Quantum fluctuations and curvature singularities in Jackiw-Teitelboim gravity

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Jackiw-Teitelboim gravity with matter degrees of freedom is considered. The classical model is exactly solvable and its solutions describe nontrivial gravitational scattering of matter wave packets. For a huge amount of the solutions the scattering space-times are free of curvature singularities. However, the quantum corrections to the field equations inevitably cause the formation of curvature singularities, vanishing only in the limit $\hbar \to 0$. The singularities cut the space-time and disallow propagation to the future. The model is inspired by the dimensional reduction of 4D pure Einstein gravity, restricted to space-times with two commuting spacelike Killing vectors. The matter degrees of freedom also stem from the 4D ansatz. The measures for the continual integrations are judiciously chosen and one-loop contributions (including the graviton and the dilaton ones) are evaluated. For the number of the matter fields $N = 24$ we obtain even the exact effective action, applying the David-Distler-Kawai procedure. The effective action is nonlocal, but the semiclassical equations can be solved by using some theory of the Hankel transformations.

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I. INTRODUCTION

Einstein gravity is undoubtedly a beautiful and physically relevant theoretical construction which, at the same time, has brought many novel mathematical structures into theoretical physics. Notions such as the black hole, the horizon, the curvature singularities, or the gravitational collapse enriched our conceptual world, but they also posed new challenging problems, yet to be understood. For instance, what is the fate of matter in the last part of the gravitational collapse? Following the overwhelming success of quantum theory in describing the world of subatomic distances, physicists feel that an appropriate theory of quantum gravity has to provide the correct solutions to the problems and improve our understanding of the phenomena occurring in very strong gravitational fields near the curvature singularities. Unfortunately, such a generally accepted and technically applicable quantum theory of gravity does not exist yet. A considerable breakthrough was reached due to string theory, where the consistent perturbative S matrix for scattering of gravitons and excitations of other fields can be obtained [1]. However, the dynamics of a gravitational collapse or the status of quantum black holes are still not understood. Though there is a common belief that quantum effects should smear the singular behavior of classical theory, there is no sufficiently established quantitative evidence for such a conjecture.

The string boom had also an indirect, but very important influence on the subject of quantum gravity. Models of $(1 + 1)$-dimensional theories of gravity coupled to a dilaton field ϕ have arisen in string theory [2,3]. These models possess black hole solutions, and they motivated Callan, Giddings, Harvey, and Strominger (CGHS) [4] to investigate the model of two-dimensional (2D) dilaton gravity coupled to conformal matter. The model is of interest as a "toy" model of quantum gravity in two dimensions which contains gravitational collapse, black holes, cosmic censorship, and Hawking radiation. Moreover, the model is very similar to that obtained by the dimensional reduction of the spherically symmetric gravitational system in $3 + 1$ dimensions; hence, one may expect the relevance of the $1 + 1$ results to the $3 + 1$ physics. Recently, many authors have been investigating the quantum dynamics of black holes by using the CGHS model [5–13]. The issues of particular interest are the back reaction of Hawking radiation on the metric and the end point of black hole evaporation. The problem is far simpler than the original $(3 + 1)$-dimensional one, and powerful methods of conformal field theory in two dimensions can be used.

The CGHS model and its variants [14–16] and also other 2D dilaton gravities have been studied in the literture [17], particularly in the context of noncritical string theory [18]. The models can be typically obtained by the dimensional reduction of higher-dimensional pure metric gravities. This fact suggests the following CGHS-like scenario for "addressing" four-dimensional quantum problems: One finds the corresponding dilaton gravity model in 1+1 dimensions and attempts to quantize it. Though 1+1 quantum theory may still be complicated enough to prevent exact solvability (as CGHS is), usually it is far simpler than its 4D counterpart. In this contribution, we adopt the scenario and address the quantum dynamics of colliding gravitational waves. The fact that the nonlinear character of the Einstein equations results in the formation of curvature singularities after collisions of gravitational waves is known only since the 1970s [19,20], and perhaps it is less familiar to nonspecialists than the fact that a black hole is formed as a consequence of gravitational collapse. However, the colliding-wave problem keeps attracting many relativists [21–24] without an interruption, since the discoveries of the first colliding-wave space-times by Szekeres [19] and Khan and Penrose [20].

The problem of main interest for us will constitute the following: What is the quantum status of those scattering space-times? As we have mentioned above, one usually expects that curvature singularities should be smeared by the effects of quantum fluctuations. Our quantitative analysis will show, however, the surprising result that the quantum curvature singularities are even worse than the classical ones and even classicaly nonsingular space-times are destabilized by quantum curvature singularities.

Apart from the physical questions which our analysis will try to answer, the model to be considered in this paper is of interest also for some more theoretical reasons. Indeed, while in CGHS and related theories [17,18] matter degrees of freedom are added by hand *after* the dimensional reduction, in our model matter degrees of freedom also come from four-dimensional theory. This fact should even increase the relevance of our results for the 4D case. There is another pleasant thing, namely, not only the matter loops, but also one-loop dilaton and graviton contributions can be evaluated, yielding the one-loop effective action. Moreover, for the critical number of matter fields ($N = 24$) our result will be nonperturbative and exact. But the good news is not exhausted by that; it turns out, moreover, that the semiclassical equations can be solved and the behavior of curvature singularities is under control.

The plan of the paper is as follows. In Sec. II we introduce the 2D matter-dilaton model motivated by the dimensional reduction of the $(3+1)$-dimensional system with two commuting spacelike Killing vectors. Then we find the classical equations of motion in the conformal gauge. The dilaton field turns out to satisfy the standard d'Alambert wave equation; hence, we introduce a sort of "light-cone" gauge. In this gauge the matter fields obey the Gowdy cylindrical wave equation [25], the general solution of which can be given by the decomposition into the Fourier-Bessel and Fourier-Neumann modes. The corresponding metric we find explicitly by integrating the remaining equations. We show that the Neumann modes cause the formation of the (classical) curvature singularities which close space-time to the future, while the appropriate superpositions of the Bessel modes describe the collisions of the wave packets traveling against each other with the velocity of light. The corresponding space-times are everywhere regular with the out region in which the scattered wave packets travel to the opposite space infinities. In Sec. III we discuss the quantization of the model. We choose the standard Polyakov measure for functional integration over the metrics and reparametrization invariant measures for the dilaton and the matter field integration. Then we compute the one-loop effective action. The effective action is nonlocal even in the conformal gauge, due to the presence of the direct matter-dilaton coupling in the action. The one-loop effective field equations are localized by going to the dilaton "light-cone" gauge. The renormalization requires a purely dilatonic counterterm, the contribution of which makes finite one infinite constant in the semiclassical field equations. The actual computation requires knowledge of the functional derivatives of the determinant of the (Gowdy) wave operator with respect to the dilaton and metric. They are evaluated by using heat kernel regularization and some theory of Hankel transformations in the Appendix. In Sec. IV we solve the semiclassical field equations. We perform a detailed analysis of the scalar curvature of the space-times, obtained by solving the semiclassical equations. We show that the contribution of the quantum fluctuations to the effective action inevitably generates curvature singularities in the semiclassical space-times. These singularities may disappear only in the limit $\hbar \to 0$, thus indicating that the classical regular scattering space-times are in fact unstable from the quantum point of view. We end up with short conclusions and an outlook.

II. THE MODEL AND ITS CLASSICAL DYNAMICS

A. Dimensional reduction

The form of the four-dimensional metric describing the collisions of collinear gravitational waves is given by [24]

$$ds^2 = -2\phi^{-\frac{1}{2}} e^{\mu} du dv + \phi(e^{\sqrt{2\kappa}Q} dx^2 + e^{-\sqrt{2\kappa}Q} dy^2), \quad (1)$$

where the metric functions μ, ϕ, and Q are invariant on the (x, y) plane of symmetry. The 4D vacuum Einstein equations for the metric (1) consist of the constraints

$$-\phi_{uu} + \mu_u \phi_u = \kappa \phi Q_u^2 , \quad (2)$$

$$-\phi_{vv} + \mu_v \phi_v = \kappa \phi Q_v^2 , \quad (3)$$

and the evolution equations

$$\phi_{uv} = 0 , \quad (4)$$

$$(\phi Q_u)_v + (\phi Q_v)_u = 0 , \quad (5)$$

$$-\mu_{uv} = \kappa Q_u Q_v. \quad (6)$$

It is not difficult to demonstrate that the same set of constraints and evolution equations follows from the 2D action

$$S = \frac{1}{2\kappa} \int d^2\xi \sqrt{-g} \phi (R - \kappa g^{\alpha\beta} \partial_\alpha Q \partial_\beta Q), \quad (7)$$

after fixing the conformal gauge

$$ds^2 = -2e^{\mu} du dv. \quad (8)$$

The action (7) can be interpreted as Jackiw-Teitelboim gravity [26] where the cosmological constant is replaced with the kinetic term of matter. The matter is coupled to the dilaton and possesses all dynamical degrees of freedom of the theory. We note that the generalization differs from the generalizations of Jackiw-Teitelboim gravity considered previously [47,48].

