Parametrized post-Newtonian gravitational redshift

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A derivation of the gravitational redshift effect to order c^{-4} is presented. The calculation is performed within the framework of the parametrized post-Newtonian formalism for analyzing metric theories of gravity, which includes corrections to second order in the Newtonian potential, gravitomagnetic contributions, and preferred-frame terms. We briefly discuss how to generalize our results to include possible violations of local Lorentz invariance or local position invariance which can arise in nonmetric theories. Our results are useful for analyzing possible new redshift experiments which may be sensitive to second-order effects, such as a space mission involving a close solar flyby.

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I. INTRODUCTION

The gravitational redshift effect is the observed shift to a lower frequency of an oscillator near a massive body relative to its frequency at infinity. This is a fundamental result of the Einstein equivalence principle (EEP), upon which general relativity and all other metric theories of gravity are based [1]. If the EEP is valid, then the laws of physics governing the operation of an oscillator or a clock should be locally Lorentz invariant and position invariant in a gravitational field. By adopting only these two requirements, it is possible to derive the redshift effect to first order in the Newtonian potential without specifying a particular theory of gravity (for example, see Ref. [2]).

To order c^{-2}, the frequency shift of a photon propagated between two points $\mathbf{x}_1$ and $\mathbf{x}_2$ is given in an inertial reference frame by the expression

$$f_2=f_1[1-\hat{\mathbf{n}}\cdot(\mathbf{v}_2-\mathbf{v}_1)/c-\tfrac{1}{2}(v_1^2-v_2^2)/c^2 -(\hat{\mathbf{n}}\cdot\mathbf{v}_1)(\hat{\mathbf{n}}\cdot\mathbf{v}_2)/c^2 +(\hat{\mathbf{n}}\cdot\mathbf{v}_1)^2/c^2-(U_1-U_2)/c^2]\,, \tag{1.1}$$

where $\mathbf{v}_1$ is the velocity of the emitter at $\mathbf{x}_1$, $\mathbf{v}_2$ is the velocity of the receiver at $\mathbf{x}_2$, $\hat{\mathbf{n}}$ is a unit vector pointing from $\mathbf{x}_1$ to $\mathbf{x}_2$, and U_1 and U_2 are the total Newtonian gravitational potentials at each point, respectively, defined positively. Equation (1.1) is consistent with the EEP to first order in U. Any metric theory of gravity, such as general relativity, must make the same prediction to this order, provided the theory yields the correct Newtonian equation of motion.

The first-order prediction has been tested to highest precision in a 1976 NASA experiment called Gravity Probe A (GP-A), in which a hydrogen maser oscillator was flown on a Scout rocket in the gravitational field of the Earth [3]. Additional spacecraft experiments have been performed at Saturn [4] and in the solar gravitational field [5], but with less stable crystal oscillators. A close solar probe mission has been studied by NASA for many years, in which a spacecraft would fly by the Sun at a heliocentric distance of only 4 solar radii (for a recent review, see Ref. [6]). Similar missions have been considered by the European Space Agency (ESA) and by the Russian Institute for Space Research (IKI). If an atomic frequency standard, such as a hydrogen maser oscillator, were included on the spacecraft, then it might be possible to test the redshift effect to second order in the Newtonian potential of the Sun at an interesting level of precision. At second order, the experiment would test not only the EEP, but also specific theories of gravity. A small group of scientists has recently been funded by NASA to investigate this possibility. A group at the Jet Propulsion Laboratory (JPL) is performing a mission simulation and a detailed covariance analysis. A second group at the Smithsonian Center for Astrophysics (CfA) is investigating requirements on the maser flight unit. At this point, however, NASA has made no definite commitment to proceed with the mission.

In a previous study, general relativistic effects on the equation of motion of the spacecraft were analyzed to post-Newtonian accuracy in the PPN formalism [7]. Only coherently transponded Doppler and range data were included in the analysis; noncoherent one-way Doppler data were not considered [8]. In this work, we present a derivation of the gravitational redshift to order c^{-4} in support of the new study. We will adopt the parametrized post-Newtonian (PPN) framework for analyzing metric theories of gravity. However, we will restrict our analysis to semiconservative theories of gravity, in which the PPN parameters $\{\alpha_3,\zeta_1,\zeta_2,\zeta_3,\zeta_4\}$ are identically zero. Furthermore, we will assume a single, stationary body whose center of mass is at rest, and which is rotating slowly enough that we may also assume nearly spherical symmetry; appropriate assumptions for analyzing a close solar flyby mission. Unlike the first-order calculation, we will see that it is necessary to include corrections to the photon equation of motion in order to model the redshift consistently to order c^{-4}. Although our main focus in this paper is on PPN effects in the redshift, we will briefly consider how to include possible violations of local Lorentz invariance and local position invariance.

