

Green function for metric perturbations due to cosmological density fluctuations

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We study scalar perturbations to a Robertson-Walker cosmological metric in terms of a pseudo-Newtonian potential, which emerges naturally from the solution of the field equations. This potential is given in terms of a Green function for matter density fluctuations of arbitrary amplitude whose time and spatial dependence are assumed known. The results obtained span both the linearized and Newtonian limits, and do not explicitly depend on any kind of averaging procedure, but make the valid assumption that the global expansion rate is that of a Friedmann-Robertson-Walker model. In addition, we discuss the similarity to diffusive processes in the evolution of the potential, and possible applications.

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I. INTRODUCTION

Most observers would agree that the geometry of our Universe is well described almost everywhere by small perturbations to a Robertson-Walker metric. These perturbations directly affect our observations of the Universe, and so a great deal of effort has been devoted to understanding their effects and evolution.

Here we study how a given matter distribution generates the metric, and hence affects the radiation that reaches us. Our goal is to find a way of expressing these perturbations directly in terms of the matter variables, in a way that may be useful for interpreting or modeling observations. Most simulations of observable effects—gravitational lensing, redshift, and so on—use some kind of simplifying assumption, such as nearest-neighbor Newtonian gravitation, weak density contrast (i.e., linearized theory), etc. Here we attempt to provide an expression which takes account of all cosmologically relevant effects, and which applies over a wide range of density contrasts.

This paper expands on the results presented briefly in Ref. [1]. We show how to derive the Green function presented there for any value of the curvature parameter, as well as its reduction to the Newtonian limit and the similarity to simple diffusion. The principal results are summarized by Eqs. (11), (24), (25), and (28) for the perturbed line element, pseudo-Newtonian potential, and Green function.

II. CONVENTIONS AND DEFINITIONS

Units are chosen with $G = c = 1$ so that all quantities can be expressed as powers of a length. The metric is written as a perturbation on the static part γ_{ab} of the Robertson-Walker background, with signature +2 and

curvature parameter $k \in \{-1, 0, 1\}$:

$$ds^2 = a^2(\eta)[\gamma_{ab}(\mathbf{x}) + h_{ab}(\eta, \mathbf{x})]dx^a dx^b. \quad (1)$$

The coefficients and coordinates are dimensionless, and so the scale factor $a(\eta)$ (which we write as a function of conformal time) is a length. Indices a, b, c, \dots run 0–3, except for i, j, \dots, n (the Fortran integers) which range over the spatial indices 1–3. An extremely useful source of formulas for the background metric and perturbations, in both conformal and proper time, is Appendixes A–D of Kodama and Sasaki [2]. Our notation is similar; in particular a prime (e.g., a') always means derivative with respect to conformal time.

Naturally the perturbations h_{ab} are assumed small compared to γ_{ab} , which are of order unity. Of course this does not imply that the corresponding matter density fluctuations are small compared to the background. We use the scheme of Futamase [3] for parametrizing the size of the perturbations and their derivatives, which is very much like the one used in the short wave approximation for studying gravitational waves in a curved background (see Isaacson [4]). With this scheme, orders of magnitude in our problem are

$$\gamma_{ab} \sim 1, \quad (2a)$$

$$h_{ab} \equiv O(\epsilon^2), \quad (2b)$$

$$\nabla_0 h_{ab} = O(\epsilon^2), \quad (2c)$$

$$\nabla_j h_{ab} \equiv O(\epsilon^2/\kappa). \quad (2d)$$

These equations serve to define ϵ and κ . Since h_{ab} are considered perturbations, their magnitude is parametrized by $\epsilon^2 \sim \text{Newtonian potential}/c^2 \ll 1$. Their derivatives are parametrized by κ , which gives the length scale relative to the particle horizon length L : $\kappa = l/L$. In principle κ may be either greater or less than 1; in practice it is usually small. Specific restrictions will apply when we consider orders of magnitude more carefully in Sec. IV.

In addition to taking $\epsilon^2 \ll 1$, it is also assumed that $\epsilon^2/\kappa \ll 1$. As shown below, this implies that the matter

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inhomogeneities always move slowly (nonrelativistically), and also that the effective stress-energy of the metric perturbations is negligible.

