Exponential-potential scalar field universes. II. Inhomogeneous models

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We obtain exact solutions for the Einstein equations with an exponential-potential scalar field ($V = \Lambda e^{k\phi}$) which represent simple inhomogeneous generalizations of Bianchi type I cosmologies. Studying these equations numerically we find that in most of the cases there is a certain period of inflationary behavior for $k^2 < 2$. We as well find that for $k^2 > 2$ the solutions homogenize generically at late times. Yet, *none of the solutions* isotropize. For some particular values of the integration constants we find a multiple inflationary behavior for which the deceleration and the inflationary phases interchange with each other several times during the history of the model.

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I. INTRODUCTION

In our previous paper [1] (hereafter paper I) we have obtained a general exact solution describing anisotropic Bianchi type I universes filled with an exponential-potential scalar field and studied their behavior. These studies are relevant in order to clarify how sensitive inflationary cosmologies are to the preinflationary epoch characterized by different initial conditions. For the case of Bianchi type I models we confirm previous results based on numerical, approximate, and qualitative techniques obtained by various authors [2] predicting a power-law inflationary behavior for a wide range of initial conditions.

Some of our latest studies [3,4] with more complicated geometries, however, cast the suspicion that the inflationary phenomena is not that generic and probably requires some special initial conditions. The Bianchi type I cosmological models, studied in our previous paper, are too simple to derive definite conclusions related to this question. Moreover, if one assumes [5] the existence of the gravitational radiation background, one may have to allow for large amplitude inhomogeneities during the early stages of the Universe. In such scenarios the influence of gravitational waves on the evolution of the Universe increases as one goes back in time. Once the wavelength of the gravitational radiation is comparable to the size of the horizon, these waves may not be described as homogeneous relativistic fluid but rather must be seen as large amplitude shear and rotation inhomogeneities. This and the desire to clarify how generic inflation is lead one to consider more general inhomogeneous spacetimes.

Another issue of interest, if the models inflate, is whether they approach a homogeneous and isotropic regime at late times. Some of the latest qualitative studies [6] as well as those based on exact anisotropic scalar field solutions [3] show that there are difficulties with isotropization in some model Bianchi universes. It is quite possible that inflation is not that effective of a device when isotropization is considered. Simple models considered previously using gravitational pulses (solitons) might be competing or complementary in order to resolve this question. In these models solitons [7,8] act as flattening devices in inhomogeneous cosmological models which start highly irregularly but approach Bianchi type I universes filled with gravitational radiation in a finite time. The Bianchi type I universes in turn are known to isotropize quite easily.

Some numerical and qualitative work about the existence of inflationary phases in inhomogeneous cosmologies has been done previously. Kurki-Suonio *et al.* [9] have shown that in some cases the inhomogeneities may prevent the Universe from ever entering the inflationary stage. Goldwirth and Piran [10] have studied both new and chaotic inflation, concluding that the duration of inflation may be significantly reduced by the inhomogeneity, while Calzetta and Sakellariadou [11] have looked at inhomogeneous, but asymptotically Friedmann-Robertson-Walker (FRW) models, and have concluded that the Cauchy data must be homogeneous over several horizons lengths in order for inflation to occur. These works point in the direction that sufficiently irregular initial data may cause problems for inflation.

Recently two of us have found an exact solution describing an inhomogeneous exponential-potential scalar field cosmological model [4]. The behavior of the solution was persistently noninflationary. Since this is only a particular solution of the Einstein field equations, it is interesting to see wether the behavior depicted by the model is shared by a larger class of inhomogeneous cosmological solutions or the solution we have found is quite untypical.

In this paper we restrict ourselves to study the effects of one-dimensional inhomogeneities on the evolution of the exponential-potential scalar field cosmologies. These inhomogeneities may be induced either by the irregularities of the scalar field itself or by the initial inhomogeneities in the geometry due, for example, to the presence of primordial gravitational waves as mentioned above.