B. Solutions of the field equations

In what follows, we shall consider the model (7) with an arbitrary number of matter fields. The classical dynamics does not change dramatically, but the properties of quantum theory will depend on that number. The action reads

$$S = \frac{1}{2\kappa}\int d^2\xi\sqrt{-g}\phi(R - \kappa g^{\alpha\beta}\partial_\alpha Q^j\partial_\beta Q^j) , \qquad (9)$$

and the constraints and the evolution equations in the conformal gauge get obviously modified:

$$-\phi_{uu} + \mu_u\phi_u = \kappa\phi Q^j_u Q^j_u , \qquad (10)$$

$$-\phi_{vv} + \mu_v\phi_v = \kappa\phi Q^j_v Q^j_v , \qquad (11)$$

$$\phi_{uv} = 0 , \qquad (12)$$

$$(\phi Q^j_u)_v + (\phi Q^j_v)_u = 0 , \qquad (13)$$

$$-\mu_{uv} = \kappa Q^j_u Q^j_v. \qquad (14)$$

The general solution of (12) reads

$$\phi = f(u) + g(v) . \qquad (15)$$

If $\phi_v = 0$ (or $\phi_u = 0$), then from (11) [or (10)] it follows that $Q^j_v = 0$ ($Q^j_u = 0$) and from (14) $\mu_{uv} = 0$. Since the scalar curvature R is given by

$$R = 2e^{-\mu}\mu_{uv} \qquad (16)$$

and in two dimensions the curvature tensor reads

$$R_{\alpha\beta\gamma\delta} = \tfrac{1}{2}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})R , \qquad (17)$$

we may conclude that the arbitrary functions $\phi(u)$ and $Q^j(u)$ [or $\phi(v)$ and $Q^j(v)$] are solutions of the field equations and the corresponding space-time is flat. Such solutions obviously describe the matter excitations propagating in one direction with the velocity of light.

If neither $\phi_v = 0$ nor $\phi_u = 0$, we can (at least locally) perform the conformal transformation

$$U = \sqrt{2}f(u), \quad V = \sqrt{2}g(v) . \qquad (18)$$

Such a transformation is obviously the symmetry transformation of the set of the field equations (10)–(14); hence, we may fix this residual symmetry by the claim

$$\phi = \frac{U+V}{\sqrt{2}} \equiv t . \qquad (19)$$

We may call this gauge fixing the "dilaton" gauge. In the gauge the field equations become

$$\mu_U = \kappa(U+V)Q^j_U Q^j_U , \qquad (20)$$

$$\mu_V = \kappa(U+V)Q^j_V Q^j_V , \qquad (21)$$

$$Q^j_{tt} + \frac{1}{t}Q^j_t - Q^j_{\sigma\sigma} = 0 , \qquad (22)$$

$$-\mu_{UV} = \kappa Q^j_U Q^j_V , \qquad (23)$$

where

$$\sigma \equiv \frac{V-U}{\sqrt{2}} . \qquad (24)$$

We observe that the linear equation for the matter fields Q^j does not contain the metric function μ. We may call this equation by the name of Gowdy, who studied cosmological models with plane symmetry [25], governed locally by (22). The general solution of the Gowdy equation (which tend to zero at the spatial infinities) is given by the mode expansion [24,28]

$$Q^j(t,\sigma) = \int_0^\infty d\omega \mathrm{Re}\big[A^j(\omega)J_0(\omega t)e^{-i\omega\sigma} + B^j(\omega)N_0(\omega t)e^{-i\omega\sigma}\big], \qquad (25)$$

where A^j and B^j are (complex) distributions ensuring the proper behavior at space infinity and J_0 and N_0 are the Bessel and Neumann functions of zero order, respectively. This mode expansion can be easily found by using the Fourier tranformation in the variable σ in (22). The resulting ordinary differential equation in the variable t is then the Bessel equation. We should note, at this place, that in higher dimensions some additional (so called "solitonic") terms are considered on the right-hand side (RHS) of (25). They do not vanish at space infinity and in the limit of weak gravitational coupling $\kappa \to 0$ those solutions diverge and do not approach the noninteracting matter solutions [30]. We shall not consider this "solitonic" sector in this paper and prescribe the boundary conditions, mentioned above.

It remains to solve Eqs. (20), (21), and (14), which determine the metric function μ. Combining (20) with (21), we obtain

$$\mu_\sigma = 2\kappa t Q^j_t Q^j_\sigma , \qquad (26)$$

$$\mu_t = \kappa t(Q^j_t Q^j_t + Q^j_\sigma Q^j_\sigma) . \qquad (27)$$

We use the fact [27] that for F and G, satisfying the Bessel equations

$$x^2\frac{d^2F}{dx^2} + x\frac{dF}{dx} + (\lambda^2x^2 - n^2)F = 0 ,$$
$$x^2\frac{d^2G}{dx^2} + x\frac{dG}{dx} + (\nu^2x^2 - n^2)G = 0 ,$$

it holds that

$$\int_a^b dx(\lambda^2 - \nu^2)xF\,G = \left[x\left(F\frac{dG}{dx} - G\frac{dF}{dx}\right)\right]_a^b . \qquad (28)$$

This formula enables us to integrate the products of the Bessel functions. The result of the integration gives the explicit form of the metric function[1] μ:

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[1]It appears that this result is new even from the point of view of 4D theory of colliding waves [23,24].

$$\mu = \kappa \int_0^\infty d\omega_1 d\omega_2 \omega_1 \omega_2 \; t \; \mathrm{Re}\Bigg(\frac{-1}{\omega_1+\omega_2} G_1^j(\omega_1 t) G_0^j(\omega_2 t) e^{-i(\omega_1+\omega_2)\sigma} + \frac{1}{2}\frac{1}{\omega_1-\omega_2}\left[G_1^{j*}(\omega_1 t)G_0^j(\omega_2 t)e^{i(\omega_1-\omega_2)\sigma} - G_1^{j*}(\omega_2 t)G_0^j(\omega_1 t)e^{-i(\omega_1-\omega_2)\sigma}\right]\Bigg), \tag{29}$$

where

$$G^j_{0(1)}(\omega t) = A^j(\omega) J_{0(1)}(\omega t) + B^j(\omega) N_{0(1)}(\omega t)\ . \tag{30}$$

We note that the classical equations (12)–(14) turn out to be "iteratively" linear. Indeed, solving the linear equation (12) and inserting its solution ϕ into Eq. (13), we get again the linear equation. After solving it, we insert Q^j into Eq. (14) and get the linear equation for μ. Such a structure of the equations gives the classical integrability and will be also important later for the quantization.

C. Curvature singularities and the global structure

We start our analysis of the curvature singularities with the formula for the scalar curvature. Following from Eqs. (16) and (23), we have

$$R = \kappa e^{-\mu}(-Q^j_t Q^j_t + Q^j_\sigma Q^j_\sigma). \tag{31}$$

Near $t \to 0^+$ we have

$$J_0(t) \sim 1 - \frac{t^2}{4} + \cdots\ , \tag{32}$$

$$N_0(t) \sim \left(1 - \frac{t^2}{4}\right)\ln t + \cdots\ ; \tag{33}$$

hence,

$$Q^j_t \sim \frac{1}{t}\left(\int_0^\infty d\omega\omega \mathrm{Re}[B^j(\omega)e^{-i\omega\sigma}]\right) + \text{bounded} \equiv \frac{E^j}{t} + \text{bounded}, \tag{34}$$

$$Q^j_\sigma \sim \ln t\left(\int_0^\infty d\omega\omega \mathrm{Re}[-iB^j(\omega)e^{-i\omega\sigma}]\right) + \text{bounded} \equiv H^j \ln t + \text{bounded}, \tag{35}$$

$$\mu \sim \kappa \ln t\ E^j E^j + \text{bounded}. \tag{36}$$

Inserting Eqs. (34), (35), and (36) into Eq. (31), we have

$$R \sim t^{-\kappa E^j E^j}\left[-\frac{E^j E^j}{t^2} + H^j H^j (\ln t)^2 + \cdots\right]. \tag{37}$$

We conclude that the regularity of the (classical) space-times requires both E^j and H^j to be equal zero, or, equivalently,

$$B^j(\omega) = 0. \tag{38}$$

Consider now (regular) space-times, given by

$$A^j(\omega,\omega_0,\rho) = |a_j|e^{i\phi_j}\sqrt{\frac{\omega}{4\rho}}e^{-\frac{(\omega-\omega_0)^2}{2\rho}},\quad B^j(\omega) = O, \tag{39}$$

where ϕ_j is real and ρ and ω_0 are real positive parameters. Note that, for $\rho \to 0$,

$$A^j(\omega,\omega_0,\rho) \to |a_j|e^{i\phi_j}\sqrt{\frac{\omega_0\pi}{2}}\delta(\omega-\omega_0). \tag{40}$$

