^{(3)} = -(\delta m/m)\sin i[\tfrac{1}{96}(57+60c^2-c^4)\sin\phi + \tfrac{9}{64}(73+40c^2-9c^4)\sin 3\phi + \tfrac{625}{192}(1-c^4)\sin 5\phi] \ , \tag{4.8d}$$

$$\zeta_{+(\eta)}^{(2)} = -\tfrac{1}{3}[(-19+11c^2+6c^4)\cos 2\phi - 24(1-c^4)\cos 4\phi] \ , \tag{4.8e}$$

$$\zeta_{+(\eta)}^{(3)} = -(\delta m/m)\sin i[\tfrac{1}{336}(-391+84c^2+7c^4)\sin\phi + \tfrac{1}{32}(-225+72c^2-81c^4)\sin 3\phi - \tfrac{625}{96}(1-c^4)\sin 5\phi] \ , \tag{4.8f}$$

$$\zeta_{+(\mathrm{tail})}^{(3)} = (1+c^2)\{4\pi\cos 2\phi + 8[\gamma - 11/12 - \ln(P/8\pi s)]\sin 2\phi\} \ . \tag{4.8g}$$

Here, $\sin i$ is the sine of the inclination of the orbit relative to the line of sight from the observer; we have used the shorthand c for the cosine of the inclination angle; $\delta m = m_1 - m_2$ is the difference in mass of the two bodies; ϕ is the orbital phase; ω is the circular orbit velocity. [The corresponding equations for the "$\times$" polarization are given below.] Notice that $\zeta_+^{(0)}$, $\zeta_+^{(1)}$, $\zeta_+^{(2)}$, and $\zeta_+^{(3)}$, which represent the test-mass behavior of the waveform, are all in agreement with Poisson [27]. See his Eqs. (6.1)–(6.3). In the test-mass limit the quantity $\delta m/m = -1$. [Also notice that Poisson's equation for $\zeta_+^{(3)}$ should read $(57 + 60\cos^2\theta + \cdots)$ instead of $(57 + 20\cos^2\theta + \cdots)$]. The non-test-mass part of the waveform is represented by the contributions $\zeta_{+(\eta)}^{(2)}$ and $\zeta_{+(\eta)}^{(3)}$. The perturbative technique ($\eta = \mu/m << 1$) used by Poisson is unable to obtain this η-dependent part of the waveform. The tail contribution to the waveform, $\zeta_{+(\mathrm{tail})}^{(3)}$, can also be identified in Poisson's result. Note that his $(3\ln 2v) = -\ln(16\pi m/P)$ can be identified with our $\ln(P/8\pi s)$, and thus our "arbitrary" scale factor is set to $2m$ in Poisson's calculation. The tail portion has no additional η dependence, and therefore Poisson's perturbative calculation gives exactly the correct answer for this term.

The fact that such different approaches to the construction of the waveform give similar results is very reassuring; however, there is one subtle, but important, difference between Poisson's result and our result. The ubiquitous factor 11/12 that gets carried along from our basic formula Eq. (2.2b) and ends up in our final result Eq. (4.8g) is different than Poisson's result; Poisson gets 17/12, not 11/12. This discrepancy is very likely due to using different boundary conditions in the two approaches. Poisson works with perturbations of the geometry around a Schwarzschild black hole; he establishes a strictly in-going radiation boundary condition at the horizon. In the Blanchet-Damour [6] construction of Eq. (2.2b) there is no horizon present, and therefore no such condition. The implication of this discrepancy between the tail portion of Poisson's waveform and the tail portion of the waveform presented here is that the waveform produced by an object orbiting a black hole is fundamentally different [at the $(\mathrm{post})^{3/2}$-Newtonian level] than the waveform of an object orbiting other compact objects, such as neutron stars. If the waveform can be measured accurately enough during the slow adiabatic orbital decay to obtain the amplitude of $\zeta_{(\mathrm{tail})}^{(3)}$, then, in principle, it may be possible to directly determine whether one of the binary constituents is a black hole. This could be done without having to indirectly infer whether one of the constituents is a black hole from the measured masses, or from the signature of the final coalescence. The observational consequences of this feature of the waveform are currently under study and will be addressed in a future publication.