We will first treat exactly the coupled Einstein-Klein-Gordon equations, reducing them to a single nonlinear ordinary differential equation similarly to that discussed in paper I. In contrast with the Bianchi type I case we are not able to find the general solution to this equation. Still, we can resolve it in some particular cases. These particular exact solutions then serve as a test for the numerical analysis.

In Sec. II we present the Einstein field equations and the way to solve them. In Sec. III several particular solutions are given and discussed. Section IV is devoted to the qualitative and numerical analysis. We conclude and discuss our results in Sec. V.

II. EINSTEIN EQUATIONS

We will concentrate on solutions with one-dimensional inhomogeneity. These can be described by the generalized Einstein-Rosen spacetimes which admit an Abelian group of isometries G_2 and include the Bianchi models of type I–VII as particular cases and, therefore, the flat and open FRW solutions. The line element is

$$ds^2 = e^f(-dt^2 + dz^2) + g_{ab}\, dx^a dx^b, \qquad a,b = 1,2\,. \quad (1)$$

The functions f and g_{ab} depend on t and z.

Assuming that the two Killing vectors are hypersurface orthogonal, the line element (1) may be cast in the diagonal form

$$ds^2 = e^f(-dt^2 + dz^2) + G(e^p dx^2 + e^{-p} dy^2). \quad (2)$$

It is now apparent that (2) is a straightforward generalization of the models considered in paper I but, here, the metric functions are allowed to depend on t and z variables.

To simplify the equations we shall only consider in this paper the class of solutions for which the element of the transitivity surface is homogeneous:

$$G = G(t). \quad (3)$$

This choice assures that the gradient of the transitivity surface area is globally timelike ($G_\mu G^\mu < 0$) and hence is appropriate for a description of cosmological models [12].

The matter source for the metric is that of a minimally coupled scalar field with potential $V(\phi)$ for which the stress-energy tensor is given by

$$T_{\alpha\beta} = \phi_{,\alpha}\phi_{,\beta} - g_{\alpha\beta}\left[\tfrac{1}{2}\phi_{,\gamma}\phi^{,\gamma} + V(\phi)\right]\,. \quad (4)$$

As in paper I the potential is taken as

$$V(\phi) = \Lambda\, e^{k\phi}. \quad (5)$$

One may rewrite this stress-energy tensor in a perfect fluid form (as long as the gradient of the hypersurface ϕ =const is timelike) with the kinematical and dynamical quantities of the fluid given in paper I.

For the line element given by Eq. (2) and the matter described by the stress-energy tensor (4), the Einstein equations can be written in the form

$$\frac{\ddot G}{G} = 2\, e^f\, V\,, \quad (6)$$

$$\ddot p - p'' + \frac{\dot G}{G}\dot p = 0\,, \quad (7)$$

$$\frac{1}{2}\dot p\, p' + \dot\phi\,\phi' - \frac{1}{2}\, f'\, \frac{\dot G}{G} = 0, \quad (8)$$

$$\frac{\ddot G}{G} - \frac{1}{2}\left(\frac{\dot G}{G}\right)^2 - \frac{\dot G}{G}\dot f + \frac{1}{2}\dot p^2 + \frac{1}{2}p'^2 + \dot\phi^2 + \phi'^2 = 0\,. \quad (9)$$

The Klein-Gordon equation for the scalar field is

$$\ddot\phi - \phi'' + \frac{\dot G}{G}\dot\phi + e^f\,\frac{\partial V}{\partial\phi} = 0\,. \quad (10)$$

Without any loss of generality we write the scalar field as

$$\phi = -\frac{k}{2}\ln G + \Phi(t,z). \quad (11)$$

Substituting Eq. (11) into Eq. (10) and using the form of the potential given by Eq. (5) along with Eq. (6), we get the following equation for the function Φ:

$$\ddot\Phi - \Phi'' + \frac{\dot G}{G}\dot\Phi = 0\,. \quad (12)$$

Note that again, as in the case of the Bianchi type I models, the scalar field Φ and the transversal gravitational degree of freedom p verify the same differential equation. This property is quite surprising and holds apparently only in two cases: (i) when the scalar field is massless and (ii) when the scalar field has an exponential potential.