From Eq. (25) for $B^j = 0$ we obtain

$$Q^j = \int_0^\infty d\omega \mathrm{Re}\left[|a_j|e^{i\phi_j}\sqrt{\frac{\omega}{4\rho}}e^{-\frac{(\omega-\omega_0)^2}{2\rho}}J_0(\omega t)e^{-i\omega\sigma}\right]. \tag{41}$$

From the well-known formula for the asymptotic behavior of Bessel functions for $t \to \pm\infty$ [27],

$$J_0(\omega t) = \sqrt{\frac{2}{\pi\omega|t|}}\cos\left(\omega t \mp \frac{\pi}{4}\right) + \cdots, \tag{42}$$

we have, for $t \to \pm\infty$,

$$Q^j = \frac{|a_j|}{\sqrt{|t|}}\left\{e^{-\frac{\rho}{2}(t-\sigma)^2}\cos\left[\omega_0(t-\sigma)+\phi_j \mp \frac{\pi}{4}\right] + e^{-\frac{\rho}{2}(t+\sigma)^2}\cos\left[\omega_0(t+\sigma)-\phi_j \mp \frac{\pi}{4}\right]\right\}. \tag{43}$$

Now the physical interpretation of this solution is obvious. At $t \to -\infty$ we have two wave packets propagating against each other by the velocity of light; at $t \to \infty$ the two scattered packets propagate apart from each other with the gained phase shift, indicated in Eq. (43). Because $J_0(\omega t)$ and its derivatives are bounded functions [27], we may use the Riemann-Lebesgue lemma and from Eq. (41) conclude that for t = const and $\sigma \to \pm\infty$, Q^j and all its derivatives with respect to t and σ vanish. Hence, by using the constraints (26) and (27) and the formula (31) for the scalar curvature, we observe that the space-time is flat in this limit. For the cases σ = const, $t \to \pm\infty$, $t+\sigma$=const, $t-\sigma \to \pm\infty$, and $t-\sigma$=const, $t+\sigma \to \pm\infty$, we use the asymptotic expression (43), the constraints (26), (27), and the formula (31) to arrive at the same conclusion. Therefore, for the "wave packet" choice (39) the corresponding space-time is asymptotically flat (see Fig. 1), it has the same topology as the two-dimensional Minkowski space-time, and is free from curvature singularities. We shall not need the explicit

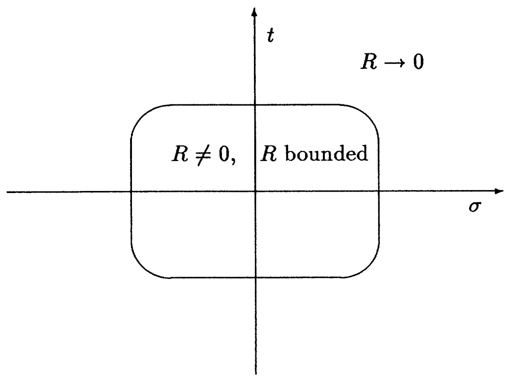

FIG. 1. The scalar curvature of the regular classical space-times.

form of the metric, which, nevertheless, can be obtained by performing the integration (29) with the choice (39). We should end up the classical analysis with some important comments.

First of all, it does not seem unexpected that for collisions of the localized wave packets traveling with the velocity of light, the space-time is asymptotically flat in the spacelike and timelike directions. What looks more surprising is the fact that the same is true for the null infinities. The reason is simple: In two dimensions a *single* propagating wave does not curve the space-time [cf. the analysis after Eq. (17)]. In higher dimensions this is not true [29,30], but in that case the curvature is given by the shape of the wave front in the *transverse* directions. Since there are no transverse directions in two dimensions, our result could be anticipated.

The second comment is closely related to the first one. It concerns the regularity of the initial data. In the higher-dimensional case the following problem was studied [23,30]. If initially regular gravitational waves interact, will a curvature singularity be formed? The criterion for the regularity of the incoming data can be naturally formulated: One requires the boundedness of the amplitude of the wave, which itself is defined by means of the components of the Riemann tensor of the corresponding metric in the so-called parallel propagated orthonormal frame [23,30]. However, in the two-dimensional case, incoming waves do not curve the space-time and this criterion of regularity fails. But certainly we should not consider all incoming waves as regular (with bounded amplitude). One might require that the scalar field Q^j itself should be bounded, but, on the other hand, we could rescale it by an arbitrary function of another scalar field, the dilaton ϕ, and we would get the classically equivalent dynamical theory with the different condition of incoming regularity. Fortunately, it appears to be a natural candidate for the amplitude of the wave. The matter part of the action (9) suggests the following inner product on the space of fields Q [see also Eq. (50)]:

$$(Q_1, Q_2) = \int d^2\xi \sqrt{-g}\phi Q_1 Q_2. \tag{44}$$

Hence, our condition of the incoming regularity reads

$$\lim_{U\to\infty (V\to\infty)} \phi Q^j Q^j = \text{finite}. \tag{45}$$

In the case of the wave packets (39) we get

$$\lim_{U\to\pm\infty} \phi Q^j Q^j = \sum_j |a_j|^2 e^{-2\rho V^2} \cos^2\left[\sqrt{2}\omega_2 V - \phi_j \mp \frac{\pi}{4}\right] = \text{finite} \tag{46}$$

and similarly for $V \to \pm\infty$.

III. QUANTIZATION

A. Functional measures and the effective action

Define the generating functional of the model $W[J_g, J_\phi, J_j]$ by

$$W[J_g, J_\phi, J_j] \equiv \int \frac{D_g g_{\alpha\beta} D_g \phi D_{g,\phi} Q^j}{\text{Vol(Diff)}} \times \exp\left\{ \frac{i}{\hbar} \int d^2\xi \left[\sqrt{-g}\frac{\phi}{2\kappa}(R - \kappa g^{\alpha\beta}\partial_\alpha Q^j \partial_\beta Q^j) + \sqrt{-g} J_g R + J_\phi \phi + J_j Q_j \right] \right\}, \tag{47}$$

where J_g is a scalar source, J_ϕ and J_j are scalar densities, and Vol(Diff) is the volume of the group of diffeomorphisms. We define the functional measures by the norms

$$||\delta g||^2 = \int d^2\xi \sqrt{-g} g^{\alpha\gamma} g^{\beta\delta} \delta g_{\alpha\beta} \delta g_{\gamma\delta}, \tag{48}$$

$$||\delta\phi||^2 = \int d^2\xi \sqrt{-g}\delta\phi^2, \tag{49}$$

$$||\delta Q^j||^2 = \int d^2\xi \sqrt{-g}\phi \delta Q^j \delta Q^j. \tag{50}$$

Equation (48) defines the usual de Witt–Polyakov norm [31], and Eq. (49) gives the standard reparametrization invariant measure for a scalar field. The norm (50) is given by the form of the matter part of the classical action, much in the same way as the norm for the quantization of the standard nonlinear σ model with coordinates X^A of the target and a metric $H_{AB}(X)$, i.e. [32],

$$||\delta X^j||^2 = \int d^2\xi \sqrt{-g} H_{AB}(X(\xi)) \delta X^A(\xi) \delta X^B(\xi). \tag{51}$$

We return to Eq. (47), and we fix the conformal gauge

$$ds^2 = -2e^{\mu}dudv. \tag{52}$$

By using the standard Faddeev-Popov procedure, we obtain

$$W[J] = \int D_\mu\mu D_\mu\phi D_{\mu,\phi}Q^j \exp\left\{i\frac{26}{48\pi}\int d^2\xi\frac{1}{2}\mu\partial^2\mu\right\} \times \exp\left\{\frac{i}{\hbar}\int d^2\xi\left[\frac{\phi}{2\kappa}(-\partial^2\mu - \kappa\partial Q^j\partial Q^j) - J_g\partial^2\mu + J_\phi\phi + J_jQ_j\right]\right\}, \tag{53}$$

where ∂^2 is the Minkowski d'Alembertian and $\partial Q^j\partial Q^j$ means the Minkowski metric scalar product. The Weyl anomaly term comes from the Faddeev-Popov determinant, and the measure $D_\mu\mu$ is given by the norm

$$\|\delta\mu\|^2 = \int d^2\xi e^{\mu}\delta\mu^2. \tag{54}$$

We suppose in a standard manner that the exponential term in the Weyl anomaly is eventually (after including all contributions) canceled by tuning of the 2D cosmological constant counterterm.