The remainder of this paper is organized as follows. In the next section, we will derive a general form for the redshift which includes contributions from the photon wave vector. In Sec. III, we will integrate the photon equation of motion. Our final results are presented in Sec. IV, and conclusions in Sec. V. For convenience, units in which $G=c=1$ will be used. Greek indices range over 0,1,2,3, whereas Latin indices range over 1,2,3. Partial derivatives are denoted by a comma, and covariant derivatives with respect to the metric connection by a semicolon.

II. THE MEASURED FREQUENCY SHIFT

For metric theories of gravity, the frequency f_m of a photon measured by an observer at a point (t,x^i) can be found by projecting the photon wave vector k onto the observer's four-velocity u at that point:

$$\omega_m=-k_\mu u^\mu=-(k_0u^0+k_iu^i)\ , \tag{2.1}$$

where $\omega_m=2\pi f_m$. We can expand the components of k about their flat spacetime values according to

$$k_0(x)=-\omega[1-\hat{\mathbf{k}}_0(x)]\ , \tag{2.2a}$$

$$k_i(x)=\omega[\hat{\mathbf{n}}_i+\hat{\mathbf{k}}_i(x)]\ , \tag{2.2b}$$

where $\hat{\mathbf{n}}_i$ is a unit vector along the direction of propagation. Equation (2.1) then becomes

$$\omega_m=\omega u^0[1-\hat{\mathbf{k}}_0(\mathbf{x})-\hat{\mathbf{n}}\cdot\mathbf{v}-\hat{\mathbf{k}}(\mathbf{x})\cdot\mathbf{v}]\ , \tag{2.3}$$

where $\mathbf{v}$ is the observer's three-velocity. Equation (2.3) can be used to calculate the ratio of the frequencies which would be measured at two points $x_1=(t_1,x_1^i)$ and $x_2=(t_2,x_2^i)$. Expanded to order v^4, the result is

$$\frac{\omega(\mathbf{x}_2)}{\omega(\mathbf{x}_1)}=\frac{u^0(\mathbf{x}_2)}{u^0(\mathbf{x}_1)}\{1-\hat{\mathbf{n}}\cdot(\mathbf{v}_2-\mathbf{v}_1)-\hat{\mathbf{n}}\cdot(\mathbf{v}_2-\mathbf{v}_1)[\hat{\mathbf{n}}\cdot\mathbf{v}_1+(\hat{\mathbf{n}}\cdot\mathbf{v}_1)^2+(\hat{\mathbf{n}}\cdot\mathbf{v}_1)^3]$$
$$-[\hat{\mathbf{k}}(\mathbf{x}_2)\cdot\mathbf{v}_2-\hat{\mathbf{k}}(\mathbf{x}_1)\cdot\mathbf{v}_1]+2(\hat{\mathbf{n}}\cdot\mathbf{v}_1)\hat{\mathbf{k}}(\mathbf{x}_1)\cdot\mathbf{v}_1-\hat{\mathbf{k}}(\mathbf{x}_2)\cdot\mathbf{v}_2(\hat{\mathbf{n}}\cdot\mathbf{v}_1)-\hat{\mathbf{k}}(\mathbf{x}_1)\cdot\mathbf{v}_1(\hat{\mathbf{n}}\cdot\mathbf{v}_2)\}\ , \tag{2.4}$$

where we have neglected contributions from $\hat{\mathbf{k}}_0$ (to be justified in the next section). Equation (2.4) provides a general form for the measurable frequency shift. To make Eq. (2.4) more useful, we will have to provide specific expressions for u^0 and the perturbed wave-vector components $(\hat{\mathbf{k}}_0,\hat{\mathbf{k}}_i)$.

With proper time defined by $d\tau^2=-g_{\mu\nu}dx^\mu dx^\nu$, then

$$u^0=\frac{dx^0}{d\tau}$$
$$=(-g_{00})^{-1/2}\left[1+2\frac{g_{0j}}{g_{00}}v^j+\frac{g_{ij}}{g_{00}}v^iv^j\right]^{-1/2}\ , \tag{2.5}$$