Case (i) was studied thoroughly by several authors in connection with the quantum description of matter fields in an anisotropic background of the early Universe [13]. By identifying the scalar field with the velocity potential of the irrotational stiff fluid [14], Liang [15] as well as Carmeli *et al.* [15] analyzed within a fully nonlinear relativistic approach the development of inhomogeneities on the spatially homogeneous background.

For the scalar field with an exponential potential this observation of the similarity between the two equations is new and is helpful not only to construct exact solutions but to see the effects in the separation of each of the fields on the dynamics of the models. Not only do these fields follow similar equations but they contribute equally to the inhomogeneity as we shall see later.

We now suppose one may separate the functions p and Φ in the following way:

$$p = \Pi(t) + P(z), \quad \Phi = \chi(t) + \psi(z). \quad (13)$$

One may separate the solutions yet in a different way as products; however, one may prove that this only leads to a particular case of the solutions obtained by the separation (13). This basically happens because of the restrictive conditions imposed by Eq. (8) and the form of the scalar field potential.

Substituting expressions (13) into Eqs. (7) and (12), one obtains

$$p = \Pi(t) + \tfrac{1}{2}\lambda z^2 + \gamma z, \quad \Phi = \chi(t) + \tfrac{1}{2} l z^2 + g z, \quad (14)$$

where λ, γ, l, and g are constants.

Substituting these equations into Eq. (8) and using f' obtained on differentiating Eq. (6), one gets the condition

$$\tfrac{1}{2}\dot{\Pi}(\lambda z + \gamma) + \dot{\chi}(l z + g) = 0. \quad (15)$$

Using Eq. (14) and taking now the time derivative of Eq. (6) to get $\dot{f}$ and substituting all these into Eq. (9), we obtain

$$\frac{\ddot{G}}{G} - \frac{\dddot{G}}{\ddot{G}}\frac{\dot{G}}{G} - K\frac{\dot{G}}{G} + \frac{1}{2}\dot{\Pi}^2 + \dot{\chi}^2 + \frac{1}{2}(\lambda z + \gamma)^2 + (l z + g)^2 = 0, \quad (16)$$

where $K = k^2/4 - 1/2$.

It follows from this equation that the sum of the last two terms must be a constant; therefore, $\lambda = l = 0$, which in its turn leads, using Eq. (15) to the relation

$$\dot{\Pi} = -\frac{2g}{\gamma}\dot{\chi}. \quad (17)$$

The last step before getting to the final equation is to substitute the form of the function $p = \Pi(t) + \gamma z$ into Eq. (7) which gives

$$\dot{\Pi} = \frac{a}{G}, \quad (18)$$

where a is a constant.

Finally, substituting Eqs. (17) and (18) into Eq. (16), we obtain a single nonlinear equation for the evolution of the function G:

$$G\ddot{G}^2 - \dddot{G}\dot{G}G - K\ddot{G}\dot{G}^2 + M^2\ddot{G} + A^2 G^2 \ddot{G} = 0, \quad (19)$$

where we have introduced the constants

$$M^2 = \frac{a^2}{2} + \frac{\gamma^2 a^2}{4g^2}, \quad A^2 = \frac{\gamma^2}{2} + g^2. \quad (20)$$

To summarize up to here, Eq. (19) provides a key to solving the Einstein equations. Once the function G is found, the rest of the functions describing the geometry and the matter are obtained by the expressions

$$\begin{aligned} p &= a\int\frac{dt}{G(t)} + \gamma z, \\ \phi &= -\frac{k}{2}\ln G - \frac{\gamma a}{2g}\int\frac{dt}{G(t)} + g z, \\ f &= -k\phi + \ln\frac{\ddot{G}}{G} - \ln 2\Lambda. \end{aligned} \quad (21)$$

The function f is derived from Eq. (6).