Define now the effective action Γ by the prescription

$$\frac{h}{i}\ln W[J] \equiv Z[J] \equiv \Gamma[\mu_c, \phi_c, Q_c^j] - J_g\partial^2\mu_c + J_\phi\phi_c + J_jQ_c^j, \tag{55}$$

where

$$\partial^2\mu_c \equiv -\frac{\delta Z}{\delta J_g}, \quad \phi_c \equiv \frac{\delta Z}{\delta J_\phi}, \quad Q_c^j \equiv \frac{\delta Z}{\delta J_j}. \tag{56}$$

We wish to compute the one-loop effective action Γ_1. In order to do that, we have first to determine Z_1 (the generating functional for the connected Green's functions) from (53) and then to perform the Legendre transformation (55) and (56). We note that the dependences of the measures on the fields μ and ϕ are of $O(\hbar)$ with respect to the classical action in the exponent. Hence, the loop diagrams with the vertices coming from the measures will be of $O(\hbar^2)$ and may be neglected in the one-loop approximation. Therefore, we may write

$$W_{\rm semicl}[J] = \int D_{\mu_J}\mu D_{\mu_J}\phi D_{\mu_J,\phi_J}Q^j \exp\left\{i\frac{26}{96\pi}\int d^2\xi\ \mu_J\partial^2\mu_J\right\} \times \exp\left\{\frac{i}{\hbar}\int d^2\xi\left[\frac{\phi}{2\kappa}(-\partial^2\mu - \kappa\partial Q^j\partial Q^j) - J_g\partial^2\mu + J_\phi\phi + J_jQ^j\right]\right\}, \tag{57}$$

where μ_J and ϕ_J are the saddle point values of the exponent, given by the equations

$$-\partial^2\mu_J - \kappa\partial Q^j\partial Q^j + 2\kappa J_\phi = 0, \tag{58}$$

$$\phi_J + 2\kappa J_g = 0, \tag{59}$$

$$\partial(\phi_J\partial Q_J^j) + J_j = 0. \tag{60}$$

We observe that in the one loop approximation we can consider the measures to be independent of the field integration variables (but, of course, dependent on the Schwinger currents). Now we evaluate the integral (59). In the (second) exponent, there stands the quadratic form in the variables ϕ and μ. Moreover, the norms defining the measures have the same μ_J dependence, i.e.,

$$\|\delta\mu\|^2 = \int d^2\xi e^{\mu_J}\delta\mu^2, \quad \|\delta\phi\|^2 = \int d^2\xi e^{\mu_J}\delta\phi^2. \tag{61}$$

Therefore, we can easily perform the Gaussian integration over μ and ϕ with the result

$$W_{\rm semicl}[J] = \int D_{\mu_J,\phi_J}Q^j \exp\left\{i\frac{24}{96\pi}\int d^2\xi\ \mu_J\partial^2\mu_J\right\}\exp\left\{\frac{i}{\hbar}\int d^2\xi(-2\kappa J_gJ_\phi + \kappa J_g\partial Q^j\partial Q^j + J_jQ^j)\right\}. \tag{62}$$

The integration over Q^j is again Gaussian; hence, we obtain the closed expression for the semiclassical generating functional:

$$W_{\rm semicl}[J_g, J_\phi, J_j] = \det{}^{-N/2}\left[-i\frac{\kappa}{\hbar}e^{-\mu_J}\frac{1}{2\kappa J_g}\partial(J_g\partial)\right] \times \exp\left\{i\frac{24}{96\pi}\int d^2\xi\mu_J\partial^2\mu_J + \frac{i}{\hbar}\int d^2\xi\left(-2\kappa J_gJ_\phi + J_j\frac{1}{4\kappa\partial(J_g\partial)}J_j\right)\right\}, \tag{63}$$

where we used the definition (50) of the measure $D_{\mu,\phi}Q^j$ and Eq. (59). The Legendre transformation (55) and (56) can be easily performed, and we obtain the following expression for the one-loop effective action

$$\Gamma_1(\mu_c,\phi_c,Q_c^j) = \int d^2\xi \frac{\phi_c}{2\kappa}(-\partial^2\mu_c - \kappa\partial Q_c^j\partial Q_c^j) + \hbar\frac{24}{96\pi}\int d^2\xi\mu_c\partial^2\mu_c + i\hbar\frac{N}{2}\ln\det\left[-\frac{i}{2\hbar}e^{-\mu_c}\frac{1}{\phi_c}\partial(\phi_c\partial)\right] + O(\hbar^2). \tag{64}$$

We recognize the classical action and two quantum corrections. The first one is the Weyl anomaly, while the second one is the nonlocal term depending on μ_c and ϕ_c. The expression (64) can also be written in a manifestly covariant way: i.e.,

$$\Gamma_1(g_{c,\alpha\beta},\phi_c,Q_c^j) = \int d^2\xi\sqrt{-g_c}\,\frac{\phi_c}{2\kappa}(R_c - \kappa g_c^{\alpha\beta}\partial_\alpha Q_c^j\partial_\beta Q_c^j) + \hbar\frac{24}{96\pi}\int d^2\xi\sqrt{-g_c}R_c\left(\frac{1}{\sqrt{-g_c}}\partial_\alpha\sqrt{-g_c}g_c^{\alpha\beta}\partial_\beta\right)^{-1}R_c + i\hbar\frac{N}{2}\ln\det\left[-\frac{i}{2\hbar}\frac{1}{\sqrt{-g_c}\phi_c}\partial_\alpha(\sqrt{-g_c}\phi_c g_c^{\alpha\beta}\partial_\beta)\right] + O(\hbar^2). \tag{65}$$

B. Case $N = 24$

For $N = 24$, the quadratic term of the Weyl anomaly vanishes. The dilaton gravities usually become simplified, and more precise results can be obtained in that case [33]. This happens also in our model. We show that the semiclassical effective action Γ_1 (64) is the exact quantum effective action of the theory for $N = 24$. We use the David-Distler-Kawai (DDK) approach [34,35] to establish this result. The dependences of the functional measures on the field μ read

$$D_\mu\mu = (D_0\mu)\exp\left[-\frac{i}{48\pi}\int d^2\xi\frac{1}{2}\mu\partial^2\mu\right], \tag{66}$$

$$D_\mu\phi = (D_0\phi)\exp\left[-\frac{i}{48\pi}\int d^2\xi\frac{1}{2}\mu\partial^2\mu\right], \tag{67}$$

$$D_{\mu,\phi}Q^j = (D_{0,\phi}Q^j) \times\exp\left[-\frac{i}{48\pi}\int d^2\xi\{\tfrac{1}{2}\mu\partial^2\mu + \tfrac{3}{2}\mu[2\partial^2\ln|\phi| + (\partial\ln|\phi|)^2]\}\right]. \tag{68}$$

The formulas (66) and (67) are fairly standard [36,37]; however, the relation (68) deserves some comment. Indeed, it can be explicitly derived by computing the Jacobian, which relates both measures, with some regularization procedure. We shall use the heat kernel regularization and use the defining formula (50) to write

$$D_{\mu,\phi}Q = D_{0,\phi}Q\sqrt{\det L}, \tag{69}$$

where L is the diagonal operator,

$$L(\xi_1,\xi_2) = e^{\mu(\xi_1)}\delta(\xi_1,\xi_2). \tag{70}$$

Note that the δ function $\delta(\xi_1,\xi_2)$ is to be understood in the sense of the scalar product (50) with $\mu = 0$, i.e.,

$$\delta(\xi_1,\xi_2) = \frac{1}{\phi(\xi_1)}\delta(\xi_1 - \xi_2). \tag{71}$$

Clearly

$$\delta\ln\det L = \delta\mathrm{Tr}\ln L = \int d^2\xi\phi(\xi)\delta(\xi,\xi)\delta\mu(\xi), \tag{72}$$

where $\delta(\xi,\xi)$ is a meaningless quantity. As in [36,37], we replace it by the heat kernel of the covariant Laplacian, but in our case with respect to the scalar product (50), i.e.,

$$\delta_\varepsilon(\xi,\xi) = \frac{1}{\phi(\xi)}\left\langle\xi\left|\exp\left\{-\varepsilon\left[-\frac{i}{\hbar}\frac{1}{\sqrt{-g}\phi}\partial_\alpha(\sqrt{-g}\phi g^{\alpha\beta}\partial_\beta)\right]\right\}\right|\xi\right\rangle = \frac{1}{\phi(\xi)}(-i)\left\{\frac{1}{24\pi}\partial^2\mu + \frac{1}{16}[2\partial^2\ln|\phi| + (\partial\ln|\phi|)^2]\right\}. \tag{73}$$

We obtained the last equality in the conformal gauge, by combining the formulas (A1) and (A7) of the Appendix. Now we insert (73) into (72) and in a straightforward way we arrive at the formula (68).

We may also check the validity of the formula (68) for a particular integrand. Indeed, let us compute the integral

$$\int D_{\mu,\phi}Q \exp\left[\frac{i}{2\hbar}\int d^2\xi Q\partial(\phi\partial)Q\right] = \det{}^{-1/2}\left[-\frac{i}{2\hbar}e^{-\mu}\frac{1}{\phi}\partial(\phi\partial)\right]. \tag{74}$$

We have [see Appendix, Eqs. (A8) and (A9)]

$$\det{}^{-1/2}\left[-\frac{i}{2\hbar}e^{-\mu}\frac{1}{\phi}\partial(\phi\partial)\right] = \det{}^{-1/2}\left[-\frac{i}{2\hbar}\frac{1}{\phi}\partial(\phi\partial)\right]\exp\left\{-\frac{i}{48\pi}\int d^2\xi\{\tfrac{1}{2}\mu\partial^2\mu + \tfrac{3}{2}\mu[2\partial^2\ln|\phi| + (\partial\ln|\phi|)^2]\}\right\}. \tag{75}$$

Because

$$\det{}^{-1/2}\left[-\frac{i}{2\hbar}\frac{1}{\phi}\partial(\phi\partial)\right] = \int D_{0,\phi}Q\exp\left[\frac{i}{2\hbar}\int d^2\xi Q\partial(\phi\partial)Q\right], \tag{76}$$

Eqs. (74), (75), and (76) obviously match with the formula (68).

