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Exact primordial black strings in four dimensions

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A solution of effective string theory in four dimensions is presented which admits interpretation of a rotating black cosmic string. It is constructed by taking the tensor product of the three-dimensional black hole, extended with the Kalb-Ramond axion, with a flat direction. The physical interpretation of the solution is discussed, with special attention to the axion, which is found to play a role very similar to a Higgs field. Finally, it is pointed out that the solution represents an exact Wess-Zumino-Witten-Novikov (WZWN) σ model on the string world sheet, to all orders in the inverse string tension α'.

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In recent years we have witnessed a very rapid growth of the family of "black" configurations, representing gravitational fields with event horizons of various topology. They have ranged from various black holes [1–9], over stringlike configurations [1,10–13], to p-branes [7,13–14].

In this paper, I will attempt to expand this family by constructing a solution of the four-dimensional effective action of string theory which admits the interpretation as a black cosmic string inside a domain of axion field gradient. The configuration is primordial, in the sense that the domain is essential for its existence, because the "axion charge" (i.e., the gradient of the pseudoscalar axion) is what stabilizes the string. Its geometric structure is that of the three-dimensional black hole recently found by Banados, Teitelboim, and Zanelli (BTZ) [3], extended with a flat line which is interpreted as the string axis. The solution could also be viewed in light of toroidal black holes investigated by Geroch and Hartle [15], and could be understood as such a black hole, which is bigger than the cosmological horizon of the Universe in which it is embedded.

The dynamics of the background field formulation of string theory is defined with the effective action which, in the Einstein frame and to order $O(\alpha'^0)$, is

$$S = \int d^4x\sqrt{g}\,\left(\frac{1}{2\kappa^2}R - \frac{1}{6}e^{-2\sqrt{2}\kappa\Phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda} - \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \Lambda e^{\sqrt{2}\kappa\Phi}\right) \quad (1)$$

where R is the Ricci scalar, $H_{\mu\nu\lambda} = \partial_{[\lambda}B_{\mu\nu]}$ is the antisymmetric field strength associated with the Kalb-Ramond field $B_{\mu\nu}$, Φ is the dilaton field, and Λ the cosmological constant. The metric is of signature $+2$, the Riemann tensor is defined according to $R^{\mu}{}_{\nu\lambda\sigma} = \partial_\lambda\Gamma^{\mu}{}_{\nu\sigma} - \cdots$, and the cosmological constant is defined with the opposite sign from the more usual general relativity (GR) conventions: $\Lambda > 0$ denotes a negative cosmological constant. It has been included to represent the central charge deficit. In the remainder of this paper, I will work in the Planck mass units: $\kappa^2 = 1$.

The simplest way to find the black string solutions is to investigate the Einstein equations together with the equations of motion for the axion and dilaton. However, it is also instructive to investigate dimensional reduction of the action (1) to three dimensions in order to establish a further relationship of the four-dimensional black string to the three-dimensional black hole, and illustrate the roles played by the degrees of freedom present in the problem. Hence I will first obtain the solution from inspecting the equations of motion, and later I will also discuss the properties of the action (1) in its various forms.

The variational equations of motion derived from the action (1) are

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \partial_\mu\Phi\partial_\nu\Phi & - \frac{1}{2}g_{\mu\nu}\partial_\lambda\Phi\partial_\lambda\Phi + g_{\mu\nu}\exp(-\sqrt{2}\Phi)\Lambda \\ & + \exp(-2\sqrt{2}\Phi)\, H_{\mu\lambda\rho}H_{\nu}{}^{\lambda\rho} \\ & - \frac{1}{6}g_{\mu\nu}\exp(-2\sqrt{2}\Phi)\, H_{\lambda\alpha\rho}H^{\lambda\alpha\rho}, \end{aligned} \quad (2)$$

$$\nabla^2\Phi = \sqrt{2}\exp(-\sqrt{2}\Phi)\Lambda - \frac{\sqrt{2}}{3}\exp(-2\sqrt{2}\Phi)H^2,$$
$$\nabla_\mu \exp(-2\sqrt{2}\Phi)H^{\mu\nu\lambda} = 0$$

along with the Bianchi identity for the axion field strength, $\partial_{[\mu}H_{\nu\lambda\sigma]} = 0$. The background ansatz is that of a stationary axially symmetric metric:

$$ds^2 = \mu^2(r)dr^2 + G_{jk}(r)\; dx^j dx^k + \eta^2(r)dz^2 \quad (3)$$

where the 2×2 matrix $G_{jk}(r)$ is of signature 0 as the metric (3) is Lorentzian and one of the coordinates $\{x^k\}$ is timelike. The z coordinate in (3) is noncompact, whereas the spacelike x^k is compact. The "lapse" function μ^2 is kept arbitrary as its variation in (1) yields the constraint equation. The dilaton Φ is a function of r only, and the axion equations $\partial_{[\mu}H_{\nu\lambda\sigma]} = 0$ and $\partial_\mu\left(\sqrt{g}\exp(-2\sqrt{2}\Phi)H^{\mu\lambda\sigma}\right) = 0$ are solved in terms of the dual vector field V^μ by

$$H_{\nu\lambda\sigma} = \exp(2\sqrt{2}\Phi)\sqrt{g}\epsilon_{\mu\nu\lambda\sigma}V^{\mu} \tag{4}$$

and $V = Qdz$ (the topological charge term). This solution has been discussed at greater length in [16] (see also [11]). There, it has been argued that the axion equations of motion can be solved by topological charge terms in cylindrical backgrounds, since such topologies include a nontrivial first cohomology from a noncontractible loop S^1 in the manifold. The charge Q above can therefore be thought of as associated with such a loop of string, except that the string size is bigger than the cosmological horizon.

In order to obtain the four-dimensional black string solution by adding the stringy version of the BTZ black hole [8–9] a flat direction, one further requires that the dilaton and the coefficient of the fourth direction η are constant. The equations of motion for these two modes can be written as

$$\nabla^2\Phi = \frac{\partial V_{\rm eff}(\Phi,\eta)}{\partial\Phi} \tag{5}$$

and

$$2\nabla^2\ln\eta = \frac{\partial V_{\rm eff}(\Phi,\eta)}{\partial\ln\eta} \tag{6}$$

where the effective potential is

$$V_{\rm eff}(\Phi,\eta) = \frac{Q^2}{\eta^4}\,e^{2\sqrt{2}\Phi} - \frac{\Lambda}{\eta^2}\,e^{\sqrt{2}\Phi}. \tag{7}$$

Interestingly, the system of equations (5), (6) is simultaneously solved by a "vacuum" $\Phi = \Phi_0$, $\eta = \eta_0$ provided that

$$Q^2 = \frac{\Lambda}{2}\eta_0^2\,e^{-\sqrt{2}\Phi_0}. \tag{8}$$

This equation can always be satisfied, and actually can be viewed as the definition of the dilaton vacuum expectation value given the other parameters.

Substituting this ansatz back into the remaining field equations (2), it is easy to verify that the complete rotating black string solution of (1) is (after setting $\eta_0 = 1$)

$$\begin{aligned} ds^2 &= \frac{d\rho^2}{\lambda_{3D}(\rho^2-\rho_+^2)} + R^2(d\theta + N^\theta dt)^2 \\ &\quad -\frac{\rho^2}{R^2}\frac{\rho^2-\rho_+^2}{\lambda_{3D}}dt^2 + dz^2, \\ H_{t\rho\theta} &= \frac{\rho}{Q}, \\ \Phi &= \Phi_0, \end{aligned} \tag{9}$$

with $\rho_+^2 = M[1-(J/M)^2]^{1/2}$, $R^2 = (\sqrt{\lambda_{3D}}/2)(\rho^2 + M - \rho_+^2)$, and $N^\theta = -J/2R^2$, with $\lambda_{3D} = (\Lambda/2\eta_0^2)\,e^{\sqrt{2}\Phi_0}$ and where the identity (8) has been used. The physical black strings should also satisfy the constraint $|J| \le M$. If this were not satisfied, one would end up with a singular structure, manifest by the appearance of closed timelike curves in the manifold accessible to an external observer, crossing the point $R = 0$. Such a voyage has been investigated in [11] for the spinless case, and also in [10] for the vacuum. Moreover, it has been argued that, although the solution (9) does not have curvature singularities $[R_{\mu\nu\lambda 3} = 0,\ R_{\mu\nu\lambda\sigma} = -\lambda_{3D}(g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda})]$, they can develop if the metric is slightly perturbed by a matter distribution [3]. Thus the singularities are hidden by a horizon if the spin is bounded above by the mass. By analogy with the BTZ black hole, the solution with $J = M$ is understood as the extremal black string, and $J = M = 0$ as the vacuum. There is a local correspondence between these two cases, as discussed in [9].