Equation (19) is very similar to that for the Bianchi type I case studied in the previous paper [Eq. (23) of I], the only difference being that the last term is nonlinear in G. While in principle one may reduce the order of this equation it leads to no simplification since we could not find a first integral like in the homogeneous case. This complicates somewhat the search for exact solutions for; one cannot integrate this equation in general. Yet one may find some particular solutions to this equation and to study their behavior.

Before describing some particular cases we should note that all the exact solutions described in paper I remain solutions of Eq. (19) when the inhomogeneity term vanishes.

The inhomogeneity of the spacetime is influenced by both the scalar field inhomogeneous mode related to the constant g and pure gravitational inhomogeneity coming from the transversal degree of the gravitational field and related to the constant γ. These terms act on the dynamics of the transitivity surface area given by the function G precisely through the last nonlinear term of Eq. (19).

III. EXPLICIT EXACT SOLUTIONS

As mentioned previously all the models given in paper I can be considered as particular solutions of Eqs. (19)–(21) and therefore we will not return to them here.

By inspection one may see that

$$G = e^t \quad (22)$$

and

$$G = \sinh\omega t \quad (23)$$

are both solutions of Eq. (19) for particular values of the constants. We will now look at these solutions separately.

A. $G = e^t$

For this case we obtain for the metric functions and the scalar field the expressions

$$\begin{aligned} p &= \gamma z, \\ \phi &= -\frac{k}{2}t + g z, \\ f &= \frac{k^2}{2}t - g k z - \ln 2\Lambda, \end{aligned} \quad (24)$$

where the constants are related by

$$2\gamma^2 + 4g^2 = k^2 - 2, \quad a = 0. \quad (25)$$

Note that $k^2 \geq 2$.

To see whether the models defined by Eqs. (24) and (25) inflate at some stage of their history we will define a four-velocity field u^α normal to the hypersurfaces $\phi =$ const:

$$u_\alpha = \frac{\phi_{,\alpha}}{\sqrt{-\phi_{,\gamma}\phi^{,\gamma}}} \,. \tag{26}$$

Note that for the spacetimes in question the inequality $-\phi_{,\gamma}\phi^{,\gamma} \geq 0$ always holds.

We may now compute the expansion Θ and the deceleration parameter q. After some algebra one gets

$$\Theta = \sqrt{2\Lambda}\,\frac{k}{4}\,\frac{3+\gamma}{\sqrt{\frac{1+\gamma}{2}}}\,e^{-\frac{k^2}{4}t}e^{\frac{gk}{2}z}, \tag{27}$$

together with q:

$$q = \gamma^2\,\frac{\gamma^2+3}{\gamma^2+1}\,\frac{k^2}{4}\,2\Lambda\,\Theta^2\,e^{-\frac{k^2}{2}t}e^{gkz} \,. \tag{28}$$

It is easy to see from Eq. (28) that $q \geq 0$. Hence, these solutions do not undergo an inflationary phase, but for $q = 0$ ($\gamma = 0$) which defines a kind of a "weak" inflation. When $k^2 = 2$ ($\gamma = g = 0$), one obtains

$$\begin{aligned} ds^2 &= \frac{1}{2\Lambda}e^t(-dt^2+dz^2) + e^t(dx^2+dy^2), \\ \phi &= -\frac{1}{\sqrt{2}}t. \end{aligned} \tag{29}$$

Transforming the solution into synchronous coordinates ($t = 2\ln T$), one obtains

$$\begin{aligned} ds^2 &= -dT^2 + T^2(dx^2+dy^2+dz^2), \\ \phi &= -\sqrt{2}\,\ln T, \end{aligned} \tag{30}$$

representing an isotropic and homogeneous FRW solution first obtained by Ellis and Madsen [16].