III. THE PERTURBATION EQUATIONS

The homogeneity and isotropy of the background γ_{ab} allows separation of space and time dependence in the field equations, and so it is useful to write the perturbations not as spacetime functions h_{ab} but as a harmonic decomposition. (See, e.g., Harrison [5] and Sasaki [6].) The perturbations' spatial dependence is expanded in eigenfunctions, or normal modes, of the covariant Laplacian $\nabla_i \nabla^i$ on the three-dimensional (3D) static background γ_{ij} , reducing the gravitational field equations to a set of equations for the time-dependent amplitudes of the modes. The Appendix gives some general references to the (extensive) literature on the eigenfunctions.

In this paper we focus on the scalar modes, representing perturbations to the metric and matter variables that can be written entirely in terms of solutions to the equations

$$\nabla_i \nabla^i Q(\mathbf{x}, \mathbf{q}) + q^2 Q = 0, \quad (3a)$$

$$k = +1 : q^2 = 0, 3, 8, \dots, (n^2 - 1), \quad (3b)$$

$$k = 0 : q^2 \geq 0, \quad (3c)$$

$$k = -1 : q^2 \geq 1. \quad (3d)$$

This describes the sort of fields resulting from density fluctuations. Spatial derivatives are longitudinal with order of magnitude $\nabla_i Q = O(q) = O(\kappa^{-1})$. Vector (rotational) and tensor (transverse) modes are not included, since it is reasonable to suppose that most observables (such as lensing and redshift) are dominated by scalar effects. However, inflation-induced gravitational waves could have a significant effect on the anisotropy of the cosmic microwave background; see, e.g., Davis *et al.* [7].

Representing the perturbations this way both simplifies and complicates the interpretation of κ . It simplifies in that for a given scalar mode Q , κ is just the reduced wavelength: $\kappa = q^{-1}$. It complicates in that for a real condensed object like a galaxy or cluster, κ must be thought of as characterizing those wavelengths which contribute the largest physical effect at a point.

$$G_{00} = 3 \left[\left(\frac{a'}{a} \right)^2 + k \right] + 2 \int d\mu Q \left[q^2 H + 3 \frac{a'}{a} H' + 3k(A - H) \right], \quad (6a)$$

$$G_{0i} = 2 \int d\mu \nabla_i Q \left[\frac{a'}{a} A - H' \right], \quad (6b)$$

$$G_{ij} = \left[\left(\frac{a'}{a} \right)^2 - 2 \left(\frac{a''}{a} \right) - k \right] \gamma_{ij} - 2\gamma_{ij} \int d\mu Q \left\{ \frac{1}{2} q^2 (A + H) + \left[\left(\frac{a'}{a} \right)^2 - 2 \left(\frac{a''}{a} \right) \right] (A - H) - \frac{a'}{a} (A' - 2H') + H'' \right\} - \int d\mu \nabla_i \nabla_j Q (A + H). \quad (6c)$$

This includes all terms linear in h_{ab} and its derivatives. It does not include nonlinear terms, which are $O(\epsilon^4)$, $O(\epsilon^4/\kappa)$, $O(\epsilon^4/\kappa^2)$, or smaller, the so-called pseu-

dotensor terms discussed at the end of Sec. IV. The stress-energy tensor T_{ab} for the matter is constructed by defining variables in the matter rest frame and then per-

$$h_{00} = 2\gamma_{00} \int d\mu(\mathbf{q}) Q(\mathbf{x}, \mathbf{q}) A(\eta, \mathbf{q}), \quad (4a)$$

$$h_{0i} = \int d\mu(\mathbf{q}) q^{-1} \nabla_i Q(\mathbf{x}, \mathbf{q}) B(\eta, \mathbf{q}), \quad (4b)$$

$$h_{ij} = 2\gamma_{ij} \int d\mu(\mathbf{q}) Q(\mathbf{x}, \mathbf{q}) [H(\eta, \mathbf{q}) + \frac{1}{3} H_T(\eta, \mathbf{q})] + 2 \int d\mu(\mathbf{q}) q^{-2} \nabla_i \nabla_j Q(\mathbf{x}, \mathbf{q}) H_T(\eta, \mathbf{q}). \quad (4c)$$

Integrals over the measure $\mu(\mathbf{q})$ stand for whatever operation expresses the completeness of the Q 's, depending on whether the eigenvalue spectra in Eqs. (3) give a discrete or continuous set of modes. Most linearized calculations omit the integrals as being implicit, but in this paper we write them explicitly for clarity in what follows.