After this digression, we now compute the effective action for the case $N = 24$. We use the defining formula (53) for the generating functional in the conformal gauge and insert the field dependences of the measures (66), (67), and (68) in it. We obtain

$$W[J] = \int D_0\mu\, D_0\phi\, D_{0,\phi}Q^j \exp\left\{-\frac{24i}{32\pi}\int d^2\xi\mu[2\partial^2\ln|\phi_c| + (\partial\ln|\phi_c|)^2]\right\}$$
$$\times\exp\left\{\frac{i}{\hbar}\int d^2\xi\left[\frac{\phi}{2\kappa}(-\partial^2\mu - \kappa\partial Q^j\partial Q^j) - J_g\partial^2\mu + J_\phi\phi + J_jQ_j\right]\right\}. \tag{77}$$

The integration over Q^j is Gaussian and over μ gives the δ function; therefore,

$$W[J] = \int D_0\phi\,\delta\left(\partial^2\phi + 2\kappa\partial^2J_g + \frac{3\hbar\kappa}{2\pi}[2\partial^2\ln|\phi| + (\partial\ln|\phi|)^2]\right)$$
$$\times\det{}^{-12}\left[-\frac{i}{2\hbar}\frac{1}{\phi}\partial(\phi\partial)\right]\exp\left[\frac{i}{\hbar}\int d^2\xi\left(J_\phi\phi - J_j\frac{1}{2\partial(\phi\partial)}J_j\right)\right]$$
$$= \det{}^{-12}\left[-\frac{i}{2\hbar}\frac{1}{\phi(J_g)}\partial(\phi(J_g)\partial)\right]\exp\left[\frac{i}{\hbar}\int d^2\xi\left(J_\phi\phi(J_g) - J_j\frac{1}{2\partial(\phi(J_g)\partial)}J_j\right)\right], \tag{78}$$

where the dependence of $\phi(J_g)$ on J_g is dictated by the δ function in (78). We stress that the formula (78) gives the *exact* generating functional. Performing the Legendre transformation (55) and (56) we obtain the *exact* effective action

$$\Gamma(\mu_c,\phi_c,Q_c^j) = \int d^2\xi\frac{\phi_c}{2\kappa}(-\partial^2\mu_c - \kappa\partial Q_c^j\partial Q_c^j)$$
$$-\frac{3\hbar}{4\pi}\int d^2\xi\mu_c[2\partial^2\ln|\phi_c| + (\partial\ln|\phi_c|)^2] + 12i\hbar\ln\det\left[-\frac{i}{2\hbar}\frac{1}{\phi_c}\partial(\phi_c\partial)\right]. \tag{79}$$

Comparing the result with Eq. (64), we conclude that for $N = 24$ the semiclassical approximation is, in fact, exact.

IV. QUANTUM CURVATURE SINGULARITIES

A. Semiclassical field equations

We obtain the semiclassical field equations by varying the one loop effective action Γ_1 with respect to the classical fields μ_c, ϕ_c, and Q_c^j. We have

$$-\frac{1}{2\kappa}\partial^2\phi_c + \hbar\frac{24}{48\pi}\partial^2\mu_c$$
$$+\hbar\frac{iN}{2}\frac{\delta}{\delta\mu_c}\ln\det\left[-\frac{i}{2\hbar}e^{-\mu_c}\frac{1}{\phi_c}\partial(\phi_c\partial)\right] = 0, \tag{80}$$

$$\partial(\phi_c\partial Q_c^j) = 0, \tag{81}$$

$$-\partial^2\mu_c - \kappa\partial Q_c^j\partial Q_c^j$$
$$+\hbar i\kappa N\frac{\delta}{\delta\phi_c}\ln\det\left[-\frac{i}{2\hbar}e^{-\mu_c}\frac{1}{\phi_c}\partial(\phi_c\partial)\right] = 0. \tag{82}$$

The solutions of the semiclassical equations have the expansions

$$\mu_c = \mu_{c,0} + \hbar\mu_{c,1} + O(\hbar^2), \tag{83}$$

$$\phi_c = \phi_{c,0} + \hbar\phi_{c,1} + O(\hbar^2), \tag{84}$$

$$Q_c^j = Q_{c,0}^j + \hbar Q_{c,1}^j + O(\hbar^2). \tag{85}$$

Because we know just the first loop corrections to the effective action, the $O(\hbar^2)$ terms in the field expansions (83), (84), and (85) are irrelevant in the one-loop approximation. Our next task will consist of the determination of $\mu_{c,1}$, $\phi_{c,1}$, and $Q_{c,1}^j$ from the semiclassical equations (80), (81), and (82), when $\mu_{c,0}$, $\phi_{c,0}$, and $Q_{c,0}^j$ is a given classical solution. Since the "ln det" terms in the one-loop field equations are already of $O(\hbar)$, it is enough to compute the functional derivatives of ln det at the *classical* solution $\mu_{c,0}$ and $\phi_{c,0}(= t)$. The actual calculation requires knowledge of the heat kernels of elliptic operators, some theory of the Hankel transformations, and some integration of the Bessel functions. The details are presented in the Appendix; here we list only the final result:

$$\frac{\delta}{\delta\mu_c}\ln\det\left[-\frac{i}{2\hbar}e^{-\mu_c}\frac{1}{\phi_c}\partial(\phi_c\partial)\right]$$

$$= \frac{i}{24\pi}\partial^2\mu_c + \frac{i}{16\pi}[2\partial^2\ln|\phi_c| + (\partial\ln|\phi_c|)^2], \tag{86}$$

$$\frac{\delta}{\delta\phi_c(\xi)}\ln\det\left[-\frac{i}{2\hbar}e^{-\mu_c}\frac{1}{\phi_c}\partial(\phi_c\partial)\right]\Bigg|_{\phi_c=t}$$

$$= \frac{i}{8\pi t}[\partial^2\mu_c - \partial(\mu_c\partial\ln|t|)] - \frac{i}{8\pi t^3}\ln\left(\frac{\hbar t^2}{\varepsilon\Omega}\right), \tag{87}$$

where Ω is a finite constant and $\varepsilon \to 0$ is the ultraviolet cutoff. The following counterterm is needed for the cancelation of the UV divergence:

$$\mathcal{L}_{\text{count}} \sim \hbar\sqrt{-g}g^{\alpha\beta}\phi^{-2}\partial_\alpha\phi\partial_\beta\phi. \tag{88}$$

Its appropriate tuning replaces the product $\varepsilon\Omega$ in the semiclassical equations by some finite constant, but it cannot remove the logarithmical dependence on t^2.

Inserting the evaluated functional derivatives (86) and (87) and the $\hbar$ expansion (83), (84), and (85) of the fields into the semiclassical equations (80), (81), and (82), we obtain the equations for $\mu_{c,1}$, $\phi_{c,1}$, and $Q_{c,1}^j$

$$-\frac{1}{2\kappa}\partial^2\phi_{c,1} + \frac{24-N}{48\pi}\partial^2\mu_{c,0} - \frac{N}{32\pi t^2} = 0, \tag{89}$$

$$\partial(\phi_{c,1}\partial Q_{c,0}^j) + \partial(t\partial Q_{c,1}^j) = 0, \tag{90}$$

$$(-\partial^2\mu_{c,1} - 2\kappa\partial Q_{c,1}^j\partial Q_{c,0}^j) = \frac{N\kappa}{8\pi}\left(\frac{1}{t}\partial^2\mu_{c,0} + \frac{1}{t^2}\partial_t\mu_{c,0} - \frac{1}{t^3}\mu_{c,0} - \frac{1}{t^3}\ln\frac{\hbar t^2}{\text{const}}\right). \tag{91}$$

This system of equations can be solved in a similar way as the classical system (12), (13), and (14) was solved. Indeed, because we know the Green's function of the Minkowski d'Alembertian, we find from Eq. (89) the general form of $\phi_{c,1}$, by adding an arbitrary solution of the homogeneous equation to one particular solution of the full equation. Inserting $\phi_{c,1}$ into (90), we obtain the linear (inhomogeneous) Gowdy equation for $Q_{c,1}^j$. Since we know the eigenvalues and the eigenfunctions of the Gowdy operator, we know also its Green's function and, eventually, the general form of $Q_{c,1}^j$. Finally, putting $Q_{c,1}^j$ into (91), we obtain the linear inhomogeneous d'Alembertian equation for $\mu_{c,1}$, the general solution of which can be easily found. We conclude that our semiclassical equations (89), (90), and (91) are exactly solvable. For our purposes, there is no need to write down the explicit (and somewhat cumbersome) formulas. Instead of that we shall concentrate on the behavior of the general solution near $t \sim 0$. We shall show, somewhat surprisingly, that even when we consider a regular classical solution $\mu_{c,0}$, $\phi_{c,0}$, and $Q_{c,0}^j$, the corresponding solution $\mu_{c,1}$, $\phi_{c,1}$, and $Q_{c,1}^j$ possesses necessarily a curvature singularity at $t = 0$. Such space-times are therefore classically regular, but the quantum fluctuations induce the scalar curvature singularity, proportional to $\hbar$. Hence, quantum effects not only do not smear the classical curvature singularities, they even destabilize the regular space-times. We present the corresponding quantitative analysis in the next subsection.