One may look at these solutions yet from a different point of view. Choosing the four-velocity as given by Eq. (26), one may show that the scalar field stress-energy tensor takes the perfect fluid form

$$T_{\mu\nu} = (p+\rho)u_\mu u_\nu + p g_{\mu\nu}, \tag{31}$$

with the density and the pressure given by

$$\begin{aligned} \rho &= -\tfrac{1}{2}\phi_{,\gamma}\phi^{,\gamma} + V(\phi), \\ p &= -\tfrac{1}{2}\phi_{,\gamma}\phi^{,\gamma} - V(\phi). \end{aligned} \tag{32}$$

Substituting V, ϕ, and u^α into these expressions, we readily find that the fluid has a simple adiabatic equation of state $p = n\rho$, where $n = \frac{\gamma^2-1}{\gamma^2+3}$ ($-\frac{1}{3} \leq n \leq 1$) and

$$\rho = \Lambda\,\left(3+\gamma^2\right)\,e^{-\frac{k^2}{2}t}e^{kgz} \,. \tag{33}$$

While at first sight this solution looks inhomogeneous, in fact it is not and after some coordinate transformations, using the scalar field as a new time coordinate, the line element may be transformed into an explicit Bianchi type VI form.

B. $G = \sinh\omega t$

For this case we have the following relations between the constants:

$$\begin{aligned} g^2 &= \frac{\omega^2}{4}(k^2-2) - \frac{\gamma^2}{2}, \\ a^2 &= 2g^2\frac{k^2+2}{k^2-2}. \end{aligned} \tag{34}$$

The rest of the metric functions and the scalar field are given by

$$\begin{aligned} p &= \frac{a}{\omega}\ln\left(\tanh\frac{\omega t}{2}\right) + \gamma z, \\ \phi &= -\frac{k}{2}\ln\sinh\omega t - \frac{\gamma}{2\omega}\sqrt{2\frac{k^2+2}{k^2-2}}\ln\left(\tanh\frac{\omega t}{2}\right) + gz, \\ f &= -k\phi + \ln\frac{\omega^2}{2\Lambda}. \end{aligned} \tag{35}$$

Let us note again that $k^2 > 2$.

To see whether these solutions inflate one has to proceed as in the previous model. Technically, however, the expressions in this case start to be quite long so we shall consider a particular representative case of this solution by choosing particular values of the free constants. We have chosen

$$\gamma = \frac{k^2}{2}, \quad \omega^2 = \frac{k^2}{2}\frac{k^2+2}{k^2-2}. \tag{36}$$

After some lengthy calculations we get that the sign of q is determined by the sign of the polynomial

$$\sum_{n=0}^{n=4} c_n \cosh^n \omega t, \tag{37}$$

where the coefficients c_n are messy functions of the parameter k. One may show that for any k all the coefficients c_n are strictly positive. We therefore conclude that this representative solution never inflates.

We may have as well a look at the asymptotic behavior of the above model at $t \to \infty$. It is easy to see that in general these models tend to homogeneous (but anisotropic) universes of Bianchi type VI. For large values of the parameter k the solutions approach also the Bianchi type VI anisotropic models whereas for values of k close to $\sqrt{2}$ the metric tends to that of Bianchi type III. After a coordinate transformation the metric can be cast for $k \sim \sqrt{2}$ into the form

$$ds^2 \sim -dT^2 + T^2(e^z dx^2 + dy^2 + dz^2) \,. \tag{38}$$

It would be interesting of course to get a general solution of Eq. (19) as we did in the Bianchi type I case. We are afraid, however, that technically this task may turn out to be very difficult. One may probably look for more particular solutions of Eq. (19); yet we feel at this stage that one may proceed to study the evolution of these models numerically since we have got enough analytic exact solutions against which the numerical results can be tested.

IV. LATE-TIME BEHAVIOR OF THE GENERIC SOLUTION

In this section we present the results of the qualitative and numerical [17] study of the asymptotic behavior of the generic solution described by the line element (2).

A. Qualitative analysis

Before presenting the results of the numerical analysis of the evolution equation (19), we can try to apply to it a qualitative analysis similar to that of paper I. As we will see in the following, the results will be rather different.