The $\times$ polarization can be similarly written by replacing $+$ with $\times$ in Eq. (4.7) and then using the quantities

$$\zeta_\times^{(0)} = 4c\sin 2\phi \ , \tag{4.9a}$$

$$\zeta_\times^{(1)} = -\tfrac{3}{2}(\delta m/m)c\sin i[\cos\phi + 3\cos 3\phi] \ , \tag{4.9b}$$

$$\zeta_\times^{(2)} = -\tfrac{2}{3}c[(17-4c^2)\sin 2\phi + 8(1-c^2)\sin 4\phi] \ , \tag{4.9c}$$

$$\zeta_\times^{(3)} = -(\delta m/m)c\sin i[-\tfrac{1}{48}(63+5c^2)\cos\phi - \tfrac{9}{32}(67-15c^2)\cos 3\phi - \tfrac{625}{96}(1-c^2)\cos 5\phi] \ , \tag{4.9d}$$

$$\zeta_{\times(\eta)}^{(2)} = -\tfrac{2}{3}c[(-13+12c^2)\sin 2\phi - 24(1-c^2)\sin 4\phi] \ , \tag{4.9e}$$

$$\zeta_{\times(\eta)}^{(3)} = -(\delta m/m)c\sin i[\tfrac{1}{168}(185-35c^2)\cos\phi + \tfrac{1}{16}(171+135c^2)\cos 3\phi + \tfrac{625}{48}(1-c^2)\cos 5\phi] \ , \tag{4.9f}$$

$$\zeta_{\times(\mathrm{tail})}^{(3)} = c\{8\pi\sin 2\phi - 16[\gamma - 11/12 - \ln(P/8\pi s)]\cos 2\phi\} \ . \tag{4.9g}$$

Again, the test-mass and tail portions agree with Poisson (modulo the same discrepancy of 11/12 vs 17/12 in the tail portion).

Now that we have a waveform which is fully accurate to the (post)$^{3/2}$-quadrupole order, we may use it to compute an additional correction to the energy flux carried away by the radiation [i.e., a correction which is $O((Gm/rc^2)^{3/2})$ beyond the usual quadrupole energy loss formula, or $O((Gm/rc^2)^{1/2})$ beyond the first post-quadrupole correction published by Wagoner and Will [25]]. We use the general expression, for the energy flux [17],

$$L = \frac{R^2}{32\pi} \int \dot{h}^{ij}_{\rm TT} \dot{h}^{ij}_{\rm TT} d\Omega \ . \tag{4.10}$$

The correction we are looking for comes from the cross terms between the lowest-order quadrupole radiation and the tail of the radiation. Assuming circular orbits we have the first "tail" contribution to the luminosity,

$$L_{\rm tail} = \tfrac{32}{5}\eta^2 (m/r)^5 \left[4\pi (m/r)^{3/2}\right], \tag{4.11}$$

where $\eta = \mu/m$. We combine this with the quadrupole energy loss formula and the first post-Newtonian correction to obtain

$$L = \frac{32}{5}\eta^2 (m/r)^5 \left[1 - \frac{2927 + 420\eta}{336}\frac{m}{r} + 4\pi (m/r)^{3/2}\right] \ . \tag{4.12}$$

See, e.g., Junker and Schäfer [28] for the first two terms. Also in the language of Junker and Schäfer, we refer to the leading term in Eq. (4.12) as $L_{5/2}$ and the next correction as $L_{7/2}$. The last term we refer to as $L_{\rm tail}$. The subscript refers to the order of the near-zone damping force which presumably gives rise to the corresponding energy loss. Figure 4 shows the luminosity as a function of r. Although it is impossible to say whether or not including the $L_{\rm tail}$ correction in Eq. (4.12) makes the luminosity formula substantially more accurate, at least it now does not turn over and go negative as it does with just the $L_{7/2}$ correction.