B. Scalar curvature of the semiclassical space-times

Let us study the behavior of the scalar curvature R near $t \sim 0$ for the space-times which solve the semiclassical field equations. In this subsection we omit the index c of the fields μ_c, ϕ_c, and Q_c^j. We choose a classical metric field μ_0 such that the classical space-time is nonsingular. From Eqs. (25) and (29) for $B^j = 0$, it follows, for $t \sim 0$,

$$\mu_0 \sim t^2 f(\sigma) + \cdots, \tag{92}$$

$$Q_0 = g(\sigma) + t^2 h(\sigma) + \cdots, \tag{93}$$

where $f(\sigma)$, $g(\sigma)$, and $h(\sigma)$ are functions, the concrete forms of which are not relevant for our purposes, and the ellipses denote the subleading terms, also irrelevant for our analysis of the curvature singularities. Hence, from Eq. (89) we find the behavior of ϕ_1 near $t \sim 0$:

$$\phi_1 \sim -\frac{N\kappa}{16\pi} \ln|t| + \text{const} \times f(\sigma)t^2 + F(U) + G(V) + \cdots. \tag{94}$$

The functions $F(U)$ and $G(V)$ cannot be specified from this equation; however, we may change our "dilaton" gauge condition (19) by the prescription

$$t + \hbar[F(U) + G(V)] \to t. \tag{95}$$

The one-loop effective action (65) is invariant under this transformation; hence, the semiclassical equations (80), (81), and (82) remain unchanged. Moreover, the classical solution at which the functional derivatives of the determinants are to be evaluated changes just by the terms of order $\hbar$. This effect, of course, remains unseen in the one-loop approximation, because the "ln det" terms are already of the first order in the $\hbar$-expansion. Thus, all subsequent analysis goes through and we can omit the $F(U) + G(V)$ term and write, without a loss of generality,

$$\phi_1 \sim -\frac{N\kappa}{16\pi} \ln|t| + \text{const} \times t^2 + \cdots . \tag{96}$$

Now we insert ϕ_1 into Eq. (90). We obtain

$$\left(-\frac{\partial^2}{\partial t^2} - \frac{1}{t}\frac{\partial}{\partial t} + \frac{\partial^2}{\partial \sigma^2}\right) Q_1 = \frac{H(\sigma)}{t} \ln|t| - \frac{N\kappa h(\sigma)}{8\pi t} + \text{bounded}, \tag{97}$$

where $H(\sigma)$ is some function of irrelevant shape. It is easy to determine the behavior of a particular solution of (97) near $t \sim 0$. It is given by

$$Q_{1,\text{par}} = -H(\sigma)t \ \ln|t| + \left[2H(\sigma) + \frac{N\kappa h(\sigma)}{8\pi}\right] t + \cdots . \tag{98}$$

A general solution of the Q_1 equation near $t \sim 0$ is then given by Eq. (98) plus an arbitrary solution (25) of the homogeneous Gowdy equation. From (98), (25), and (93) then it easily follows that

$$\partial Q_1^j \partial Q_0^j \sim C(\sigma) \ln|t| + \text{bounded}. \tag{99}$$

The function $C(\sigma)$ vanishes if the Neumann modes are absent in the "homogeneous" part of Q_1. Inserting (99) into the remaining semiclassical evolution equation (91), we obtain

$$-\partial^2 \mu_1 = 2\kappa C(\sigma) \ln|t| - \frac{N\kappa}{8\pi}\left[\frac{f(\sigma)}{t} + \frac{1}{t^3} \ln \frac{\hbar t^2}{\text{const}}\right] + \text{bounded}; \tag{100}$$

hence

$$\mu_1 = -\frac{N\kappa}{16\pi t} \ln \frac{\hbar t^2}{\text{const}} + \rho(U) + \nu(V) + \text{bounded}. \tag{101}$$

In the classical case, the arbitrary integration functions $\rho(U)$ and $\nu(V)$ are determined from the constraints (20) and (21). In the semiclassical case they have to be determined from the given boundary conditions. The situation is fully analogous to that occurring in the CGHS model [4], where the semiclassical contribution to the constraints come from the Polyakov nonlocal action

$$S_P = +\hbar\frac{24 - N}{96\pi} \int d^2\xi \sqrt{-g} R \times \left(\frac{1}{\sqrt{-g}} \partial_\alpha \sqrt{-g} g^{\alpha\beta} \partial_\beta\right)^{-1} R_c. \tag{102}$$

We can make the generally covariant action (102) local at the cost of introducing a new auxiliary field Z (in a similar but not identical way as in [46]). It reads

$$S_P = \hbar\frac{24 - N}{48\pi} \int d^2\xi \sqrt{-g} \left[\tfrac{1}{2} g^{\alpha\beta} \partial_\alpha Z \partial_\beta Z + RZ\right]. \tag{103}$$

The contributions to the constraints are then obtained from δg^{uu} and δg^{vv} variations of the action, in the conformal gauge. We have

$$\frac{1}{\sqrt{-g}} \frac{\delta S_P}{\delta g^{uu}} = \hbar\frac{24 - N}{48\pi} (\tfrac{1}{2} Z_u Z_u - Z_{uu} + \mu_u Z_u), \tag{104}$$

$$\frac{1}{\sqrt{-g}} \frac{\delta S_P}{\delta g^{vv}} = \hbar\frac{24 - N}{48\pi} (\tfrac{1}{2} Z_v Z_v - Z_{vv} + \mu_v Z_v). \tag{105}$$

We get rid of the auxiliary field Z using the equation of motion

$$\partial^2(\mu + Z) = 0; \tag{106}$$

hence

$$\mu = -Z + \mu^+(u) + \mu^-(v) \tag{107}$$

and

$$\frac{1}{\sqrt{-g}} \frac{\delta S_P}{\delta g^{uu}} = \hbar\frac{24 - N}{48\pi} (\mu_{uu} - \tfrac{1}{2}\mu_u \mu_u) + T^+(u), \tag{108}$$

$$\frac{1}{\sqrt{-g}} \frac{\delta S_P}{\delta g^{vv}} = \hbar\frac{24 - N}{48\pi} (\mu_{vv} - \tfrac{1}{2}\mu_v \mu_v) + T^-(v). \tag{109}$$

The functions $T^+(u)$ and $T^-(v)$ are undetermined, because $\mu^+(u)$ and $\mu^-(v)$ are not known.

In our present model the Polyakov nonlocal action is a part of the one-loop semiclassical action [cf. (65)]. Therefore, the unknown functions $\rho(U)$ and $\nu(V)$ can be specified only by fixing the boundary conditions. The δg^{uu} and δg^{vv} variations of the remaining ln det part of the effective action of our model cannot influence this conclusion, and we shall not consider them.

Finally we are ready to write down the scalar curvature of the semiclassical space-times. It reads [see Eq. (16)]

$$\begin{aligned} R(\hbar) &= -e^{-\mu(\hbar)}\partial^2\mu(\hbar) = R(0) + \hbar[-R(0)\mu_1 - e^{-\mu_0}\partial^2\mu_1] + O(\hbar^2) \\ &= R(0) + \hbar\left\{R(0)\frac{N\kappa}{16\pi t}\ln\frac{\hbar t^2}{\text{const}} - R(0)[\rho(U)+\nu(V)]\right\} \\ &\quad +\hbar\left\{-e^{-\mu_0}2\kappa C(\sigma)\ln|t| + e^{-\mu_0}\frac{N\kappa}{8\pi t}\left(f(\sigma)+\frac{1}{t^2}\ln\frac{\hbar t^2}{\Omega}\right)\right\} + \text{bounded} + O(\hbar^2). \end{aligned} \tag{110}$$

Clearly, whatever the functions $\rho(U)$ and $\nu(V)$ may be, the semiclassical space-time is obviously singular. The singularity occurs at $t = 0$, and all timelike observers will run into it. We arrived at the remarkable conclusion that while at the classical level there existed the nonsingular space-times, at the semiclassical level *all* space-times are necessarily singular. From Eqs. (37) and (110) we also learn that the singular behavior of classical and quantum curvatures is different; hence no cancellation of a classical curvature singularity due to quantum effects may occur.[2] If the classical space-time is regular, then the formula (110) says that the corresponding semiclassical space-time is plagued by a curvature singularity proportional to $\hbar$. Schematically

$$R = \text{regular} + \hbar \text{ singular} \cdots . \tag{111}$$

We conclude that the quantum effects destabilize the classical space-times and lead to even more severe curvature singularities than the classical dynamics does. There remains only one possibility to avoid this conclusion in the framework of the present model, which may seem quite unnatural, however. It consists in introducing by hand into the effective action several *finite* counterterms of a new type, which would be fine-tuned to cancel the divergent terms in (100). But also keeping this possibility in mind we may conclude that the quantum instabilities in our model are generic.