By using

$$x = \ln G, \qquad y = \dot{G} e^{(K-1)x}, \tag{39}$$

Eq. (19) reduces to

$$y'' + (1-K)\, y' = e^{2Kx}\left(A^2 + e^{-2x} M^2\right) \frac{y' + (1-K)y}{y^2}. \tag{40}$$

For $-1/2 < K < 0\, (k^2 < 2)$ the right-hand side of (40) vanishes when $y \to E$ and $x \to \infty$. So one can expect the same asymptotic behavior as in the homogeneous case: $G \sim (Ct + D)^{1/K}$. Our numerical experiments have shown that this is the case.

However, from Eq. (40) we see that one cannot expect the same behavior when $0 < K < 1\, (2 < k^2 < 6)$.

In terms of the new variables

$$x = \ln G, \qquad y = \dot{G}, \tag{41}$$

Eq. (19) reduces to the equation

$$y'' + (K-1)\, y' = \left(M^2 + A^2 e^{2x}\right) \frac{y'}{y^2}. \tag{42}$$

If $K > 1$ (i.e., if $k^2 > 6$), the right-hand side of Eq. (42) does not vanish when $y \to C$ and $x \to \infty$. Therefore one does not expect, in this case, that the asymptotic behavior is of the form $G \sim Ct + D$.

To analyze the cases $K > 0$, $(k^2 > 2)$, in which we should expect an asymptotic behavior different from that of the homogeneous case, let us consider the variables

$$u = \frac{\dot{G}}{G}, \qquad v = \frac{\ddot{G}}{\dot{G}}. \tag{43}$$

The evolution equation (19) can be written in the form of a nonlinear first order system:

$$\begin{aligned} \dot{G} &= G\, u, \\ \dot{u} &= u\,(v-u), \\ \dot{v} &= -K\, u\, v + \left(A^2 + \frac{M^2}{G^2}\right) \frac{v}{u}. \end{aligned} \tag{44}$$

If $M = 0$, the last two equations form an autonomous system:

$$\begin{aligned} \dot{u} &= u\,(v-u), \\ \dot{v} &= -K\, u\, v + A^2 \frac{v}{u}, \end{aligned} \tag{45}$$

which for $K > 0$ has a single equilibrium point $u = v = A/\sqrt{K}$. Furthermore, the characteristic exponents of this equilibrium point are $-A(1 \pm \sqrt{1-8K})/2\sqrt{K}$ and their real parts are always negative. Consequently, the equilibrium point is asymptotically stable. This attractor corresponds to the solutions of the form $G \propto \exp(At/\sqrt{K})$. The corresponding phase space is depicted in Fig. 1 for $K = 1/2$ and $A = 1/2$.

When $M \neq 0$ there is an additional term $M^2 v/G^2 u$, but we see from Eq. (44) that it will decrease exponentially as (u, v) approaches the equilibrium point. One thus expects the same asymptotic behavior even in this case. For instance, we can see in Fig. 2 the same case as in Fig. 1 but with $M = 0.001$ (the same behavior is observed for larger values of M). The solutions corresponding to the same initial conditions with $M = 0$ are displayed as dotted lines. As expected, we see that both cases are rather different for small values of t (which correspond to small G), but tend asymptotically to the equilibrium point. Note that some lines appear to cross because with $M \neq 0$ the plane (u, v) is a projection of the three-dimensional phase space (G, u, v). As described in the next subsection, we have found this behavior in all numerical experiments.