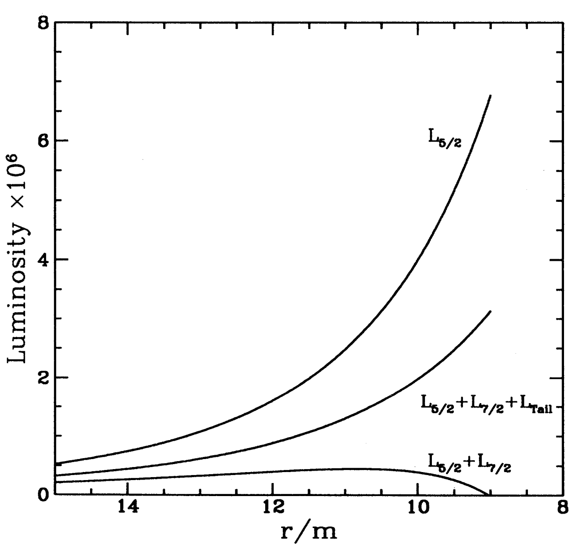

FIG. 4. Luminosity plotted against orbital separation for decaying circular orbit with $m_1 = m_2$.

The "tail" contribution (the 4π term) to Eq. (4.12) has also been discussed by other authors, Cutler *et al.* [16] and Cutler *et al.* [29].

It has often been assumed, but by no means rigorously verified, that the higher-order corrections to the luminosity in Eq. (4.12) exactly compensate for the orbital energy loss due to near-zone radiation back-reaction forces. (However, see Damour's warning about equating far-zone energy flux with near-zone energy loss [30].) Using this assumption of energy balance, Iyer and Will [31] have derived a general expression for the (post)$^{7/2}$-Newtonian radiation reaction force. Here we continue with this assumption that the corrections to the energy flux given by Eq. (4.12) exactly compensate for the post-Newtonian orbital energy loss due to the (post)$^{7/2}$ damping force and the tail transported radiation reaction force. (The latter force is known [5], and the energy balance assumption has been explored in [6].)

The post-Newtonian expression for the orbital energy for bodies in circular motion is

$$\frac{E_{\rm circular}}{\mu} = -\frac{1}{2}\frac{m}{r} + \frac{7-\eta}{8}\left(\frac{m}{r}\right)^2 + O((m/r)^4). \tag{4.13}$$

We now equate

$$\frac{dE_{\rm circular}}{dt} = -L \tag{4.14}$$

and solve for $\dot{r}$. This yields

$$\dot{r} = -\frac{64}{5}\eta (m/r)^3 \left[1 - \frac{1751 + 588\eta}{336}\frac{m}{r} + 4\pi (m/r)^{3/2}\right] \ . \tag{4.15}$$

The first two terms in Eq. (4.15) have been investigated by Junker and Schäfer [28]. The last term is the contribution which comes from the tail part of the luminosity.

Although the correction terms in Eq. (4.15) are quite small they can have the following observational effect. When attempting to analyze LIGO signals, it will be crucial to have accurate "template" waveforms to compare against the observed signal. Since the sensitivity range of the LIGO will be roughly 10–1000 Hz, it will be necessary (or at least desirable) to predict the phase of the waveform over many orbits as the signal sweeps through this frequency range. In order to compute the total number of orbits in this regime we use

$$N = \frac{1}{2\pi}\int \omega dt = \frac{1}{2\pi}\int_{r_{10\ \rm Hz}}^{r_{1000\ \rm Hz}} \frac{\omega}{\dot{r}} dr \ . \tag{4.16}$$

For two 1.4-solar-mass neutron stars a gravitational wave frequency of 10 Hz (1000 Hz) corresponds to an orbital separation of $r_{10\ \rm Hz} = 174m$ ($r_{1000\ \rm Hz} = 8m$). The number of orbits as the binary sweeps through this regime is obtained by substituting the post-Newtonian expression for the orbital frequency,

$$\omega = \left(\frac{m}{r^3}\right)^{1/2}\left[1 - \frac{3-\eta}{2}\frac{m}{r} + O((m/r)^2)\right] \qquad (4.17)$$

and Eq. (4.15) into Eq. (4.16) and integrating. [There is no term of $O((m/r)^{3/2})$ in Eq. (4.17).] We get