Choosing the longitudinal gauge, $B = H_T = 0$, simplifies these expressions greatly. With only scalar perturbations this gauge fully specifies the metric and lets us think of the perturbations as local length contractions and time dilations, making for easy comparison with Newtonian theory because it leaves the metric diagonal:

$$h_{00} = -2 \int d\mu Q A, \quad (5a)$$

$$h_{0i} = 0, \quad (5b)$$

$$h_{ij} = +2\gamma_{ij} \int d\mu Q H. \quad (5c)$$

Note that the amplitudes $A(\eta, \mathbf{q})$ and $H(\eta, \mathbf{q})$ are supposed to have the same order of magnitude (ϵ^2) as the metric perturbations themselves.

(Other common choices for the gauge include harmonic and synchronous. Harmonic gauge is the usual choice for gravitational waves, and eliminates many higher-order perturbation terms in the Einstein tensor, but the form of the metric is more complicated. Synchronous gauge is also common, and has the advantage of leaving the time coordinate unchanged. But it suffers from being underspecified, even when there are only scalar modes, allowing spurious gauge-mode solutions which can be difficult and annoying to identify and remove.)

In longitudinal gauge the Einstein tensor is

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forming a Lorentz boost into the coordinate frame [8]. The perfect-fluid background model is given (scalar) perturbations to density, pressure, and velocity as

$$\bar{\rho}(\eta, \mathbf{x}) = \rho(\eta) + \rho(\eta) \int d\mu(\mathbf{q}) Q(\mathbf{x}, \mathbf{q}) \Delta(\eta, \mathbf{q}), \quad (7a)$$

$$\bar{P}(\eta, \mathbf{x}) = P(\eta) + \rho(\eta) \int d\mu(\mathbf{q}) Q(\mathbf{x}, \mathbf{q}) \Pi(\eta, \mathbf{q}), \quad (7b)$$

$$\bar{v}_i(\eta, \mathbf{x}) = 0 - \int d\mu(\mathbf{q}) q^{-1} \nabla_i Q(\mathbf{x}, \mathbf{q}) v(\eta, \mathbf{q}). \quad (7c)$$

Actually, it is better to call Δ a density *fluctuation* because, as mentioned earlier, it is not necessarily small. However, the changes in velocity and pressure are always small, as shown below. (Note that the definition of the pressure perturbation Π differs from Bardeen [8]; this definition is easier to interpret when $P = 0$.)

To first order in velocity, the boost to the coordinate frame gives components of T_{ab} as

$$T_{00} = a^2 \rho \left[1 + \int d\mu Q (2A + \Delta) \right], \quad (8a)$$

$$T_{0i} = a^2 \rho \left[(1 + \sigma) \int d\mu q^{-1} \nabla_i Q v + \int d\mu q^{-1} \nabla_i Q v \int d\mu Q \Delta \right], \quad (8b)$$

$$T_{ij} = a^2 \rho \gamma_{ij} \left[\sigma + \int d\mu Q (2\sigma H + \Pi) \right]. \quad (8c)$$

(σ is the background pressure/density ratio.) Reminders are $O(v^2)$; the product term in the time-space component is kept because Δ may be greater than 1.

Most approaches to perturbation theory equate terms with equal orders of magnitude. But deferring order of magnitude arguments lets us exploit the harmonic decomposition of the field equations first, in essentially the same way as isolating coefficients in a Fourier series [see Eq. (13)]. This has the advantage that the results obtained remain valid as the relative order of magnitude of terms in the field equations change—say, as the density contrast is either diffuse (large κ) or condensed (small κ), or as its amplitude becomes large or small.