Under the assumptions stated in Sec. I, the PPN metric components are given by

$$g_{00}=-1+2U-2\beta U^2-(\alpha_1-\alpha_2)w^2U-\alpha_2U(\mathbf{w}\cdot\hat{\mathbf{r}})^2$$
$$+\tfrac{1}{2}\alpha_1\mathbf{w}\cdot(\hat{\mathbf{r}}\times\mathbf{J})/r^2\ , \tag{2.6a}$$

$$g_{0j}=\tfrac{1}{4}\Delta\frac{(\hat{\mathbf{r}}\times\mathbf{J})_j}{r^2}-\tfrac{1}{2}(\alpha_1-2\alpha_2)Uw^j-\alpha_2(\hat{\mathbf{r}}\cdot\mathbf{w})U\hat{\mathbf{r}}^j\ , \tag{2.6b}$$

$$g_{ij}=(1+2\gamma U)\delta_{ij}\ , \tag{2.6c}$$

where $\hat{\mathbf{r}}=\mathbf{x}/|\mathbf{x}|$, $\Delta=4\gamma+4+\alpha_1$, and $\mathbf{w}$ is the possible preferred-frame velocity of the PPN coordinate system (for example, see Ref. [9]). The vector $\mathbf{J}$ represents the rotational angular momentum of the body. Using these metric components in Eq. (2.5), we obtain, to order c^{-4},

$$u^0=1+U+\tfrac{1}{2}v^2+(\tfrac{3}{2}-\beta)U^2+(\tfrac{3}{2}+\gamma)Uv^2+\tfrac{3}{8}v^4$$
$$-\tfrac{1}{2}(\alpha_1-\alpha_2)w^2U-\tfrac{1}{2}\alpha_2U(\mathbf{w}\cdot\hat{\mathbf{r}})^2$$
$$+\tfrac{1}{4}\alpha_1\mathbf{w}\cdot(\hat{\mathbf{r}}\times\mathbf{J})/r^2+\tfrac{1}{4}\Delta\mathbf{v}\cdot(\hat{\mathbf{r}}\times\mathbf{J})/r^2$$
$$-\tfrac{1}{2}(\alpha_1-2\alpha_2)(\mathbf{v}\cdot\mathbf{w})U-\alpha_2U(\mathbf{v}\cdot\hat{\mathbf{r}})(\mathbf{w}\cdot\hat{\mathbf{r}})\ . \tag{2.7}$$

This expression can be used to calculate the ratio $u^0(\mathbf{x}_2)/u^0(\mathbf{x}_1)$ in Eq. (2.4). The result is

$$\frac{u^0(\mathbf{x}_2)}{u^0(\mathbf{x}_1)}=1-(U_1-U_2)-\tfrac{1}{2}(v_1^2-v_2^2)-\tfrac{3}{8}(v_1^4-v_2^4)-\tfrac{1}{4}v_1^2v_2^2+\tfrac{1}{4}v_1^4-[(\tfrac{1}{2}-\beta)U_1^2-(\tfrac{3}{2}-\beta)U_2^2+U_1U_2]$$
$$-[(\tfrac{1}{2}+\gamma)U_1v_1^2-(\tfrac{3}{2}+\gamma)U_2v_2^2+\tfrac{1}{2}(U_1v_2^2+U_2v_1^2)]$$
$$+\tfrac{1}{2}\{(\alpha_1-\alpha_2)w^2(U_1-U_2)+\alpha_2[U_1(\mathbf{w}\cdot\hat{\mathbf{r}}_1)^2-U_2(\mathbf{w}\cdot\hat{\mathbf{r}}_2)^2]+(\alpha_1-2\alpha_2)[(\mathbf{v}_1\cdot\mathbf{w})U_1-(\mathbf{v}_2\cdot\mathbf{w})U_2]$$
$$+2\alpha_2[U_1(\mathbf{v}_1\cdot\hat{\mathbf{r}}_1)(\mathbf{w}\cdot\hat{\mathbf{r}}_1)-U_2(\mathbf{v}_2\cdot\hat{\mathbf{r}}_2)(\mathbf{w}\cdot\hat{\mathbf{r}}_2)]\}$$
$$-\tfrac{1}{4}\{\Delta[\mathbf{v}_1\cdot(\hat{\mathbf{r}}_1\times\mathbf{J})/r_1^3-\mathbf{v}_2\cdot(\hat{\mathbf{r}}_2\times\mathbf{J})/r_2^2]+\alpha_1\mathbf{w}\cdot[(\hat{\mathbf{r}}_1\times\mathbf{J})/r_1^2-(\hat{\mathbf{r}}_2\times\mathbf{J})/r_2^2]\}\ . \tag{2.8}$$

