Exponential-potential scalar field universes. I. Bianchi type I Models

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We obtain a general exact solution of the Einstein field equations for the anisotropic Bianchi type I universes filled with an exponential-potential scalar field and study their dynamics. It is shown, in agreement with previous studies, that for a wide range of initial conditions the late-time behavior of the models is that of a power-law inflating Friedmann-Robertson-Walker (FRW) universe. This property does not hold, in contrast, when some degree of inhomogeneity is introduced, as discussed in our following paper.

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I. INTRODUCTION

Two of the present authors have recently studied exact model universes filled with an exponential-potential scalar field and have found [1,2] a persistent noninflationary behavior in some exact Bianchi types III and VI cosmologies, as well as in some cosmological models which may be thought as inhomogeneous generalizations of Bianchi type I cosmologies. These studies cast a doubt on the so-called "inflationary paradigm" which, while never properly formulated, vaguely states that the Universe had undergone a period of inflationary expansion which is not only a must to solve a host of problems of standard cosmology, but is rather a "typical" dynamical behavior common to a wide class of scalar field cosmologies.

One of the main reasons the inflationary universe scenario is considered to be attractive [3] is that it has brought back to life Misner's hope [4] to explain the large-scale homogeneity, implied by the measurements of the cosmic microwave background radiation, without the need to impose very special conditions on the initial expansion of the Universe. Strangely enough, however, most of the work on inflation is done in the framework of isotropic and homogeneous Friedmann-Robertson-Walker (FRW) universes (Olive [5] and references therein).

This paper deals with *exact solutions* of the Einstein field equations. We show how one may obtain a general exact solution for Bianchi type I cosmologies with an exponential-potential scalar field. Apart from the fact that one obviously needs a specific model for the potential to solve exactly the Einstein equations, it is convenient to concentrate on this sort of potential, often used in inflationary analysis, for the following main reasons.

(1) An exponential-potential scalar field introduces a rather small additional degree of nonlinearity into the Einstein field equations, as will be explained later, so that possibilities exist to solve these equations analytically in a variety of cases. In the case of cosmological models with Bianchi type I spatial symmetry one can obtain a general solution.

(2) It seems that the exponential potentials for scalar fields arise [6] in many theories such as Jordan-Brans-Dicke theory, the superstring theory, Salam-Sezgin theory, and others.

This work represents an introductory step towards our following paper, paper II, where the homogeneity will be broken in one direction to study the inhomogeneity effects on the late-time behavior of scalar field cosmological models.

The main result of this paper is to confirm, on the basis of *exact solutions* we obtain, previous qualitative and numerical studies by other authors [7] and the pattern of the late-time behavior of the exponential-potential scalar field cosmologies. This agreement is only true, however, as long as the underlying geometry of the cosmological model is as simple as that of Bianchi type I. When the spatial symmetry group is more complicated, as, for example, in Bianchi type III and VI models as well as when the spatial homogeneity is broken, the dynamics of the models is different. This, however, will be discussed in paper II.

In Sec. II we discuss the geometry and the matter content of the models. In Sec. III the Einstein equations are considered and solved. Some representative solutions are given in Sec. IV, while in Sec. V the late-time behavior of the generic solutions is studied qualitatively and numerically. In Sec. VI we conclude and summarize our results.

II. BIANCHI TYPE I EXPONENTIAL-POTENTIAL SCALAR FIELD COSMOLOGIES

The usual synchronous form for a general Bianchi type I element is given by

$$ds^2 = -dT^2 + a_1^2(T)\,dx^2 + a_2^2(T)\,dy^2 + a_3^2(T)\,dz^2\,, \quad (1)$$

representing the anisotropic generalization of the spatially flat FRW universe expanding differently in the x, y, and z directions. While the form (1) is frequently used to study the dynamical behavior of the cosmological model, we have found it convenient, in order to obtain exact analytic solutions, to cast the metric into the so-to-say, semiconformal form

$$ds^2 = e^{f(t)}(-dt^2 + dz^2) + G(t)\,(e^{p(t)}dx^2 + e^{-p(t)}dy^2)\,. \tag{2}$$

This form of the metric is of particular interest when studying the inhomogeneous generalizations of the line element (1), and is obtained from Eq. (1) by the identification

$$dt = \frac{dT}{a_3}\,, \quad e^f = a_3^2\,, \quad G = a_1 a_2\,, \quad e^p = \frac{a_1}{a_2}\,. \tag{3}$$

Since from now on we will be working with the line element (2) exclusively, it is worthwhile to specify some cases of interest corresponding to particular values the metric functions f, G, and p can take.