$$N = N_{5/2} + \Delta N_{7/2} + \Delta N_{\rm tail}, \qquad (4.18a)$$

where $N_{5/2}$, $\Delta N_{7/2}$, and $\Delta N_{\rm tail}$ represent the leading order, post-Newtonian, and tail corrections, and where

$$N_{5/2} = 7900 \text{ orbits} \ , \qquad (4.18b)$$

$$\Delta N_{7/2} = 325 \text{ orbits} \ , \qquad (4.18c)$$

$$\Delta N_{\rm tail} = -252 \text{ orbits} \ . \qquad (4.18d)$$

Therefore, we can see that neglecting the tail damping would result in a huge phase error in our template waveforms [i.e., $\approx 252(2\pi)$ rad]. Furthermore, it does not appear that the sequence is converging very fast; thus if we want to track the phase of the radiation to within 1 rad over these 7900 orbits it is highly unlikely that even including many more higher-order corrections in Eq. (4.12) will be sufficient.

V. NEAR-ZONE RADIATION-REACTION FORCE

In order to obtain the orbital decay rate in Eq. (4.15) and the subsequent phase evolution in Eq. (4.18) of the decaying binary system we used the energy balance condition Eq. (4.14). However, because of the nonlinear and nonlocal nature of the field equations and equations of motion, Damour [30] has articulated a strong admonition against naively equating the near-zone orbital energy loss rate with the far-zone energy flux. In spite of the weakness of such energy balance arguments, it is true that the lowest-order radiation-reaction force obtained by these arguments does yield the correct answer. ("correct" in the sense that the results agree with the more rigorously derived results of Damour and Deruelle [32] and Grishchuk and Kopejkin [33].) Furthermore, it is quite likely that the first correction to the radiation reaction force [the (post)$^{7/2}$-Newtonian contribution to the equations of motion] recently obtained by Iyer and Will [31] using just such an energy balance condition will agree with more rigorous calculations when they are completed. However, in this paper we are pushing beyond the (post)$^{7/2}$-Newtonian order in the equations of motion and dealing with "tail" effects. These effects are highly nonlinear and nonlocal and therefore the use of an instantaneous energy balance argument, such as Eq. (4.14), is even more suspect than it was at lower order. In order to put the results of this paper on the firmest footing possible, we show that the tail contribution to the orbital decay rate [the 4π term in Eq. (4.15)], and thus the orbital phase evolution Eq. (4.18), can be obtained directly from a rigorously computed contribution to the near-zone radiation-reaction force. This calculation is based solely on the near-zone metric, and never uses an energy balance condition such as Eq. (4.14). The arguments presented here show in the concrete case of a coalescing binary system in a nearly circular orbit precisely what Blanchet and Damour [6] showed in the general case: The near-zone, tail-transported radiation reaction force bleeds the orbital energy defined by Eq. (4.13) away from the system at precisely the same rate as the tail radiation carries away energy in the far zone. (See particularly Sec. III D of Ref. [6].) Thus, in a nonrigorous sense, the derivation of the orbital decay presented here does not *use* the energy balance condition; rather it *confirms* the consistency of the energy balance condition.

Blanchet and Damour [5] have actually obtained the first tail-transported correction to the near-zone metric component g_{00}. [See particularly Eq. (6.33) in Ref. [5].] From this metric component they construct the first tail-transported correction to the radiation-reaction potential. Their result can be written in terms of a radiation-reaction acceleration:

$$a^i_{\rm RR} = -\Phi_{{\rm RR},i} = -\tfrac{1}{5}[x^a x^b \overset{(5)}{I\!\!\!/}{}^{ij}]_{,i} = -\tfrac{2}{5}x^j \overset{(5)}{I\!\!\!/}{}^{ij} \ , \qquad (5.1)$$

where

$$I\!\!\!/^{ij} = I\!\!\!/^{ij}_{\rm quad} + \delta I\!\!\!/^{ij}_{\rm tail} \ , \qquad (5.2)$$

and the crosshatch denotes that the trace has been removed, e.g.,

$$I\!\!\!/^{ij}_{\rm quad} = Q^{ij} - \tfrac{1}{3}\delta^{ij}Q^{kk} \ . \qquad (5.3)$$