III. COMPUTATION OF THE PHOTON WAVE VECTOR

It is convenient to express the PPN metric in the form

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}\ , \tag{3.1}$$

where $\eta=\mathrm{diag}(-1,+1,+1,+1)$ is the Minkowski tensor and $h_{\mu\nu}$ is the metric perturbation. Using this definition in the equation of motion of the photon,

$$k_{\mu;\nu}k^{\nu}=0\ , \tag{3.2}$$

results in the expression

$$\frac{dk_\mu}{d\lambda}=\tfrac{1}{2}h_{\beta\nu,\mu}k^{\beta}k^{\nu}\ , \tag{3.3}$$

where λ is an affine parameter along the photon trajectory (see Ref. [1] for further details). Expanding k_μ according to Eq. (2.2) and applying the condition $k_\mu k^\mu=0$, we obtain from Eqs. (2.6) and (3.3) the result

$$\frac{d\hat{\mathbf{k}}_\mu}{dt}=(1+\gamma)U_{,\mu}+h_{0j,\mu}\hat{\mathbf{n}}^j\ , \tag{3.4}$$

to necessary order, where t is coordinate time. Equation (3.4) can be integrated to obtain $\hat{\mathbf{k}}_\mu(x)$. We note that $\hat{\mathbf{k}}_0$ is zero at this order, given no explicit time dependence of the source. This is why it was not included in Eq. (2.4).

To integrate Eq. (3.4) for $\hat{\mathbf{k}}_i$, we assume a nearly spherically symmetric source such that $U(\mathbf{x})=M/r$; the expression for h_{0j} is dictated by Eq. (2.6b). The unperturbed path of the photon is a straight line with coordinates $\mathbf{x}(t)=\hat{\mathbf{n}}(t-t_e)+\mathbf{x}_e$ as a function of t, where $(t_e,\mathbf{x}_e)$ specifies the time and place of emission. To necessary order, this relation can be used to integrate Eq. (3.4). However, the constant of integration is specified by requiring that the orthogonal projection of the actual perturbed path satisfies

$$\left.\frac{dx_\perp^i}{dt}\right|_{t_e}=0\ , \tag{3.5a}$$

$$x_\perp^i=x^i-\hat{\mathbf{n}}^i(\hat{\mathbf{n}}\cdot\mathbf{x})\ , \tag{3.5b}$$

i.e., that upon emission the photon propagates initially in the direction $\hat{\mathbf{n}}$, where

$$\frac{dx^i}{dt}=[\hat{\mathbf{n}}^i-2(1+\gamma)U\hat{\mathbf{n}}^i+\hat{\mathbf{k}}^i]\ , \tag{3.6}$$

to the required order. After evaluating several integrals and collecting terms, we find that, for a photon emitted from the point x_i in an initial direction $\hat{\mathbf{n}}$,

$$\hat{\mathbf{k}}(\mathbf{x})=(1+\gamma)\frac{M}{d^2}\left[\frac{\hat{\mathbf{n}}(\mathbf{x}\cdot\mathbf{x}_1)}{r}-\frac{\mathbf{x}_1(\hat{\mathbf{n}}\cdot\mathbf{x})}{r}+\frac{\mathbf{d}(\hat{\mathbf{n}}\cdot\mathbf{x}_1)}{r_1}\right]$$

$$-\tfrac{1}{4}\Delta\left\{\frac{(\mathbf{x}_1\times\mathbf{J})}{r_1^3}+\frac{\hat{\mathbf{n}}\times\mathbf{J}}{d^2}\left[\frac{\hat{\mathbf{n}}\cdot\mathbf{x}}{r}-\frac{\hat{\mathbf{n}}\cdot\mathbf{x}_1}{r_1}\right]+\hat{\mathbf{n}}[(\hat{\mathbf{n}}\times\mathbf{J})\cdot\mathbf{x}_1]\left[\frac{1}{r^3}-\frac{1}{r_1^3}\right]\right.$$

$$\left.+[\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{x}_1)][(\hat{\mathbf{n}}\times\mathbf{J})\cdot\mathbf{x}_1]\left[\left[\frac{\hat{\mathbf{n}}\cdot\mathbf{x}}{r^3}-\frac{\hat{\mathbf{n}}\cdot\mathbf{x}_1}{r_1^3}\right]+\frac{2}{d^2}\left[\frac{\hat{\mathbf{n}}\cdot\mathbf{x}}{r}-\frac{\hat{\mathbf{n}}\cdot\mathbf{x}_1}{r_1}\right]\right]\right\}$$

$$+\tfrac{1}{2}(\alpha_1-2\alpha_2)M\left[\frac{\mathbf{w}}{r_1}-\frac{\hat{\mathbf{n}}(\mathbf{x}\cdot\mathbf{x}_1)}{d^2r}+\frac{\mathbf{x}_1(\hat{\mathbf{n}}\cdot\mathbf{x})}{d^2r}-\frac{\mathbf{d}(\hat{\mathbf{n}}\cdot\mathbf{x}_1)}{d^2r_1}\right]$$