When the function p vanishes globally the spacetime expands equally in the x and y axes in which case the spacetime is called locally rotationally symmetric (LRS). If in addition to $p = 0$ one has $G = e^{f(t)}$, then the metric is of the FRW form. In fact the only way the metric (2) may approach the FRW solution is when $e^{f(t)} \sim G$ together with $p \sim 0$.

The line element (2) admits the important vacuum solutions

$$G = t\,, \quad p = \kappa \ln t\,, \quad f = \frac{\kappa^2 - 1}{2} \ln t\,, \tag{4}$$

which are of Kasner type and always anisotropic apart from two "degenerate cases" $\kappa = \pm 1$ which are disguised Minkowski line elements.

The energy-momentum tensor for the scalar field driven by the potential $V(\phi)$ is given by

$$T_{\alpha\beta} = \phi_{,\alpha}\phi_{,\beta} - g_{\alpha\beta}\left[\tfrac{1}{2}\phi_{,\gamma}\phi^{,\gamma} + V(\phi)\right]\,, \tag{5}$$

and as long as attention is concentrated on the homogeneous spacetimes Eq. (5) may be rewritten in the perfect fluid form

$$T_{\alpha\beta} = (p + \rho)u_\alpha u_\beta + p g_{\alpha\beta}\,, \tag{6}$$

where

$$u_\alpha = \frac{\phi_{,\alpha}}{\sqrt{-\phi_{,\gamma}\phi^{,\gamma}}}, \tag{7}$$

together with

$$\begin{aligned} \rho &= -\tfrac{1}{2}\phi_{,\gamma}\phi^{,\gamma} + V(\phi), \\ p &= -\tfrac{1}{2}\phi_{,\gamma}\phi^{,\gamma} - V(\phi). \end{aligned} \tag{8}$$

The perfect fluid interpretation of the scalar field, while not obligatory, is very useful to study the kinematical behavior of the cosmological models. It is convenient to introduce [8] the expansion

$$\Theta = u_{\mu;\nu}g^{\mu\nu} \tag{9}$$

and the deceleration parameter

$$q = -3\Theta^2\left(\Theta_{;\alpha}u^\alpha + \tfrac{1}{3}\Theta^2\right)\,. \tag{10}$$

The sign of the deceleration parameter indicates whether the cosmological model inflates. The positive sign corresponds to "standard" decelerating models whereas the negative sign indicates inflation.

For the models to isotropize at late times the shear measured with respect to the expansion rate must vanish asymptotically [9]. The models defined by the line element (2) are simple enough so that one may get a precise idea whether the spacetime isotropizes just from looking at the asymptotic behavior of the line element.

III. SOLVING THE EINSTEIN EQUATIONS

The Einstein equations for the metric given by the line element (2) and the matter specified by the stress-energy tensor (5) are given by

$$\ddot\phi + \frac{\dot G}{G}\,\dot\phi + e^f\,\frac{\partial V}{\partial \phi} = 0\,, \tag{11}$$

$$\frac{\ddot G}{G} = 2\,e^f\,V\,, \tag{12}$$

$$\ddot p + \frac{\dot G}{G}\,\dot p = 0\,, \tag{13}$$

$$\frac{\ddot G}{G} - \frac{1}{2}\left(\frac{\dot G}{G}\right)^2 - \frac{\dot G}{G}\,\dot f + \frac{1}{2}\dot p^2 = -\dot\phi^2\,, \tag{14}$$

where Eq. (11) is the Klein-Gordon equation for the scalar field.