B. Numerical analysis

As in the case of homogeneous Bianchi type I models (paper I) we look again for the following asymptotic behavior suggested by the exact solutions:

$$G \sim t^N \tag{46}$$

and

$$G \sim e^{Nt}, \tag{47}$$

which correspond to FRW (Kasner $N = 1$) or anisotropic behavior when the inhomogeneity is switched off. All the numerical solutions we ever obtained had one of the

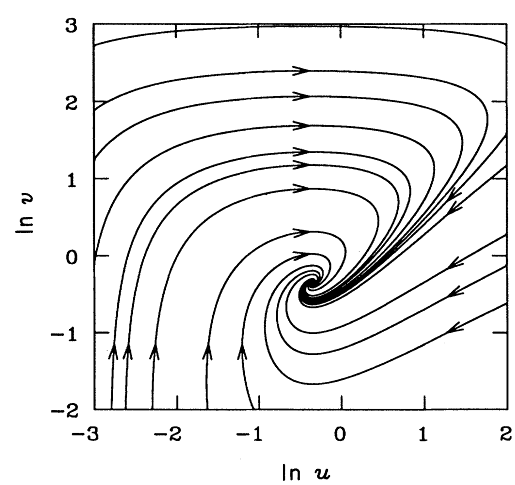

FIG. 1. Log-log plot of the phase space of Eq. (45) for $K = A = 1/2$.

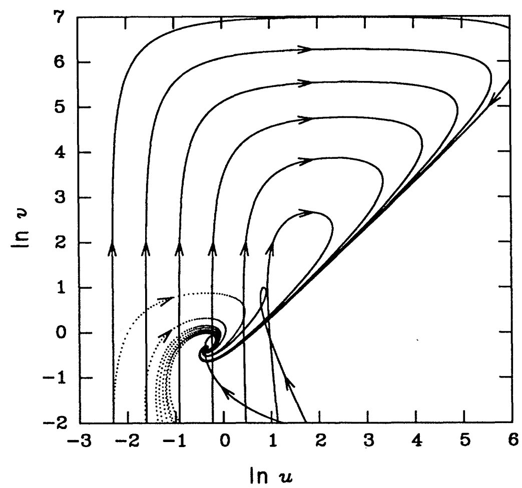

FIG. 2. Some solutions of Eq. (44) for $K = A = 1/2$ and $M = 0.001$. The solutions corresponding to the same initial conditions but $M = 0$ appear as dotted lines. All solutions decay to the equilibrium point of Fig. 1 (note that the scale has been changed).

aforementioned asymptotic behavior. As in paper I we use the quantities

$$n_1(t) = \frac{\dot{G}^2}{\dot{G}^2 - G\ddot{G}} \tag{48}$$

and

$$n_2(t) = \frac{\ddot{G}}{\dot{G}} \tag{49}$$

which tend to a constant if each of the asymptotic behaviors occurs. Equation (48) monitors the behavior given by Eq. (46), while Eq. (49) monitors that one described by Eq. (47).

We have integrated numerically Eq. (19) and summarize our results in Table I, where C and D are constants.

Different initial conditions were used during the numerical integration. We have only kept the initial condition $G(0) = 0$ fixed. During the numerical integration we have always monitored the positivity of the function G and of its second derivative.

Looking at Table I, we see that a new type of the asymptotic behavior appears when the inhomogeneity is introduced: $G \sim e^{At/\sqrt{K}} + D$. Note, as pointed out in our previous paper, that this type of behavior was structurally unstable. Surprisingly, the inhomogeneity stabilizes this asymptotic solution. The late-time exponential behavior occurs for $k^2 > 2$. In this case the models tend to those of Bianchi type VI as described in Sec. III B. These models never become isotropic, although the Universe homogenizes. For $k^2 < 2$ the function G tends to $(Ct + D)^{1/K}$ which in absence of inhomogeneity would have been that of FRW case. Yet the inhomogeneity imprints in other metric functions p and f prevent these models to homogenize, in the sense that they do not tend to a Bianchi model, let alone to isotropize.

We have also studied numerically the occurrence of inflation by computing the sign of the deceleration parameter q. We have noticed that inflation always occurs for $k^2 < 2$ although it takes some time for the model to start inflating. For $k^2 > 2$ most of the models do not inflate as long as the gradient of the scalar field remains timelike. We insist on this condition since otherwise the fluid interpretation of the matter field is problematic [18].