The first term in Eq. (5.2) will generate the usual Burke-Thorne radiation-reaction force. The second term $\delta I\!\!\!/^{ij}_{\rm tail}$ represents the tail-transported correction to the near-zone radiation-reaction force. Blanchet and Damour [5] have given the following expression for this term:

$$\delta \overset{(5)}{I\!\!\!/}{}^{ij}_{\rm tail} = 4m\int_0^\infty \overset{(7)}{I\!\!\!/}{}^{ij}_{\rm quad}(t-u)\ln(u/2s_a)du \ . \qquad (5.4)$$

Here s_a is a characteristic time scale of the problem; it is not necessarily equal to the time scale s of Eq. (2.2b); at the order we are considering it has no observable effects on the result of the calculation.

As we did in Sec. IV, we assume that the binary is in a slowly decaying circular orbit and we perform the integration using the techniques of Sec. IV. The result for the tail-transported contribution to the radiation-reaction acceleration is

$$\mathbf{a}_{\substack{\rm RR\\ \rm tail}} = \frac{256}{5}\eta\frac{m^5}{r^6}\left[[\ln(P/s_a) - \ln(8\pi) - \gamma]\hat{\mathbf{n}} - \frac{\pi}{2}\hat{\phi}\right] \ , \qquad (5.5)$$

where $\hat{\mathbf{n}}$ is the unit radial vector and $\hat{\phi}$ is a unit vector in the direction of increasing orbital phase.

The effect of a perturbing acceleration, such as Eq. (5.5) above, on the orbital evolution can be obtained directly from the Lagrange planetary equations. (See, for example, Lincoln and Will [26].) Using Eq. (2.7c) from Lincoln and Will and boiling it down to the case of circular orbits we find, for the orbital decay rate,

$$\dot{r} = 2\left(\frac{r^3}{m}\right)^{1/2}\left[a_\phi + (\text{zero})a_n\right] \ , \tag{5.6}$$

where a_n and a_ϕ represent the components of the perturbing acceleration, and "zero" represents a factor which vanishes in the special case of nearly circular orbits. In our case

$$a_\phi = \frac{128}{5}\eta\frac{m^5}{r^6}\pi \ , \tag{5.7}$$

and thus the tail contribution to the orbital decay rate is found to be

$$\dot{r}_{\text{tail}} = -\tfrac{256}{5}\eta(m/r)^{9/2}\pi \ . \tag{5.8}$$

This is in exact agreement with Eq. (4.15).

VI. CONCLUDING REMARKS

Starting with a general formula for the tail of the radiation given by Blanchet and Damour we have constructed specific formulas for the tail of the radiation for in-spiraling binaries and gravitational bremsstrahlung. We have shown that the general formulas are well defined for these two astrophysical events, and that there is no unreasonable sensitivity to the past history of the system.

In the bremsstrahlung case the tail of the radiation is suppressed by $O((v/c)^3)$ from the usual quadrupole radiation and is therefore probably undetectable in all but the most relativistic encounters. We also note that although the quadrupole radiation exhibits memory for a bremsstrahlung encounter, the tail correction does not. In the case of coalescing binaries we are able to show that the tail correction to the waveform can grow to be comparable to the quadrupole portion when the system is near coalescence. We are also able to give a new correction to the orbital decay rate for coalescing binaries.

ACKNOWLEDGMENTS

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APPENDIX: PHYSICAL INTUITION

As we described in the Introduction, the physical intuition underlying the "tail" radiation is that it is the portion of the direct radiation which scatters off the background curvature of the spacetime, and thereby takes an indirect path to the observer. Although this idea is quite picturesque, Blanchet and Damour [6] seldom appeal to such physical intuition in their development of the tail radiation formula Eq. (2.2b), i.e., Eq. (3.4a) in Ref. [6]. In this appendix we will show that the behavior of the tail radiation as predicted by the Blanchet-Damour formula and illustrated in the explicit formulas Eq. (3.4) and Eq. (4.5) and Figs. 1–5 of this paper is consistent with a number of hand-waving, qualitative arguments about what one should expect of the tail radiation. The arguments we present here are loosely based on a comparison of the field equations of general relativity with the simpler Klein-Gordon field equation. In particular we will explain the origin of the logarithmic correction to the radiative time coordinate in Eq. (2.5). We will discuss the quantitative nature of the phase lag, or time lag, of the tail radiation. [See Figs. 3 and 5 and Eq. (4.6).] Finally we will examine the absence of "memory" in the tail radiation.