Note that Eq. (13) is somewhat decoupled from the other equations in the sense that it is the same as in the vacuum case. The matter field acts indirectly on this equation through the "transitivity area" function G. Yet one may immediately integrate the equation in terms of the function G:

$$\dot p = \frac{a}{G}\,, \tag{15}$$

where a is an arbitrary integration constant.

Now we substitute Eq. (15) into Eq. (14) and use the fact that on differentiating Eq. (12) and using it again one gets, for $V \neq 0$,

$$\dot f = \frac{\dddot G}{\ddot G} - \frac{\dot G}{G} - \frac{\dot V}{V}\,. \tag{16}$$

We then are left with the two equations

$$\ddot\phi + \frac{\dot G}{G}\,\dot\phi = -\frac{1}{2}\,\frac{\ddot G}{GV}\,\frac{\partial V}{\partial\phi}, \tag{17}$$

$$\frac{\ddot G}{G} + \frac{1}{2}\,\frac{\dot G^2 + a^2}{G^2} - \frac{\dddot G\,\dot G}{G\,\ddot G} + \frac{\dot G}{G}\,\frac{\dot V}{V} = -\dot\phi^2\,, \tag{18}$$

where Eq. (17) has been obtained by substituting Eq. (12) into Eq. (11).

We now specify the potential

$$V(\phi) = \Lambda e^{k\phi}. \tag{19}$$

Note that Eqs. (17) and (18) are not valid for the case $V = 0$; in this case, however, the original system of equations is much simpler and its general solution is given by

$$\Lambda = 0, \quad G = t, \quad p = \alpha \ln t, \quad \phi = \beta \ln t,$$
$$f = \frac{\alpha^2 + 2\beta^2 - 1}{2} \ln t. \tag{20}$$

When $\alpha = 0$ and $\beta = \pm\sqrt{3}/2$ the above solution describes an isotropic spatially flat FRW universe.

Returning now to the general case $\Lambda \neq 0$, the Klein-Gordon equation (17) takes the form

$$\ddot{\phi} + \frac{\dot{G}}{G}\dot{\phi} = -\frac{k}{2}\frac{\ddot{G}}{G}. \tag{21}$$

In analogy with Eq. (13) one readily obtains, from Eq. (21),

$$\dot{\phi} = \frac{m}{G} - \frac{k}{2}\frac{\dot{G}}{G}, \tag{22}$$

where m is an arbitrary constant.

It is important to note that the exponential potential saves one much trouble with the nonlinearity of the Klein-Gordon equation. This is not only true for the homogeneous Bianchi type I models but for any homogeneous or inhomogeneous cosmological spacetime which may be cast into the "semiconformal" form (2) with the metric functions depending on the z coordinate as well (technically these are called spacetimes with two commuting orthogonal spacelike Killing vectors and include Bianchi types I, III, V, VI and their unidirectional inhomogeneous generalizations).

Substituting $\dot{\phi}$ given by Eq. (22) and the form of the potential into Eq. (18) we are left with a single equation

$$G\ddot{G}^2 - \dddot{G}\dot{G}G + \left(\frac{1}{2} - \frac{k^2}{4}\right)\ddot{G}\dot{G}^2 + \left(m^2 + \frac{a^2}{2}\right)\ddot{G} = 0. \tag{23}$$

The substitution

$$\dot{G} = y(G) \tag{24}$$

reduces the degree of the last equation, and after redefining some of the constants we finally get

$$y'' + \left(K - \frac{M}{y^2}\right)\frac{1}{G}y' = 0, \tag{25}$$

where

$$K = \frac{k^2}{4} - \frac{1}{2}, \quad M = m^2 + \frac{a^2}{2}. \tag{26}$$

and $y' = dy/dG$.