V. CONCLUSIONS AND OUTLOOK

We attempted to give a detailed description of the classical and quantum dynamics of the Jackiw-Teitelboim gravity with the cosmological constant replaced by the kinetic term of matter fields. We showed that the classical solutions of the model have a natural physical interpretation: Namely, they describe the collisions of the wave packets of matter. For a huge class of such solutions the corresponding classical space-times are topologically trivial, asymptotically flat, and free of curvature singularities. Then we computed the semiclassical effective action of the model; for the case $N = 24$ we, in fact, obtained the exact expression. The effective field equations turned out to be manageable from the technical point of view. We have solved them and provided a simple analysis of the semiclassical solutions near $t \sim 0$. A surprising result followed: The scalar curvature acquires the quantum correction which is necessarily singular. Hence, quantum fluctuations do not smear classical curvature singularities; in fact, they do just the opposite: They plague the regular classical space-times with quantum curvature singularities. Because for $N = 24$ we obtained a result starting from the exact effective action, our conclusion does not seem to be an artifact of the semiclassical approximation.

We believe that the model which we investigated is also interesting from a field theoretical point of view. At the classical level it is completely integrable and iteratively linear in the sense of Sec. II B. This kind of "linearity" played a decisive role in the evaluation of the continual integral, in a similar way as was reported recently in the context of 2+1 Chern-Simons theory [38]. The fact that for $N = 24$ that computation gives an exact result suggests the existence of a deeper *algebraic* structure in the model.[3] Moreover, it turns out that the model possesses an unexpected and interesting *geometrical* structure. Indeed, the action (7) with the included matter field can be interpreted as the Jackiw-Teitelboim action (without a cosmological constant) in the noncommutative geometry of the "two sheet" manifolds $Y \times Z_2$, where Y is the 2D space-time and Z_2 is the internal space containing just two points [39,40]. The matter field plays the geometrical role of the distance between the two points in internal space.

Our present model also has connections to string theory on curved backgrounds and to exact 2D conformal field theories. Indeed, in the conformal gauge, the classical action reads

$$S = -\frac{1}{2\kappa}\int d^2\xi(-\partial\mu\partial\phi + \kappa\phi\partial Q^j\partial Q^j). \tag{112}$$

[2] I am grateful to R. Jackiw for a comment on this point.

[3] This comment is due to K. Gawędzki.

This is obviously an action of the nonlinear σ model where μ, ϕ, and Q^j are the coordinates of the target manifold with the metric

$$ds^2 = -d\mu d\phi + \kappa\phi \, dQ^j dQ^j. \tag{113}$$

It is not difficult to see that the metric (113) (it is written in the so-called Rosen coordinates) describes a *single* gravitational plane wave propagating on a $(N+2)$-dimensional target [29,30]. In other words, a *single* gravitational wave in a $(N+2)$-dimensional target yields the σ-model action describing collisions of the *two* gravitational waves in two dimensions. Generalizing this work [41] Brooks has shown that by adding the target dilaton background in the critical target dimension to such a σ model, one obtains an exact conformal field theory [42]. It would be interesting to study our present model from this point of view. All the mentioned features of the model look quite promising for further investigations, and we shall certainly return to those problems elsewhere.

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APPENDIX

In this appendix, we evaluate the functional derivatives of the determinant in (64), which were needed for obtaining the explicit form of the semiclassical field equations. In evaluating the traces we carefully keep in mind the definition of the scalar product (50).

Start with the derivative with respect to μ_c:

$$\begin{aligned}\frac{\delta}{\delta\mu_c(\xi)}\ln\det\left[-\frac{i}{2\hbar}e^{-\mu_c}\frac{1}{\phi_c}\partial(\phi_c\partial)\right] &= \frac{\delta}{\delta\mu_c(\xi)}(-1)\int_\varepsilon^\infty\frac{d\tau}{\tau}\mathrm{Tr}\exp\left\{-\tau\left[-\frac{i}{2\hbar}e^{-\mu_c}\frac{1}{\phi_c}\partial(\phi_c\partial)\right]\right\}\\ &= \int_\varepsilon^\infty d\tau\left\langle\xi\left|\frac{i}{2\hbar}e^{-\mu_c}\frac{1}{\phi_c}\partial(\phi_c\partial)\exp\left\{-\tau\left[-\frac{i}{2\hbar}e^{-\mu_c}\frac{1}{\phi_c}\partial(\phi_c\partial)\right]\right\}\right|\xi\right\rangle\\ &= -\left\langle\xi\left|\exp\left\{-\varepsilon\left[-\frac{i}{2\hbar}e^{-\mu_c}\frac{1}{\phi_c}\partial(\phi_c\partial)\right]\right\}\right|\xi\right\rangle.\end{aligned} \tag{A1}$$

We have used the heat kernel regularization [43,44] based on the representation

$$\ln x = -\int_\varepsilon^\infty\frac{d\tau}{\tau}e^{-\tau x} + (\text{an } x \text{ independent constant}) + O(\varepsilon x). \tag{A2}$$

The "bras" and "kets" in Eq. (A1) have to be understood in the standard sense. The asymptotic expression for the heat kernel for small ε was obtained for an arbitrary elliptic operator in two dimensions [44]. If M has the form

$$M = -\frac{1}{\sqrt{-g}}(\nabla_\alpha + B_\alpha)\sqrt{-g}g^{\alpha\beta}(\nabla_\beta + B_\beta) - B_0, \tag{A3}$$

then[4]

$$\langle\xi|e^{-M\varepsilon}|\xi\rangle = \frac{1}{4\pi\varepsilon}\sqrt{-g} + \frac{1}{24\pi}R\sqrt{-g} + \frac{1}{4\pi}B_0\sqrt{-g} + O(\varepsilon). \tag{A4}$$

Performing the Wick rotation to the Minkowski time, we can use Eq. (A4) for evaluating the heat kernel (A1). In our case

$$B_\alpha = \tfrac{1}{2}\partial_\alpha\ln|\phi_c|, \tag{A5}$$

$$B_0 = e^{-\mu_c}\left[-\tfrac{1}{2}\partial^2\ln|\phi_c| - \tfrac{1}{4}(\partial\ln|\phi_c|)^2\right], \tag{A6}$$

hence

$$\frac{\delta}{\delta\mu_c}\ln\det\left[-\frac{i}{2\hbar}e^{-\mu_c}\frac{1}{\phi_c}\partial(\phi_c\partial)\right] = i\frac{1}{24\pi}\partial^2\mu_c + \frac{i}{16\pi}\left[2\partial^2\ln|\phi_c| + (\partial\ln|\phi_c|)^2\right]. \tag{A7}$$

Note that the functional derivative with respect to μ_c is the local expression. We also did not consider the first term on the RHS of Eq. (A4), which is eventually to be canceled by the two-dimensional cosmological constant counterterm.

Next we compute the variation with respect to ϕ_c. First of all we note that Eq. (A7) implies

$$\ln\det\left[-\frac{i}{2\hbar}e^{-\mu_c}\frac{1}{\phi_c}\partial(\phi_c\partial)\right] = i\int d^2\xi\left(\frac{1}{48\pi}\mu_c\partial^2\mu_c + \frac{1}{16\pi}\mu_c\left[2\partial^2\ln|\phi_c| + (\partial\ln|\phi_c|)^2\right]\right) + F(\phi_c), \tag{A8}$$

[4]We put the sign + in front of R [see also Alvarez [45], Eq. (4.38)], because R is given by Eq. (16).

where $F(\phi_c)$ is some μ_c-independent functional. But Eq. (A8) itself gives

$$\ln\det\left[-\frac{i}{2\hbar}\frac{1}{\phi_c}\partial(\phi_c\partial)\right] = F(\phi_c). \tag{A9}$$

Hence,

$$\frac{\delta}{\delta\phi_c(\xi)}\ln\det\left[-\frac{i}{2\hbar}e^{-\mu_c}\frac{1}{\phi_c}\partial(\phi_c\partial)\right] = \frac{i}{8\pi\phi_c}[\partial^2\mu_c - \partial(\mu_c\partial\ln|\phi_c|)] + \frac{\delta}{\delta\phi_c(\xi)}\ln\det\left\{-\frac{i}{2\hbar}[\partial^2 + (\partial\ln|\phi_c|)\partial]\right\}. \tag{A10}$$