We present this hand-waving development of the behavior of the tail radiation with a word of caution: These arguments are in no way a substitute, or shortcut, for the rigorous solution to the field equations of Blanchet and Damour, which is represented by Eq. (2.2). At present it seems that the multipolar post-Minkowski (MPM) formalism of Blanchet, Damour, and Iyer [4–8] culminating in Eq. (2.2) is the only formalism that is both sufficiently rigorous and sufficiently general to tackle the problem of coalescing binaries of compact objects. For example, the formalism of Thorne and Kovacs [15, 22, 24] is well suited for the case of high-speed, low-deflection gravitational bremsstrahlung, but it is unclear how this technique would have to be modified in order to solve the low-speed, high-deflection problem of coalescing binaries. The retarded time expansion of the Epstein-Wagoner formalism [10], which uses a flat-space Green's function, is not adequate for analyzing curved-space effects, such as the Shapiro time delay, or the curvature-induced heredity effects. Other perturbative techniques which assume a uniform background curvature are not well suited for compact sources.

The physical effects that we are discussing all have their origin in the gravitational field equations, which we write

$$\Box h^{\alpha\beta} = -16\pi\tau^{\alpha\beta} \ , \tag{A1}$$

where $\Box \equiv -\partial^2/\partial t^2 + \nabla^2$ is the flat-space wave operator, and $h^{\alpha\beta}$ is the potential which is related to the metric by

$$h^{\alpha\beta} \equiv -(-g)^{1/2}g^{\alpha\beta} + \eta^{\alpha\beta} \ . \tag{A2}$$

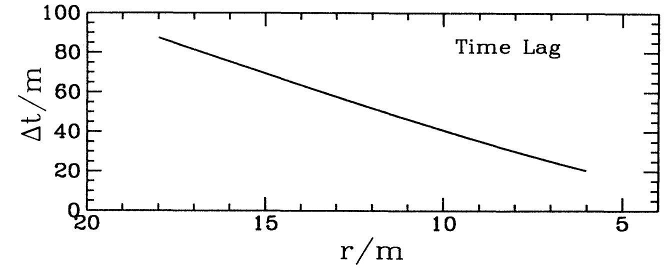

FIG. 5. Lag time of tail radiation as compared to direct radiation. Note the almost linear relationship.

In writing the field equations in this form we have also chosen the de Donder gauge condition

$$h^{\alpha\beta}{}_{,\beta} = 0 \ . \tag{A3}$$

The source in Eq. (A1) is the effective stress-energy pseudotensor $\tau^{\alpha\beta}$ which is given by

$$\tau^{\alpha\beta} = (-g)(T^{\alpha\beta} + t_{\rm LL}{}^{\alpha\beta}) + (1/16\pi)[h^{\alpha\mu}{}_{,\nu}\, h^{\beta\nu}{}_{,\mu} - h^{\alpha\beta}{}_{,\mu\nu}\, h^{\mu\nu}] \ , \tag{A4}$$

where $T^{\alpha\beta}$ is the stress-energy tensor of the matter, the commas denote partial differentiation, and $t_{\rm LL}{}^{\alpha\beta}$ is the Landau-Lifshitz pseudotensor [Ref. [18], Eq. (20.22)], which is composed essentially of terms quadratic in $h^{\alpha\beta}_{,\mu}$. Using the flat-space wave operator in Eq. (A1) masks the curved-space nature of the field equations. When written in this form, many of the curved-space effects in the field equations are hidden in the definition of the source. Notice that all the terms in Eq. (A4) involve only first derivatives of the field except the last term $h^{\alpha\beta}{}_{,\mu\nu}\, h^{\mu\nu}$. In a mathematical sense this second derivative term "belongs" on the left side of Eq. (A1) with the other second derivative terms that are incorporated in the wave operator. We will show that this second derivative source term is responsible for many of the curved-space effects, such as the tail of the radiation and the Shapiro time delay.