$$+\alpha_2M\left\{\frac{(\mathbf{x}_1\cdot\mathbf{w})\mathbf{x}_1}{r_1^3}+\frac{\hat{\mathbf{n}}}{d^2}\left[\frac{(\hat{\mathbf{n}}\cdot\mathbf{w})(\mathbf{x}\cdot\mathbf{x}_1)}{r}-\frac{(\mathbf{w}\cdot\mathbf{x}_1)(\hat{\mathbf{n}}\cdot\mathbf{x})}{r}+\frac{(\mathbf{w}\cdot\mathbf{d})(\hat{\mathbf{n}}\cdot\mathbf{x}_1)}{r_1}\right]+\frac{\mathbf{w}}{d^2}\left[\frac{\mathbf{x}\cdot\mathbf{x}_1}{r}-\frac{(\hat{\mathbf{n}}\cdot\mathbf{x}_1)(\hat{\mathbf{n}}\cdot\mathbf{x})}{r}\right]\right.$$

$$-(\hat{\mathbf{n}}\cdot\mathbf{w})\hat{\mathbf{n}}\left[\frac{3}{r}-\frac{r_1^2}{r^3}-\frac{2}{r_1}-\frac{2}{r}(\hat{\mathbf{n}}\cdot\mathbf{x})^2\left[\frac{1}{r^2}-\frac{1}{d^2}\right]+\frac{2}{r_1}(\hat{\mathbf{n}}\cdot\mathbf{x}_1)^2\left[\frac{1}{r_1^2}-\frac{1}{d^2}\right]-\frac{2}{r^3}(\hat{\mathbf{n}}\cdot\mathbf{x})^2\right.$$

$$\left.+\frac{2}{r_1^3}(\hat{\mathbf{n}}\cdot\mathbf{x}_1)^2-\frac{2}{d^2r}(\hat{\mathbf{n}}\cdot\mathbf{x})^3(\hat{\mathbf{n}}\cdot\mathbf{x}_1)\left[\frac{1}{r^2}+\frac{2}{d^2}\right]+\frac{2}{d^2r_1}(\hat{\mathbf{n}}\cdot\mathbf{x}_1)^4\left[\frac{1}{r^2}+\frac{2}{d^2}\right]\right]$$

$$-\left[\frac{\hat{\mathbf{n}}\cdot\mathbf{x}}{r}\left[\frac{1}{r^2}-\frac{1}{d^2}\right]-\frac{\hat{\mathbf{n}}\cdot\mathbf{x}_1}{r_1}\left[\frac{1}{r_1^2}-\frac{1}{d^2}\right]-\frac{2}{r^3}(\hat{\mathbf{n}}\cdot\mathbf{x}_1)+\frac{2}{r_1^3}(\hat{\mathbf{n}}\cdot\mathbf{x}_1)\right.$$

$$\left.-\frac{1}{d^2r}(\hat{\mathbf{n}}\cdot\mathbf{x})(\hat{\mathbf{n}}\cdot\mathbf{x}_1)^2\left[\frac{1}{r^2}+\frac{2}{d^2}\right]+\frac{1}{d^2r_1}(\hat{\mathbf{n}}\cdot\mathbf{x}_1)^3\left[\frac{1}{r_1^2}+\frac{2}{d^2}\right]\right]$$

$$\times[(\hat{\mathbf{n}}\cdot\mathbf{w})\mathbf{x}_1+(\mathbf{x}_1\cdot\mathbf{w})\hat{\mathbf{n}}+(\mathbf{x}_1\cdot\hat{\mathbf{n}})(\hat{\mathbf{n}}\cdot\mathbf{w})\hat{\mathbf{n}}]$$

$$-\left[\frac{1}{r^3}-\frac{1}{r_1^3}+\frac{(\hat{\mathbf{n}}\cdot\mathbf{x})(\hat{\mathbf{n}}\cdot\mathbf{x}_1)}{d^2 r}\left[\frac{1}{r^2}+\frac{2}{d^2}\right]-\frac{(\hat{\mathbf{n}}\cdot\mathbf{x}_1)^2}{d^2 r_1}\left[\frac{1}{r_1^2}+\frac{2}{d^2}\right]\right]$$

$$\times[(\mathbf{x}_1\cdot\mathbf{w})\mathbf{x}_1+(\hat{\mathbf{n}}\cdot\mathbf{w})(\hat{\mathbf{n}}\cdot\mathbf{x}_1)\mathbf{x}_1+(\hat{\mathbf{n}}\cdot\mathbf{x}_1)(\mathbf{x}_1\cdot\mathbf{w})\hat{\mathbf{n}}]$$

$$+\left[\frac{\hat{\mathbf{n}}\cdot\mathbf{x}}{d^2 r}\left[\frac{1}{r^2}+\frac{2}{d^2}\right]-\frac{\hat{\mathbf{n}}\cdot\mathbf{x}_1}{d^2 r_1}\left[\frac{1}{r_1^2}+\frac{2}{d^2}\right]\right](\hat{\mathbf{n}}\cdot\mathbf{x}_1)(\mathbf{w}\cdot\mathbf{x}_1)\mathbf{x}_1\Bigg\}, \tag{3.7}$$

where $\mathbf{d}=(\hat{\mathbf{n}}\times\mathbf{x}_1)$.