Equation (25) has the first integral

$$Gy' + (K-1)y + \frac{M}{y} = B, \tag{27}$$

where B is an arbitrary constant. Integrating Eq. (27) one finally obtains

$$G = \left[\frac{2(K-1)y + B - \sqrt{\Delta}}{2(K-1)y + B + \sqrt{\Delta}}\right]^{B/2(K-1)\sqrt{\Delta}} \times \left\{N[(K-1)y^2 + By + M]\right\}^{-1/2(K-1)}, \tag{28}$$

where

$$\Delta = B^2 - 4(K-1)M, \tag{29}$$

and N is yet another arbitrary constant of integration. Equation (28) as it stands is valid for $\Delta \geq 0$. For $\Delta < 0$ the integral takes a different form.

This formally concludes the solution. For once $G(y)$ is given by Eq. (28) one may in principle invert the expression to get $y(G)$, then using $t = \int \frac{dG}{y(G)}$, and inverting again to obtain $G(t)$. From $G(t)$ one easily reconstructs all the metric functions: $p(t)$ from Eq. (15), $\phi(t)$ from Eq. (22) and $f(t)$ from Eq. (12) for example.

Before turning to analyze the general solution given by Eq. (28) we will present some explicit particular cases of interest in Sec. IV to illustrate the above mentioned procedure.

IV. EXPLICIT EXACT SOLUTIONS

In this section we obtain explicitly some exact solutions of interest and briefly discuss their behavior. We start with the simplest ones.

A. FRW universes

Homogeneous and isotropic universes are obtained if one specifies $M = 0$ and $B = 0$ in Eq. (27). Note that $M = 0$ automatically means $a = 0$ in the Eq. (15), which in turn excludes the transversal part of the gravitational field p restricting the class of models to LRS ones. If, moreover, the constant $B = 0$, one always finishes with an isotropic solution.

If $K = 0$ ($k^2 = 2$), one obtains the so-called "coasting solution" [10]

$$ds^2 = \frac{A^2}{2\Lambda}e^{At}(-dt^2 + dx^2 + dy^2 + dz^2). \tag{30}$$

and

$$\phi = -\frac{k}{2}At. \tag{31}$$

This Universe expands linearly in synchronous coordinates and has a zero deceleration parameter.

For $K \neq 0$ one gets, from Eq. (28),

$$G = t^\lambda, \quad \lambda = \frac{4}{k^2 - 2}. \tag{32}$$

Two different classes of solutions appear. If $\lambda = 1$, one has $\Lambda = 0$ and the solution is that described by Eq. (20) representing the massless minimally coupled scalar field FRW universe. If, however, $\lambda \neq 1$, one gets

$$G = t^\lambda, \quad e^f = \frac{\lambda(\lambda - 1)}{2\Lambda} t^\lambda, \quad \phi = -\frac{k\lambda}{2} \ln t\,, \tag{33}$$

and the metric in synchronous form is given by

$$ds^2 = -dT^2 + T^{4/k^2}(dx^2 + dy^2 + dz^2). \tag{34}$$

The sign of the deceleration parameter for the models described by Eq. (34) depends on the quantity $k^2 - 2$ so that the model inflates for $k^2 < 2$ while it decelerates for $k^2 > 2$.

B. LRS models

Assuming now $M = 0$ but $B \neq 0$ and taking $K = 0$ to obtain an analytic expression we get, from the integral (28),

$$G = e^t + B, \quad e^f = \frac{1}{2\Lambda} e^t, \quad \phi = -\frac{k}{2} \ln(e^t + B). \tag{35}$$

The metric in the synchronous form is given by

$$ds^2 = -dT^2 + T^2 dz^2 + (T^2 + B)(dx^2 + dy^2). \tag{36}$$

Note that at early times this LRS model is anisotropic while at $T \to \infty$ it approaches the linearly expanding FRW universe.

The sign of the deceleration parameter is given by

$$-B\left(e^t - \frac{B}{3}\right), \tag{37}$$

and if the constant $B < 0$, the model never inflates. Note, however, that the range of the parameter B might be restricted by the sign of the potential V. In the solution (36) the constant B measures the deviations from isotropy and as long as B is negative the model does not inflate independently of how negligible the deviation from the isotropy is.