If the second derivative term in Eq. (A4) is taken to the other side of Eq. (A1) its effect on the solution to the differential equation can be examined heuristically. First, if the term $h^{\alpha\beta}{}_{,\mu\nu}\, h^{\mu\nu}$ is thought of as entering the differential equation as

$$\Box h^{\alpha\beta} + [(\text{factor})^{\mu\nu}]h^{\alpha\beta}{}_{,\mu\nu} = -16\pi[(\text{remainder of the source})^{\alpha\beta}] \ , \tag{A5}$$

we see that it changes the coefficients of the second derivatives in this equation. The dominant contribution to the "factor" in this equation is $(-h^{00})$, which to leading order is given by

$$h^{00} \approx \frac{4m}{r} \ . \tag{A6}$$

This means that the dominant terms on the left-hand side of Eq. (A5) are

$$(-1 - 4m/r)h^{\alpha\beta}{}_{,00} + h^{\alpha\beta}{}_{,rr} \ . \tag{A7}$$

It is the coefficients of the second derivative terms in Eq. (A7) which determine the characteristic curves along which information propagates to the distant observer. For the differential equation above the equation for the characteristic curve is

$$-(1 + 4m/r)(dr/d\alpha)^2 + (dt/d\alpha)^2 = 0 \ . \tag{A8}$$

[See, e.g., Mathews and Walker [35], Eq. (8.14).] Neglecting terms of $O(m^2)$ and integrating out to the distant observer at R, the solution to this equation is

$$t - R - 2m \ln R = \text{const} \ . \tag{A9}$$

This is precisely the definition of the radiation-coordinate retarded time Eq. (2.5). The logarithmic term is the Shapiro time delay, which represents the additional coordinate time required for the signal to crawl out of the $1/r$ potential of the compact source.

The second derivative term in the source can also be thought of as entering the field equation the other way as

$$\Box h^{\alpha\beta} + K^{\alpha\beta}{}_{\mu\nu} h^{\mu\nu} = -16\pi[(\text{remainder of the source})^{\alpha\beta}] \ , \tag{A10}$$

where $K^{\alpha\beta}{}_{\mu\nu}$ represents $h^{\alpha\beta}{}_{,\mu\nu}$. Loosely speaking, $K^{\alpha\beta}{}_{\mu\nu}$ consists of two derivatives of the metric tensor, and therefore is closely related to the curvature, i.e., $||K^{\alpha\beta}{}_{\mu\nu}|| \sim m/r^3$. We see that the differential equation Eq. (A10) now has *some* resemblance to the flat-spacetime Klein-Gordon equation

$$\Box\phi + m_s^2\phi = -4\pi s \ . \tag{A11}$$

[In discussing the Klein-Gordon equation we use units such that m_s has units of $(\text{length})^{-1}$.] Proceeding with more imagination and faith than rigor, we will attempt to glean some aspects of the behavior of the solution to the field equations from the behavior of the solution to the Klein-Gordon equation. We state here at the outset that we are mindful of several differences. In particular, (1) we are neglecting the fact that $K^{\alpha\beta}{}_{\mu\nu}$ is not a constant as is its counterpart m_s in the Klein-Gordon equation, and (2) we recognize that there is nothing in the scalar Klein-Gordon equation which is analogous to the mixing of polarization from the implied summation over μ and ν in Eq. (A10).

The solution to the Klein-Gordon equation can be written in terms of a two-part Green's function:

$$\phi(x) = \int G(x,x')s(x')d^4x' = \int G_{\rm direct}(x,x')s(x')d^4x' + \int G_{\rm mass}(x,x')s(x')d^4x' \ , \tag{A12}$$

where the "direct" part of the Green's function is the usual

$$G_{\rm direct} = \frac{\delta(t - t' - |\mathbf{x} - \mathbf{x}'|)}{|\mathbf{x} - \mathbf{x}'|} \tag{A13}$$

and the "mass" part of the Green's function can be written explicitly as

$$G_{\rm mass} = -\, m_s\Theta(t - t' - |\mathbf{x} - \mathbf{x}'|) \times \frac{J_1(m_s\sqrt{(t-t')^2 - |\mathbf{x} - \mathbf{x}'|^2}\,)}{\sqrt{(t-t')^2 - |\mathbf{x} - \mathbf{x}'|^2}} \ . \tag{A14}$$