IV. FINAL RESULTS

We now have all of the ingredients necessary to calculate the relativistic frequency shift to order c^{-4}. Using Eq. (2.8) in Eq. (2.4), we obtain

$$\begin{aligned}\frac{f(\mathbf{x}_2)}{f(\mathbf{x}_1)}=&1-\hat{\mathbf{n}}\cdot(\mathbf{v}_2-\mathbf{v}_1)-(U_1-U_2)-\tfrac{1}{2}(v_1^2-v_2^2)-[(\hat{\mathbf{n}}\cdot\mathbf{v}_1)+(\hat{\mathbf{n}}\cdot\mathbf{v}_1)^2+(\hat{\mathbf{n}}\cdot\mathbf{v}_1)^3]\hat{\mathbf{n}}\cdot(\mathbf{v}_2-\mathbf{v}_1)\\&-[\hat{\mathbf{n}}\cdot(\mathbf{v}_2-\mathbf{v}_1)+(\hat{\mathbf{n}}\cdot\mathbf{v}_1)^2][(U_1-U_2)+\tfrac{1}{2}(v_1^2-v_2^2)]-[(\hat{\mathbf{k}}_2\cdot\mathbf{v}_2)-(\hat{\mathbf{k}}_1\cdot\mathbf{v}_1)]\\&+2(\hat{\mathbf{n}}\cdot\mathbf{v}_1)(\hat{\mathbf{k}}_1\cdot\mathbf{v}_1)-(\hat{\mathbf{n}}\cdot\mathbf{v}_1)(\hat{\mathbf{k}}_2\cdot\mathbf{v}_2)-(\hat{\mathbf{n}}\cdot\mathbf{v}_2)(\hat{\mathbf{k}}_1\cdot\mathbf{v}_1)-\tfrac{3}{8}(v_1^4-v_2^4)-\tfrac{1}{4}(v_1^2v_2^2-v_1^4)\\&-[(\tfrac{3}{2}-\beta)(U_1^2-U_2^2)-(U_1^2-U_1U_2)]-[(\tfrac{3}{2}+\gamma)(U_1v_1^2-U_2v_2^2)+\tfrac{1}{2}(U_1v_2^2+U_2v_1^2-2U_1v_1^2)]\\&-\tfrac{1}{4}\{\Delta[\mathbf{v}_1\cdot(\hat{\mathbf{r}}_1\times\mathbf{J})/r_1^2-\mathbf{v}_2\cdot(\hat{\mathbf{r}}_2\times\mathbf{J})/r_2^2]+\alpha_1\mathbf{w}\cdot[(\hat{\mathbf{r}}_1\times\mathbf{J})/r_1^2-(\hat{\mathbf{r}}_2\times\mathbf{J})/r_2^2]\}\\&+\tfrac{1}{2}\{(\alpha_1-\alpha_2)w^2(U_1-U_2)+\alpha_2[U_1(\mathbf{w}\cdot\hat{\mathbf{r}}_1)^2-U_2(\mathbf{w}\cdot\hat{\mathbf{r}}_2)^2]\\&\quad+(\alpha_1-2\alpha_2)[(\mathbf{v}_1\cdot\mathbf{w})U_1-(\mathbf{v}_2\cdot\mathbf{w})U_2]+2\alpha_2[U_1(\mathbf{v}_1\cdot\hat{\mathbf{r}}_1)(\mathbf{w}\cdot\hat{\mathbf{r}}_1)-U_2(\mathbf{v}_2\cdot\hat{\mathbf{r}}_2)(\mathbf{w}\cdot\hat{\mathbf{r}}_2)]\}\ ,\end{aligned} \tag{4.1}$$

where $\mathbf{k}(\mathbf{x})$ is given by Eq. (3.7). A possible limitation of Eq. (4.1) is the assumption of the validity of the EEP, upon which the standard PPN formalism is based. We would like to incorporate into Eq. (4.1) possible violations of local Lorentz invariance (LLI) or local position invariance (LPI). To be rigorous, a complete nonmetric formalism should be adopted which reduces to the PPN formalism in an appropriate limit [10]. However, this would lead us too far outside the scope of this paper, in which our intent has been to focus the analysis on the standard PPN formalism [11]. Nevertheless, it is possible to generalize to a certain extent the results already at hand.