The solution (9) describes a rotating black string as is rather obvious from the metric. However, it is the axion which gives further clues regarding the nature of the string. As was mentioned above, one way to think of the solution is to imagine it as a loop of string with its length parametrized by z, which is bigger than the cosmological horizon of the Universe where it is embedded. In this sense, the solution represents an explicit example of a toroidal black hole [15]. Such an interpretation obviously puts limits on the validity of the approximations underlying the assumption that the string is straight. A more interesting picture is obtained if one retains the image of the string as infinitely long and straight. The dual axion field strength $V = Qdz = da(z)$ can be integrated between any two spacelike (t = const) hypersurfaces $z_{1,2}$ = const to give $a(z_2) - a(z_1) = Q\Delta z$. Therefore the axion solution can be understood as a constant gradient of the pseudoscalar axion field. As $z_{1,2} \to \infty$, the axion diverges. But this is easy to explain: it is merely a consequence of the assumption that the string is infinitely long. In reality, one should expect some cutoff sufficiently far away along the string. The situation is precisely analogous to that of the electrostatic potential between the plates of a parallel plate capacitor in ordinary electromagnetism. There, the cutoff occurs on the plates of the capacitor, where the potential assumes constant values. The gradient is just $\nabla V = (\Delta V/\Delta L)\mathbf{z}$. This analogy shows that the black string solution (9) should be viewed as a gravitational configuration which arose inside a transitory region separating two domains within which the axion is constant: a_1 and a_2, respectively. The axion gradient inside this region corresponds to the adiabatic change in the axion vacuum, where the adiabatic approximation is better if the transitory region (and hence the string) is bigger. The configuration (9) then evidently needs the domain of axionic gradient for its existence (because the axion gradient stops the dilaton from rolling), and thence can justifiably be labeled primordial. It is worth noting here that one can object to the analogy of the solution (9) and the toroidal black holes of [15] on the grounds that the pseudoscalar axion $a(z)$ increases linearly with z, and hence apparently is not periodic as required, even if the z direction is compactified. This problem is easily solved once one recalls that the pseudoscalar axion is by definition a coordinate on a circle U(1), and hence the translations by $2n\pi$ represent global gauge transformations. Therefore the increase of the axion is completely artificial and can be safely ignored.

The discussion of the previous paragraph illustrates only one aspect of the importance of the axion in obtaining the solution (9). In order to gain further insight for its role, as well as of the dilaton and the mode η, one can dimensionally reduce the action (1). The simple form of the axion allows that it be integrated out from the action, with the help of a Lagrange multiplier, and treating the solution (4) as a constraint. The resulting effective action for the dilaton-gravity system is

$$S = \int d^4x\sqrt{g}\left(\frac{1}{2}R - \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{Q^2}{\eta^2}\, e^{2\sqrt{2}\Phi} + \Lambda e^{\sqrt{2}\Phi}\right). \tag{10}$$

Note that the action (1) has been rewritten in the form similar to Einstein gravity with a minimally coupled self-interacting scalar field. Also note that the sign of Q^2 is negative. This is a consequence of the proper replacement of the axion with its charge using the Lagrange multiplier method, and can be verified from the inspection of the equations of motion (2). The background has three toroidal coordinates $\{x^k\}$ and z which are dynamically irrelevant. Hence the problem is effectively one dimensional and the Kaluza-Klein reduction [17] can be employed to simplify the action (10). It is instructive here to perform the Kaluza-Klein reduction in two steps, in order to isolate the dynamics of the mode η. The first step is to integrate out the coordinate z. The resulting effective action in three dimensions (3D) is, after the rescaling $S^{3D}_{\rm eff} = S/\int dz$,

$$S^{3D}_{\rm eff} = \int d^3x\sqrt{G}\,\left(\frac{1}{2}\eta\bar{R} - \frac{1}{2}\eta\partial_\mu\Phi\partial^\mu\Phi - \frac{Q^2}{\eta}\, e^{2\sqrt{2}\Phi} + \eta\Lambda e^{\sqrt{2}\Phi}\right). \tag{11}$$

The mode η (the "compacton") in this form of the action is obviously just a Lagrange multiplier. Its Euler-Lagrange equation however involves the 3D part of the metric. Thus, to investigate, it is necessary to write out the complete set of Einstein's equations in addition to it. This can be avoided with a conformal redefinition of the 3D metric such that the 3D Ricci curvature disappears from the η equation. The conformal rescaling which ensures this is $G_{\mu\nu} = (1/\eta^2)\tilde{G}_{\mu\nu}$. The resulting action is just

$$S^{3D}_{\rm eff} = \int d^3x\sqrt{\tilde{G}}\,\left(\frac{1}{2}\tilde{R} - \frac{1}{\eta^2}\partial_\mu\eta\partial^\mu\eta - \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{Q^2}{\eta^4}\, e^{2\sqrt{2}\Phi} + \frac{\Lambda}{\eta^2}\, e^{\sqrt{2}\Phi}\right). \tag{12}$$