We also find that the introduction of the inhomogeneity may introduce multiple inflation: The model starts decelerating then accelerates, then decelerates, and accelerates again. This never happens for Bianchi type I models.

V. CONCLUSIONS

We have discussed in this paper the simplest inhomogeneous generalizations of the Bianchi type I cosmological models with an exponential-potential scalar field. Restricting the geometry to be as close as possible to that of Bianchi type I anisotropic cosmological model by keeping the element of the transitivity area time dependent only and thus globally timelike, we have been able to reduce the Einstein equations to a single nonlinear differential equation. Several exact solutions to this equation, and consequently to the full set of the equations, were presented and discussed. These solutions then served us as a bench test to analyze numerically the central equation (19) and the dynamics of the cosmological models.

It is needless to say that the numerical integrations were at each stage tested against the analytic results obtained both for homogeneous and inhomogeneous cases.

For the models we have studied our results are as follows.

a. Homogeneous anisotropic case (see as well paper I).

(1) The slope of the potential given by the constant k is the key factor influencing the occurrence of inflation and late-time isotropization of the model.

(2) For $k^2 < 2$ the models always inflate and isotropize. For $2 < k^2 < 6$ the models still isotropize; however, they do not inflate in most of the cases. Yet the $G \sim t^{1/K}$ (FRW-type) behavior cannot be called an attractor in a strict technical sense, since $G \sim t^{1/K}$ becomes an exact solution of the Einstein equations only when the

TABLE I. Equation (19) integrated numerically where C and D are constants.

	$0 < k^2 < 2$	$2 < k^2 < 6$	$k^2 > 6$
$A^2 = 0$ Homogeneous case	$G \sim (Ct + D)^{1/K}$	$G \sim (Ct + D)^{1/K}$	$G \sim Ct + D$
$A^2 \neq 0$ Inhomogeneous case	$G \sim (Ct + D)^{1/K}$	$G \sim Ce^{At/\sqrt{K}} + D$	$G \sim Ce^{At/\sqrt{K}} + D$

integration constants are severely restricted.

(3) For $k^2 > 6$ the models do not isotropize and have a Kasner-like asymptotic behavior. This is an attractor solution, for it is a solution of the Einstein equations with arbitrary integration constants.

b. Inhomogeneous case .

(1) The slope of the potential is still of key importance in the behavior of the cosmological models; in this case, however, the inhomogeneity influences strongly the evolution and enriches the behavior of the models.

(2) The introduction of the inhomogeneity stabilizes the $G \sim e^{t/\sqrt{K}}$ asymptotic behavior, leading always, for $k^2 > 2$, to an anisotropic Bianchi type VI universe.

(3) The solutions $G \sim t^{1/K}$ which are asymptotically generic in the case $k^2 < 2$ are of no help for isotropization, in this case, for, unlike in the homogeneous case, these are not FRW solutions anymore. The spatial dependence of other metric functions prevents the homogenization.

(4) As to the inflation, we have found that for $k^2 < 2$ the models do generically inflate. We have also observed that the introduction of the inhomogeneity induces a new type of the dynamical behavior, not present in Bianchi type I models, multiple inflation, in which the deceleration parameter q changes its sign several times during the entire history of the Universe. Multiple inflation is, however, subject to a fine-tuning of the integration constants.

We feel that the best way to close is to call for more work on inhomogeneous inflation. To treat the generic inhomogeneous model one certainly needs to use the numerical analysis. We hope then that some of our results may be of use for further numerical studies of generic inhomogeneous models.

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[17] The numerical integrations were performed by means of *ODE Workbench* [J.M. Aguirregabiria, *ODE Workbench*, Physics Academy Software (American Institute of Physics, New York, in press)]. The quality of the numerical results was tested by using different integration codes, ranging from the very stable embedded Runge-Kutta code of eighth order due to Dormand and Prince to very fast extrapolation routines. All the codes have adaptive step size control and we checked that smaller tolerances did not change the results. Double precision was used in all calculations and different exact cases were used to test our numerical work.

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