[Equation (A14) can be obtained from formulas in Morse and Feshbach [36]. It is also given explicitly by Zaglauer [37].] The function Θ is the step function which has the value of unity on the interior of the past light cone of the field point and vanishes elsewhere, and J_1 is the Bessel function of order 1. Thus we say that the Green's function for the Klein-Gordon equation has support not only *on* the past light cone of the observer, but also *inside* the past light cone. This means that the solution to the Klein-Gordon equation will depend on the behavior of the source at the retarded time and at all times prior to the retarded time. Therefore, we can accurately refer to the second integral in Eq. (A12) as the hereditary part of the solution. The decaying behavior of the Bessel function ensures that the solution is only weakly dependent on the ancient history of the source. Notice that these are features that we have already seen in the tail radiation.

The two other features of the mass part of the Klein-Gordon Green's function that we now wish to explore are that (1) the massive portion of the Green's function G_{mass} is proportional to the mass m_s of the scalar field and (2) the function G_{mass} is an oscillatory function; the frequency is determined by the mass m_s of the scalar field. From these two features we make two observations about the solutions to the Klein-Gordon equation: (1) Obviously, if m_s is small the contribution to the second integral (the hereditary part) in Eq. (A12) is small; (2) frequency components of the source which are commensurate with the time scale established by m_s will produce the largest effect in the second integral in Eq. (A12). Other frequencies would tend to cancel when integrated against the oscillating Bessel function. With these observations we make two inferences (i.e., weakly founded guesses) about the solution to the gravitational field equations, Eq. (A1). (1) Just as small values of m_s produce small contributions to the hereditary part of the solution to the Klein-Gordon equation, we expect that regions of weak curvature (small $||K^{\alpha\beta}{}_{\mu\nu}||$) will give small contributions to the hereditary part of the solution to the field equations. In other words, regions of weak curvature will not produce much tail radiation. Thus not much scattering takes place far from the source, and therefore the tail radiation will only have weak dependence on the ancient history of the system. (2) Just as frequency components of the source which are commensurate with the time scale set by m_s will produce the dominant part of the hereditary piece of the solution to the Klein-Gordon equation, we expect that regions of spacetime where $||K^{\alpha\beta}{}_{\mu\nu}||^{-1/2}$ is commensurate with the period of the source will give the largest contributions to the tail radiation. In other words, the region of dominant scattering will be a region of spacetime where

$$\text{orbital period} \propto ||K^{\alpha\beta}{}_{\mu\nu}||^{-1/2} \ . \tag{A15}$$

This second point we wish to explore semiquantitatively. The relationship can be stated as

$$2\pi \left(\frac{r_{\text{orbit}}^3}{m}\right)^{1/2} \propto \left(\frac{r_{\text{scatter}}^3}{m}\right)^{1/2} \ , \tag{A16}$$

where r_{scatter} is the characteristic location of the scattering. Thus, r_{scatter} would also be proportional to the average path difference between the direct radiation and the tail radiation. Therefore we expect that

$$r_{\text{orbit}} \propto \Delta t_{\text{lag}} \ . \tag{A17}$$

Here Δt_{lag} is the lag time of the tail radiation as compared to the direct radiation. This linear relationship of Eq. (A17) is clearly borne out in Fig. 5.

The absence of memory in the tail radiation makes some sense by a similar argument. Memory in the tail radiation would be the extremely low frequency component of the tail portion of the radiation. Its source would be the extremely low frequency component of the quadrupole radiation, that is, the quadrupole memory. [See Fig. 2.] By the argument above we would expect these low frequency, long wavelength contributions to have been scattered in regions of spacetime where the radius of curvature of the spacetime is very large, i.e., very far from the source. However, by the first inference above, we expect these regions of spacetime to give little contribution to the tail. Therefore it is quite reasonable that Eq. (2.2) does not exhibit tail memory.

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