This action represents ordinary 3D general relativity with two minimally coupled self-interacting fields Φ and $\ln\eta$, with self-interactions described by the effective potential (7). Thus, the ansatz employed to obtain the solution (9) corresponds to the "vacuum" sector of the effective theory given by (12). The remaining effective action for gravity in three dimensions under the assumption that the "matter" modes are in "vacuum" can be obtained after substituting (8) in (12), and dropping the dynamics of the dilaton and the compacton. It is

$$S^{3D}_{\rm eff\ vac} = \int d^3x\sqrt{\tilde{G}}(\tfrac{1}{2}\tilde{R} + \lambda_{3D}) \tag{13}$$

and represents just the normal 3D Einstein-Hilbert action with an effective (negative) cosmological constant $\lambda_{3D} = (\Lambda/2\eta_0^2)\, e^{\sqrt{2}\Phi_0}$. Its unique black hole solution is the BTZ solution, as shown in [3,9]. This is why the metric part of the black string solution can be written as $ds^2 = ds^2_{\rm BTZ} + dz^2$ after setting $\eta_0 = 1$.

Thence, besides providing the extra contribution to the dilaton-compacton self-interactions, the axion also plays role of a Higgs field, which is evident from the steps leading from Eq. (1) to Eq. (10). The axion condensate Q^2 in (4) breaks the normal general covariance group GL(3,1) of (1) down to GL(2,1) which is the invariance group of (10). It should be noted, though, that the Higgs-like behavior of the axion is purely topological; indeed, in the $O(\alpha'^0)$ approximation, the axion has no self-interactions, and hence no potential to minimize. Again, this is not really a surprise. The behavior of the Peccei-Quinn (PQ) axion has been found very much the same, and at the tree level the PQ axion condensate was also purely topological. It was only after the radiative corrections were included, that its self-interaction potential arose. Hence, to investigate the Higgs aspect of the axion further it would be necessary to inspect higher order corrections to (1).

This program could be best conducted via the Wess-Zumino-Witten-Novikov (WZWN) σ model approach [2]. Namely, it was demonstrated recently that the BTZ solution can be obtained as either a nongauged WZWN model on the group $SL(2,R)/P$ or an extremely gauged WZWN model on the coset $[SL(2,R)\times R]/(R\times P)$, where P is a discrete group which represents compactification of one of the spacelike coordinates to a circle [8,9]. In this light, the solution (9) is obviously obtained by taking either of these two σ models and simply tensoring them with an additional flat direction, which will be the coordinate along the string. Thus, specifically, (9) is an extremely gauged WZWN model on the coset $[SL(2,R)\times R^2]/(R\times P)$. Higher order corrections could now be investigated following the resummation procedure established by Tseytlin [18] and by Bars and Sfetsos [19]. It turns out that the black string configuration actually survives the corrections, and appears to be an exact solution of string theory to all orders in α'. The only effect of the higher order α' corrections is finite renormalization of the parameters in (9), and in particular, renormalization of the semiclassical expression for the cosmological constant. The details will be presented elsewhere [20].

There still remains the problem of stability of solution (9) under small perturbations. Some indications can be obtained by looking at the "matter" sector of the effective action $S^{3D}_{\rm eff}$ (12), after the conformal rescaling. The dilaton and the compacton in (12) can be viewed as an O(2) doublet, with the effective potential (7) manifestly breaking O(2). As a consequence, the linear combination $\sqrt{2}\Phi - \ln\eta$ of the dilaton and compacton picks up a mass term of order Λ, whereas its orthogonal complement remains massless. It would have been preferable if both the dilaton and the compacton became massive, because their big masses would *de facto* decouple them and improve the stability of the solution (9). As is, the solution (9) could actually be spoiled by perturbations of the massless mode, which can accumulate exterior to the black hole, much like the Goldstone modes present in global cosmic string backgrounds [21]. This remains to be investigated further in the future.