We see from Eq. (2.3) that in metric theories the measured frequency of a photon depends upon the time component u^0 of the observer's four-velocity, where u^0 is defined by Eq. (2.5). This factor provides the conversion between time measured in the observer's rest frame and coordinate time. An EEP violation can result from the presence of nonmetric couplings in the equations governing the operation of an oscillator or a clock, which would thus modify Eq. (2.5). Therefore, we can account for certain possible violations by inserting the additional parameters $(\alpha,\epsilon_1,\epsilon_2)$ into Eq. (2.7) such that

$$u^0=1+\alpha U+\tfrac{1}{2}\epsilon_1 v^2+\tfrac{3}{8}\epsilon_2 v^4+\cdots\ , \tag{4.2}$$

where the ellipsis denotes the remaining terms. The PPN parameters β and γ appearing in these remaining terms should be relabeled in order to absorb other possible LLI- or LPI-violating terms having similar dependencies. We accomplish this by simply placing a "tilde" over the parameters, but leave unchanged the meanings of the gravitomagnetic parameter Δ and the preferred-frame parameters α_1 and α_2. Violations of LLI or LPI could have a different affect on the photon equation of motion. Therefore, to carry our generalization further, the parameter γ appearing in Eq. (3.7) should be given still a different name, perhaps by giving it a p subscript (for "photon"). With this in mind, our generalization of Eq. (4.1) is given by

$$\frac{f(\mathbf{x}_2)}{f(\mathbf{x}_1)}=1-\hat{\mathbf{n}}\cdot(\mathbf{v}_2-\mathbf{v}_1)-\alpha(U_1-U_2)-\frac{\epsilon_1}{2}(v_1^2-v_2^2)-[(\hat{\mathbf{n}}\cdot\mathbf{v}_1)+(\hat{\mathbf{n}}\cdot\mathbf{v}_1)^2+(\hat{\mathbf{n}}\cdot\mathbf{v}_1)^3]\hat{\mathbf{n}}\cdot(\mathbf{v}_2-\mathbf{v}_1)$$
$$-[\hat{\mathbf{n}}\cdot(\mathbf{v}_2-\mathbf{v}_1)+(\hat{\mathbf{n}}\cdot\mathbf{v}_1)^2]\left[\alpha(U_1-U_2)+\frac{\epsilon_1}{2}(v_1^2-v_2^2)\right]-[(\hat{\mathbf{k}}_2\cdot\mathbf{v}_2)-(\hat{\mathbf{k}}_1\cdot\mathbf{v}_1)]$$
$$+2(\hat{\mathbf{n}}\cdot\mathbf{v}_1)(\hat{\mathbf{k}}_1\cdot\mathbf{v}_1)-(\hat{\mathbf{n}}\cdot\mathbf{v}_1)(\hat{\mathbf{k}}_2\cdot\mathbf{v}_2)-(\hat{\mathbf{n}}\cdot\mathbf{v}_2)(\hat{\mathbf{k}}_1\cdot\mathbf{v}_1)-\epsilon_2\tfrac{3}{8}(v_1^4-v_2^4)-\frac{\epsilon_1^2}{4}(v_1^2v_2^2-v_1^4)$$
$$-[(\tfrac{3}{2}-\tilde{\beta})(U_1^2-U_2^2)-\alpha^2(U_1^2-U_1U_2)]-\left[(\tfrac{3}{2}+\tilde{\gamma})(U_1v_1^2-U_2v_2^2)+\frac{\epsilon_1\alpha}{2}(U_1v_2^2+U_2v_1^2-2U_1v_1^2)\right]$$
$$-\tfrac{1}{4}\{\Delta[v_1\cdot(\hat{\mathbf{r}}_1\times\mathbf{J})/r_1^2-\mathbf{v}_2\cdot(\hat{\mathbf{r}}_2\times\mathbf{J})/r_2^2]+\alpha_1\mathbf{w}\cdot[(\hat{\mathbf{r}}_1\times\mathbf{J})/r_1^2-(\hat{\mathbf{r}}_2\times\mathbf{J})/r_2^2]\}$$
$$+\tfrac{1}{2}\{(\alpha_1-\alpha_2)w^2(U_1-U_2)+\alpha_2[U_1(\mathbf{w}\cdot\hat{\mathbf{r}}_1)^2-U_2(\mathbf{w}\cdot\hat{\mathbf{r}}_2)^2]$$
$$+(\alpha_1-2\alpha_2)[(\mathbf{v}_1\cdot\mathbf{w})U_1-(\mathbf{v}_2\cdot\mathbf{w})U_2]+2\alpha_2[U_1(\mathbf{v}_1\cdot\hat{\mathbf{r}}_1)(\mathbf{w}\cdot\hat{\mathbf{r}}_1)-U_2(\mathbf{v}_2\cdot\hat{\mathbf{r}}_2)(\mathbf{w}\cdot\hat{\mathbf{r}}_2)]\}\ . \tag{4.3}$$

V. CONCLUSIONS

We have presented a derivation of the gravitational redshift effect to order c^{-4}. The calculation was performed within the framework of the PPN formalism for metric theories of gravity, but we also briefly considered how to include possible violations of the equivalence principle. As mentioned in the Introduction, our primary motivation was to model accurately a possible solar probe test of the redshift.