In the case $B > 0$ the cosmological model decelerates until $t < \ln \frac{B}{3}$ but inflates for later times. Another interesting observation for this model is that they are nonsingular for $B > 0$. This does not contradict the singularity theorems for, if $B > 0$, one has

$$\rho + 3p = -\frac{4B}{(B + T^2)^2} < 0\,, \tag{38}$$

and the energy condition is broken.

C. Nonsymmetric solutions

We put $M \neq 0$, $B = 0$, $N = -1/M$ as well as specify $K = 0$ again to obtain an analytic expression. With these constants the integral (28) gives

$$G = \sinh \omega t, \quad \omega = \sqrt{M}. \tag{39}$$

The transversal and longitudinal degrees of freedom are then given by

$$p = \frac{a}{\omega} \ln\left(\tanh \frac{\omega t}{2}\right) \tag{40}$$

and

$$f = \ln(\sinh \omega t) - \frac{km}{\omega} \ln\left(\tanh \frac{\omega t}{2}\right) + \ln \frac{\omega^2}{2\Lambda}, \tag{41}$$

respectively. And the scalar field evolves according to the law

$$\phi = -\frac{k}{2} \ln(\sinh \omega t) + \frac{m}{\omega} \ln\left(\tanh \frac{\omega t}{2}\right). \tag{42}$$

It is interesting to see from this solution that the anisotropy is contributed by both the scalar field ($m \neq 0$) and the transversal degree of the gravitational field ($a \neq 0$). This can be seen as well from Eq. (23) where the contribution to the anisotropy due to the transversal degree of the gravitational field and that due to the scalar field enter symmetrically.

These solutions decelerate for the times $t < t_c$, where t_c is given by

$$t_c = \frac{1}{\omega} \mathrm{arccosh}\left(\frac{\frac{11}{6}m^2 + \frac{3}{4}a^2}{m\sqrt{2m^2 + a^2}}\right), \tag{43}$$

and then after this time the solutions inflate.

V. ASYMPTOTIC LATE-TIME BEHAVIOR OF THE GENERIC SOLUTIONS

A. Numerical analysis

We now turn to study the asymptotic behavior of the generic solutions of Eq. (23). From the previous section we have seen that two different types of late-times behavior occur in exact solutions,

$$G \sim t^N \tag{44}$$

and

$$G \sim e^{Nt}\,. \tag{45}$$

These two different asymptotic behaviors correspond to FRW (Kasner if $N = 1$) and anisotropic models, respectively. Technically we have found it very difficult to study the integral (28) analytically. In most of the cases one cannot integrate Eq. (28) and further integrals in terms of elementary functions. We therefore have been forced to use numerical methods.

To ascertain whether the asymptotic behavior described by Eqs. (44) and (45) occurs in other solutions of Eq. (23) we have monitored the values of the following functions while integrating Eq. (23) numerically [11]:

$$n_1(t) = \frac{\dot{G}^2}{\dot{G}^2 - G\ddot{G}}\,, \tag{46}$$

together with

$$n_2(t) = \frac{\ddot{G}}{\dot{G}}\,. \tag{47}$$

It is obvious that if a solution of Eq. (23) asymptotically follows the power law given by Eq. (44) it is necessary and sufficient that the function n_1 defined by Eq. (46) tends to a constant as the time increases. One could use a different function, say, $n(t) = t\dot{G}/G$, but our numerical experiments have shown that the final answer is the same. The quantity in (46) was finally retained because many test cases gave the answer somewhat faster. Similarly the asymptotic behavior given by Eq. (45) is monitored by the test function n_2 (or $\dot{G}/G$ which finally gives the same answer).

We have integrated numerically Eq. (23) for different values of the constants k, m, and a. In each case different initial conditions for G, $\dot{G}$, and $\ddot{G}$ were chosen, though always remaining positive for physical reasons. Negative G would imply a change in the signature, negative $\ddot{G}$ would imply a negative cosmological constant via Eq. (12), and finally the positivity of $\dot{G}$ accounts for initial expansion rather than contraction.