In closing, it has been shown that the serendipitous BTZ 3D black hole has simple generalizations to four dimensions, where it can be interpreted as a primordial spinning black string, which is singularity-free. The most attractive generalization is where it represents a vacuum solution of tree level string theory, where the dilaton and compacton have been decoupled due to the axion charge. In this respect, the axion plays the role of a Higgs field, since it breaks the invariance group GL(3,1) of the underlying 4D theory down to the invariance group GL(2,1) of the resulting three-dimensional effective action, and modifies the effective scalar potential of the model leading to the previously mentioned decoupling of the scalar modes. The dilaton of the configuration is constant and thus (9) also represents a solution of four-dimensional Einstein gravity with a minimally coupled 3-form field strength (see also [22]). Moreover, the solution represents an exact WZWN σ model on the world sheet, and thence can be easily extended to include higher order α' corrections, as I will show elsewhere [20]. In the end, these do not affect the nature of the solution, and it remains a well behaved singularity-free string configuration with a horizon.

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[1] For a recent review of stringy aspects of black holes and strings see J.A. Harvey and A. Strominger, Fermilab Report No. EFI-92-41/hep-th/9209055, 1992 (unpublished); A. Sen, Tata Institute Report No. TIFR-TH-92-57/hep-th/9210050, 1992 (unpublished); G.T. Horowitz, University of California at Santa Barbara Report No. UCSBTH-92-32/hep-th/9210119, 1992 (unpublished), and references therein.
[2] E. Witten, Phys. Rev. D **44**, 314 (1991).
[3] M. Banados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992); M. Banados *et al.*, IAS Report No. HEP-92/81/gr-qc/9302012, 1993 (unpublished); D. Cangemi, M. Leblanc, and R.B. Mann, Phys. Rev. D **48**, 3606 (1993).
[4] G.W. Gibbons, Nucl. Phys. **B207**, 337 (1982); M.J. Bowick *et al.*, Phys. Rev. Lett. **61**, 2823 (1988); D. Garfinkle, G.T. Horowitz, and A. Stominger, Phys. Rev. D **43**, 3140 (1991); A. Shapere *et al.*, Mod. Phys. Lett. A **6**, 2677 (1991).
[5] A. Sen, Phys. Rev. Lett. **69**, 1006 (1992).
[6] R. Kallosh *et al.*, Phys. Rev. D **46**, 5278 (1992).
[7] D. Gershon, Tel Aviv University Report No. TAUP-1937-91/hep-th/9202005, 1991 (unpublished); Phys. Rev. D (to be published).
[8] G.T. Horowitz and D.L. Welch, Phys. Rev. Lett. **71**, 328 (1993).
[9] N. Kaloper, Phys. Rev. D **48**, 2598 (1993).
[10] J.H. Horne and G.T. Horowitz, Nucl. Phys. **B368**, 444 (1992).
[11] N. Kaloper, Phys. Rev. D **47**, 2403 (1993).
[12] A. Sen, Nucl. Phys. **B388**, 457 (1992); E. Raiten, Fermilab Report No. FERMILAB-PUB-91-338-T, 1991 (unpublished); S. Mahapatra, Institute of Mathematical Sciences Madras, Report No. IMSC-93-06/hep-th/9301125, 1993 (unpublished).
[13] I. Bars and K. Sfetsos, Phys. Rev. D **46**, 4495 (1992); **46**, 4510 (1992); Phys. Lett. B **301**, 183 (1993); K. Sfetsos, Nucl. Phys. **B389**, 424 (1993).
[14] S.B. Giddings and A. Strominger, Phys. Rev. Lett. **67**, 2930 (1991); G.T. Horowitz and A. Strominger, Nucl. Phys. **B350**, 197 (1991); A. Sen, Phys. Lett. B **274**, 34 (1992); Supriya K. Kar, S. Pratik Khastgir, and Gautam Sengupta, Phys. Rev. D **47**, 3643 (1993); S. Mahapatra, Mod. Phys. Lett. A **7**, 2999 (1992).
[15] R. Geroch and J. Hartle, J. Math. Phys. **23** (4), 680 (1981).
[16] E. Witten, Phys. Lett. **153B**, 243 (1985).
[17] For a review see E. Cremmer, in *Supergravity 81*, edited by S. Ferrara and J.G. Taylor (Cambridge University Press, Cambridge, England, 1982), p. 313.
[18] A. Tseytlin, Nucl. Phys. **B399**, 601 (1993); CERN Report No. CERN-TH-6804/93/hep-th/9302083, 1993 (unpublished).
[19] I. Bars and K. Sfetsos, Phys. Rev. D **48**, 844 (1993).
[20] N. Kaloper, University of Alberta Report No. Alberta-Thy-7-93 (unpublished).
[21] See, e.g., A. Vilenkin, Phys. Rep. **121**, 263 (1985).
[22] G. Grignani and G. Nardelli, Università degli Studi di Perugia Report No. DFUPG-59-1992/gr-qc/9211001, 1992 (unpublished).