Currently, our JPL group is planning to use Eq. (4.3), along with the PPN equation of motion of the spacecraft, in a covariance analysis of a solar flyby mission to determine the expected accuracy of the experiment in the presence of anticipated error sources. These sources include stochastic spacecraft accelerations and the effects of the Earth's troposphere and ionosphere on the radio signals. We are investigating the applicability of acceleration noise models to attitude control disturbances and spacecraft buffeting. Vessot at the CfA has proposed a special "four-link" radio tracking system to provide time-correlated frequency data during the mission [12]. This approach would provide a way to cancel the first-order classical Doppler shift and to calibrate the effects of the troposphere and ionosphere on the one-way downlink. Therefore, if an atomic frequency standard, such as a hydrogen maser, were flown on the spacecraft, it should be possible to limit fractional frequency uncertainties to less than 1 part in 10^{15} during the time of the flyby (about 18 h).

We can compare the significance of certain terms in Eq. (4.3) to this limiting fractional frequency uncertainty. At the periapsis of 4 solar radii, the solar Newtonian potential U is of order 5×10^{-7} and $U^2\sim3\times10^{-13}$. The spacecraft is expected to have a velocity of roughly 300 km/s at periapsis, which implies that $v^2\sim10^{-6}$ and $v^2U\sim5\times10^{-13}$. The gravitomagnetic effect arising from solar rotation is seen to be of order vJ/r^2. Based upon the observed surface rotation rate, the solar angular momentum $J=1.63\times10^{48}$ g cm^2/s, or roughly 0.4 km^2 in units of $G=c=1$ [13]. With these values, $vJ/r^2\sim8\times10^{-16}$ and thus might be barely detectable.

A primary goal of a solar probe redshift experiment would be to test the PPN parameter β. As listed in Table 14.2 of Ref. [1], $\beta=1.000\pm0.003$ from planetary motion, assuming that the solar quadrupole moment has the small value implied by helioseismic observations. These observations favor a value for the dimensionless quadrupole moment parameter of $J_2=1.7\times10^{-7}$, with an accuracy of about 10% [14]. This knowledge is sufficient to model the Newtonian potential of the Sun to the required accuracy for the experiment (see Ref. [12] for more details). From the Viking time-delay experiment, the PPN parameter $\gamma=1.000\pm0.002$, which is just at sufficient a level of accuracy to neglect any uncertainty from its contribution to Eq. (4.1). The PPN parameters α_1 and α_2 are known to an accuracy of at least 4×10^{-4} from planetary motion and Earth gravimeter data, respectively. In Eq. (4.1), these parameters multiply terms of order w^2U. For a preferred-frame velocity of roughly $w=350$ km/s, as implied by the observed dipole anisotropy of the cosmic microwave background, $w^2U\sim7\times10^{-13}$ at 4 solar radii. With this assumption, the contributions by α_1 and α_2 could be neglected. Thus, it may be possible to test β to an accuracy comparable to that provided by planetary motion. By neglecting α_1 and α_2, we can simplify Eq. (3.7) substantially.

In addition to providing an independent test of β, perhaps the most important result of a solar probe redshift experiment would be a high-precision test of the EEP in the gravitational field of the Sun. The weak equivalence principle (equality of free fall) has been tested to high accuracy in the solar gravitational field by laboratory torsion-balance experiments (to 1 part in 10^{12}; see Sec. 2.4 of Ref. [1]). However, it is possible that these experiments might have been insensitive to certain violations of local position invariance [15]. The first-order redshift, parametrized by α in Eq. (4.3), has been tested to an accuracy of 2×10^{-4} in the gravitational field of the Earth [3], but to only 10^{-2} in the solar gravitational field [5]. It may be possible to test the solar redshift to an accuracy of 10^{-4} in a future laboratory experiment [16]. However, these accuracies could be greatly exceeded by a solar probe experiment, which could test α to an accuracy of $\sim10^{-9}$ and possible second-order violations to $\sim3\times10^{-3}$. These estimates will be quantified more precisely in the planned covariance analysis, based upon the model provided by Eq. (4.3). A more rigorous theoretical model and thorough analysis of the effects of nonmetric theories awaits a future investigation.

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