Our numerical results depict a clear scenario for the asymptotic behavior of Eq. (23). We have found that for every numerical solution the constructed test function n_1 tends towards a constant value which depends only on the slope of the scalar field potential defined by the constant k. This behavior and the value of the exponent N are absolutely independent of the level of the anisotropy introduced either via a scalar field related to the parameter m or purely geometrical anisotropy related to the parameter a. For $k^2 < 6$ the generic solution behaves as

$$G \sim t^{\frac{4}{k^2-2}}, \tag{48}$$

as t goes to infinity which is an asymptotic behavior of the isotropic FRW model.

For $k^2 > 6$ the set of solutions of Eq. (23),

$$G = Ct + D, \tag{49}$$

which corresponds to an asymptotic behavior of the vacuum Kasner solutions or to the solutions with a minimally coupled massless scalar field, is an attractor of Eq. (23) and describes the asymptotic behavior of its generic solution. It is worthwhile to mention that the status of expressions (48) and (49) is different: While (48) is only an asymptotic solution, (49) is an exact solution for an arbitrary set of constants K, M, and B.

We have also found that the asymptotic behavior described by Eq. (45) happens only in the case $k^2 = 2$ and is structurally unstable, for any small deviation in the parameter k changed the asymptotic behavior of the solution to that described by Eq. (48).

B. Qualitative analysis

The outcome of our numerical calculations can be made plausible by a qualitative analysis of Eq. (23), which in terms of the new variables

$$x = \ln G, \qquad y = \dot{G} \tag{50}$$

reduces to the autonomous equation

$$y'' + (K-1)y' = M^2\,\frac{y'}{y^2}. \tag{51}$$

In the particular case in which $M = 0$, the resulting linear equation

$$y'' + (K-1)y' = 0 \tag{52}$$

has the general solution

$$y = C + Ee^{(1-K)x}. \tag{53}$$

If $K > 1$ (i.e., if $k^2 > 6$), and we assume that G, and thus x, increases with time, the asymptotic behavior of the general solution is $y \sim C$, which corresponds to the set of solutions given by Eq. (49).

In the nonlinear case, $M \neq 0$, one could expect that the asymptotic behavior is still the same because if $y \to C$ the right-hand side of Eq. (51) goes to 0. Indeed, our numerical experiments show that this is the case. This may be seen as well by direct integration of Eq. (51), as depicted in Fig. 1 for $K = 2$ and $M = 0.1$. It is clearly seen that after some time the solutions of Eq. (51) attain a high precision straight line with a slope of -1 [i.e., solutions of the linear equation (52)] and, finally, approach the attracting line $y' = 0$.

Though unchanging the asymptotic behavior, the right-hand side of Eq. (51) is, of course, important for small y. To stress it we have drawn as dashed lines some solutions of the linear equation (52) corresponding to the same initial conditions.

As for the assumption of G being an increasing function of t, we see in the same figure that solutions cor-

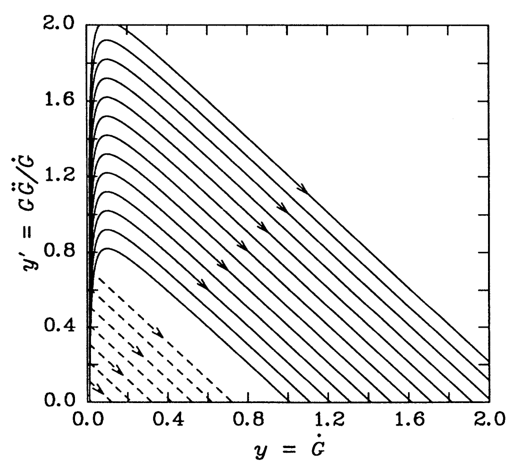

FIG. 1. Phase space of Eq. (51) for $K = 2$ and $M = 0.1$. The solutions correspond to initial conditions in the form $y_0 = 0.01, y_0' = 0.01 + 0.1n$, with $n = 0$ to 12. The dashed lines represent the solutions of the linear case (52) for the same initial conditions for $n = 1$ to 7.

responding to positive initial conditions (as required by physical reasons) never reach negative values for $y = \dot{G}$, for they never cross the line $y' = 0$. This behavior was observed in all numerical integrations.