Now we have

$$\begin{aligned}\delta\ln\det\left\{-\frac{i}{2\hbar}[\partial^2 + (\partial\ln|\phi_c|)\partial]\right\}\Bigg|_{\phi_c=t} &= -\delta\int_\epsilon^\infty \frac{d\tau}{\tau}\mathrm{Tr}\exp\left\{-\tau\left[-\frac{i}{2\hbar}[\partial^2+(\partial\ln|\phi_c|)\partial]\right]\right\}\Bigg|_{\phi_c=t}\\ &= \int_\epsilon^\infty d\tau\,\mathrm{Tr}\left\{-\frac{i}{2\hbar}\left(\partial\frac{\delta\phi_c}{t}\right)\partial\exp\left\{\tau\left[\frac{1}{2\hbar}(\alpha-i)\left(\partial_t^2+\frac{1}{t}\partial_t\right)+\frac{1}{2\hbar}(\alpha+i)\partial_\sigma^2\right]\right\}\right\},\end{aligned} \tag{A11}$$

where $\alpha > 0$ is a (small) "Euclidean" cutoff, damping the oscillatory behavior of the exponent. Now we wish to evaluate the last trace in Eq. (A11). Form the basis of the space of fields Q as follows

$$\Psi_{\pm,k,p}(t,\sigma) = \theta(\pm t)J_0(kt)\frac{1}{\sqrt{2\pi}}e^{-ip\sigma}, \quad k>0, \tag{A12}$$

where $\theta(t)$ is the usual step function. Using the theory of the Hankel transformations [27], it is easy to establish the relations of orthogonality:

$$\int_{R^2} dtd\sigma\ t\ \Psi^*_{\pm k,p}(t,\sigma)\Psi_{\pm k',p'}(t,\sigma) = (-1)^{\frac{\pm1-1}{2}}\frac{1}{k}\delta(k-k')\delta(p-p'), \tag{A13}$$

$$\int_{R^2} dtd\sigma\ t\ \psi^*_{\pm k,p}(t,\sigma)\psi_{\mp k',p'}(t,\sigma) = 0, \tag{A14}$$

and the relation of completeness

$$\int_0^\infty kdk\int_{-\infty}^\infty dp[\Psi^*_{+kp}(\xi)\Psi_{+kp}(\xi') - \Psi^*_{-kp}(\xi)\Psi_{-kp}(\xi')] = \frac{1}{t}\delta(t-t')\delta(\sigma-\sigma'). \tag{A15}$$

[Note that on the RHS of Eq. (A15) stands the δ function $\delta(\xi,\xi')$ with respect to the inner product (50) for $\mu = 0$ and $\phi = t$.] Therefore, the trace of an operator O is given by

$$\mathrm{Tr}O = \int_{R^2} dtd\sigma t\int_0^\infty kdk\int_{-\infty}^\infty dp[\Psi^*_{+kp}(\xi)O\Psi_{+kp}(\xi) - \Psi^*_{-kp}(\xi)O\Psi_{-kp}(\xi)]. \tag{A16}$$

Using Eq. (A16), we can easily evaluate the trace in Eq. (A11). We have

$$\begin{aligned}\delta\ln\det\left\{-\frac{i}{2\hbar}[\partial^2 + (\partial\ln|\phi_c|)\partial]\right\}\Bigg|_{\phi_c=t} &= \int_\epsilon^\infty d\tau\int_{R^2} dtd\sigma t\int_0^\infty kdk\int_{-\infty}^\infty dp\exp\left\{-\tau\left[\frac{1}{2\hbar}(\alpha-i)k^2+\frac{1}{2\hbar}(\alpha+i)p^2\right]\right\}\\ &\quad\times\left\{\Psi^*_{+kp}(\xi)\left[-\frac{i}{2\hbar}\left(\partial\frac{\delta\phi_c}{t}\right)\partial\right]\Psi_{+kp}(\xi) - \Psi^*_{-kp}(\xi)\left[-\frac{i}{2\hbar}\left(\partial\frac{\delta\phi_c}{t}\right)\partial\right]\Psi_{-kp}(\xi)\right\}\\ &= \int_\epsilon^\infty d\tau\int_{R^2} dtd\sigma t\int_0^\infty kdk\int_{-\infty}^\infty dp\exp\left\{-\frac{\tau}{2\hbar}[(\alpha-i)k^2+(\alpha+i)p^2]\right\}\\ &\quad\times\frac{i}{2\hbar}\left(\partial_t\frac{\delta\phi_c}{t}\right)\frac{1}{2}\partial_t[\Psi^*_{+kp}(\xi)\Psi_{+kp}(\xi) - \Psi^*_{-kp}(\xi)\Psi_{-kp}(\xi)].\end{aligned} \tag{A17}$$

Now the formula (see [27])

$$\int_0^\infty r dr e^{-\rho^2 r^2} J_0(\lambda r) J_0(\mu r) = \frac{1}{2\rho^2} e^{-\frac{(\lambda^2+\mu^2)}{4\rho^2}} J_0\left(\frac{i\lambda\mu}{2\rho^2}\right) \tag{A18}$$

and the standard Gaussian integration explicitly give the integrals over k and p. We obtain (for $t \neq 0$)

$$\delta \ln\det\left\{-\frac{i}{2\hbar}\left[\partial^2 + (\partial \ln|\phi_c|)\partial\right]\right\}\Bigg|_{\phi_c=t}$$

$$= \int_\epsilon^\infty d\tau \int_{R^2} dt d\sigma t \frac{i}{4\hbar}\partial_t\left(\frac{\delta\phi_c}{t}\right)\sqrt{\frac{\hbar}{(\alpha+i)2\tau\pi}}\partial_t\left[\frac{\hbar}{(\alpha-i)\tau}e^{-\frac{\hbar t^2}{(\alpha-i)\tau}}J_0\left(\frac{i\hbar t^2}{\tau(\alpha-i)}\right)\left[\theta^2(t)-\theta^2(-t)\right]\right]. \tag{A19}$$

We can rewrite Eq. (A19) as

$$\frac{\delta}{\delta\phi_c(t,\sigma)}\ln\det\left\{-\frac{i}{2\hbar}\left[\partial^2 + (\partial \ln|\phi_c|)\partial\right]\right\}\Bigg|_{\phi_c=t}$$

$$= -[\theta(t)-\theta(-t)]\int_0^{\frac{\hbar}{\epsilon}} d\rho \rho^{-1/2}\frac{i}{4\sqrt{2\pi(\alpha-i)}}\frac{1}{t}\partial_t\left[t\partial_t\left(e^{-\frac{\rho t^2}{\alpha-i}}J_0\left(\frac{i\rho t^2}{\alpha-i}\right)\right)\right]$$

$$= -[\theta(t)-\theta(-t)]\frac{i}{4\sqrt{2\pi(\alpha-i)}}\frac{1}{t}\partial_t t\partial_t\left(\frac{1}{|t|}\int_0^{\frac{\hbar t^2}{\epsilon}} d\rho\rho^{-1/2}e^{-\frac{\rho}{\alpha-i}}J_0\left(\frac{i\rho}{\alpha-i}\right)\right)$$

$$= -\frac{i}{4\sqrt{2\pi(\alpha-i)}}\frac{1}{t}\partial_t t\partial_t\frac{1}{t}\int_0^{\hbar t^2/\epsilon} d\rho\rho^{-1/2}e^{-\frac{\rho}{\alpha-i}}J_0\left(\frac{i\rho}{\alpha-i}\right). \tag{A20}$$

Now we can decompose the integral over ρ:

$$\int_0^{\hbar t^2/\varepsilon}\frac{d\rho}{\sqrt{\rho}}e^{-\frac{\rho}{\alpha-i}}J_0\left(\frac{i\rho}{\alpha-i}\right) = \text{const} + \int_1^{\hbar t^2/\varepsilon}\frac{d\rho}{\sqrt{\rho}}e^{-\frac{\rho}{\alpha-i}}\left[J_0\left(\frac{i\rho}{\alpha-i}\right)-\sqrt{\frac{\alpha-i}{2\pi\rho}}e^{\frac{\rho}{\alpha-i}}\right] + \int_1^{\hbar t^2/\varepsilon}\frac{d\rho}{\rho}\sqrt{\frac{\alpha-i}{2\pi}}. \tag{A21}$$

From the asymptotic behavior of the Bessel functions [27], we conclude that the first integral on the RHS of Eq. (A21) is convergent for $\varepsilon \to 0$. Hence

$$\frac{\delta}{\delta\phi_c(t,\sigma)}\ln\det\left\{-\frac{i}{2\hbar}\left[\partial^2 + (\partial \ln|\phi_c|)\partial\right]\right\}\Bigg|_{\phi_c=t} = -\frac{i}{8\pi}\frac{1}{t}\,\partial_t\, t\,\partial_t\frac{1}{t}\ln\left(\frac{\hbar t^2}{\varepsilon\ \text{const}}\right) = -\frac{i}{8\pi t^3}\ln\left(\frac{\hbar t^2}{\varepsilon\Omega}\right), \tag{A22}$$

where Ω is a finite constant.

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