The cases corresponding to $k^2 < 6\,(-1/2 < K < 1)$ can be discussed in a very similar way by using

$$x = \ln G, \qquad y = \dot{G}e^{(K-1)x}, \tag{54}$$

which leads to the equation

$$y'' + (1-K)\,y' = M^2\,\frac{y' + (1-K)y}{y^2}\,e^{2(K-1)x}. \tag{55}$$

Again, the particular case in which $M = 0$ is trivial and its general solution $y = E + Fe^{(K-1)x}$ will approach $y = E$, which corresponds to (48). If $M \neq 0$, $y = D$ is no longer a solution of Eq. (55) for a finite x, but it can still represent the asymptotic behavior of the typical solution, because the right-hand side of Eq. (55) vanishes when $y \to E$ and $x \to \infty$. We have checked by numerical integration of both Eqs. (48) and (55) that this is indeed the case.

VI. CONCLUSIONS

We have presented here a general exact solution of the Einstein field equations for the anisotropic Bianchi type I universes filled with an exponential-potential scalar field. Some of the representative cases and their behavior were explicitly considered in Sec. IV. We must stress that by considering and studying exact analytic examples one gets a good idea as to the behavior of the general model. Exact solutions, in our view, are indispensable and must be considered, if possible, before any numerical analysis is undertaken to be sure that the results based on numerical "experiments" are of any significance.

Looking at the exact examples we have found three different typical late-time behaviors of the models: (i) the Kasner-like behavior which is characteristic of the vacuum and massless scalar field models; (ii) the FRW-like behavior; and (iii) the limiting "coasting"-type behavior.

Integrating numerically the general solution we have seen that the limiting $G \sim e^{Nt}$ behavior is structurally unstable. This in fact is very interesting, for, exactly this type of behavior, as we shall see in paper II, is generic for the models with one-dimensional inhomogeneity. So, while in such simple models as Bianchi I the instability of these solutions do not cause any reason to worry about the isotropization, in more complicated models their stability causes problems.

We have seen that when the constant k^2, defining the slope of the potential, is less than 6, the generic late-time behavior is that of an isotropic FRW model. In such situations the scalar field acts similarly to a positive cosmological constant.

For $k^2 > 6$ the late-time behavior is that of a Kasner-type universe or, which is the same, of the model filled with massless minimally coupled scalar field. As long as $k^2 > 6$ one has no reason to believe that the model will isotropize.

To see whether the models inflate one may look at the sign of the deceleration parameter q. After some algebra and using the first integral given by the Eq. (27) the sign of the deceleration parameter q is given by

$$\frac{3}{2}\,K\dot{G}^2 - \left(km + \frac{B}{2}\right)\,\dot{G} + \frac{1}{6}\,(km - B)^2 + \frac{3}{2}\,M. \tag{56}$$

It is easy to see from this expression that as long as the anisotropy parameters M and B are switched off ($m = a = 0$), the inflation of the solutions depends only on the slope of the potential given by k. Nevertheless, if the anisotropy is present, the inflation is not only driven by the parameter k but depends as well on the rates of anisotropy. Studying the behavior of Eq. (50) one finds out that solutions generically inflate for $k^2 < 2$, confirming previous results. For $k^2 > 2$ most of the solutions do not inflate, yet depending on the rate of the anisotropy one may find inflating solutions.

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bench, Physics Academy Software (American Institute of Physics, New York, in press)]. The quality of the numerical results was tested by using different integration codes, ranging from the very stable embedded Runge-Kutta code of eighth order due to Dormand and Prince to very fast extrapolation routines. All the codes have adaptive step size control and we checked that smaller tolerances did not change the results. Double precision was used in all calculations and different exact cases were used to test our numerical work.