Removing the trace from the Einstein equation $G_{ij} - 8\pi T_{ij} = 0$ gives a result which looks familiar from linearized gravity (on flat spacetimes):

$$2 \int d\mu (\nabla_i \nabla_j Q + \frac{1}{3} q^2 \gamma_{ij} Q) (A + H) = 0, \quad (9)$$

which can only be true for arbitrary amplitudes A and H if

$$A(\eta, \mathbf{q}) = -H(\eta, \mathbf{q}) + O(\epsilon^4). \quad (10)$$

Using this in Eqs. (5) and (1) for the perturbations and the line element puts the metric in the linearized pseudo-Newtonian form widely used for studies of gravitational lensing, etc.:

$$ds^2 = a^2 [-(1 + 2\phi) d\eta^2 + (1 - 2\phi) \gamma_{ij} dx^i dx^j], \quad (11)$$

$$\phi(\eta, \mathbf{x}) = -\frac{1}{2} h_{00} = -\int d\mu(\mathbf{q}) Q(\mathbf{x}, \mathbf{q}) H(\eta, \mathbf{q}) + O(\epsilon^4). \quad (12)$$

To obtain equations for $H(\eta, \mathbf{q})$, and thus $\phi(\eta, \mathbf{x})$, in terms of the matter variables we use Eq. (10) in the components of the orthogonality equation

$$\int dV Q^* (G_{ab} - 8\pi T_{ab}) = 0, \quad (13)$$

where dV is the proper volume element in the static three-space γ_{ij} (see the Appendix). The density fluctuations are governed primarily by the time component $a = b = 0$. Using Eqs. (6) and (8) for the Einstein and stress-energy tensors gives

$$3 \left(\frac{a'}{a} \right) H' + (q^2 + 8\pi a^2 \rho - 6k) H = 4\pi a^2 \rho \Delta. \quad (14)$$

To this level of approximation, the scale factor obeys the usual Friedman-Robertson-Walker (FRW) equation for the background density $\rho(\eta)$ [compare Eq. (20) below]:

$$\frac{8\pi}{3} (a^2 \rho)' = \left(\frac{a'}{a} \right)^2 + k. \quad (15)$$

In a formal sense, a must obey this equation in order that the integration in Eq. (13) give a Dirac δ function in position when applied to the order 1 (background) terms in the field equations, owing to the assumed completeness of the Q 's. This allows us to make free use of substitutions from the background model for terms involving the scale factor and ρ in the following sections.

Equations relating the metric perturbations to velocity and pressure perturbations follow similarly. The equations for Π and v come most readily from the spatial and space-time components of (13), respectively:

$$H'' + 3 \left(\frac{a'}{a} \right) H' - (8\pi a^2 \rho \sigma + 2k) H = -4\pi a^2 \rho \Pi, \quad (16)$$

$$q \left[H' + \left(\frac{a'}{a} \right) H \right] = -4\pi a^2 \rho (1 + \sigma) v [1 + O(\Delta)]. \quad (17)$$

(Strictly speaking, the last equation holds only for $q \neq 0$ —no real restriction, because $q = 0$ is a constant mode, merely representing an improper definition of background.) We leave the Δ correction in (17) inexplicit since products of eigenfunctions add nontrivial complications to the formalism, but this suffices for order of magnitude assessment.

IV. ORDERS OF MAGNITUDE

Comparing the perturbations and their relative effects on the metric requires careful consideration of the size of the time derivatives. These are of order one in systems that are not gravitationally bound, $H' = O(\epsilon^2)$. However, in bound systems we expect $H' = O(\epsilon^3/\kappa)$ —for instance, one can imagine an observer stationed a fixed distance R from the center of a collapsing dust cloud; changes in the potential are $\sim \phi(V/R)$, where V is the speed of infall for the dust, or $\sim \epsilon^2(\epsilon/\kappa)$ by order of magnitude.

In Eq. (14), the size of the density contrast depends on the ratio ϵ/κ [remember that $q = O(\kappa^{-1})$]. The al-

lowed regimes are labeled as linear or nonlinear density, depending on the size of Δ :

	$O(H')$	$O(\Delta), \kappa \ll 1$	$O(\Delta), \kappa \gg 1$
LDR: $\epsilon/\kappa \ll 1$	ϵ^2	ϵ^2/κ^2	ϵ^2
NLDR: $\epsilon/\kappa \gg 1$	ϵ^3/κ	ϵ^2/κ^2	Not allowed ($\epsilon \gg 1$)

Strong density fluctuations with large (superhorizon) scales are not allowed because they create potentials that cannot be treated as perturbations on the background metric. (So the results derived here, in particular the Green function given later, should not be expected to work in a strongly “tilted” universe.) However, the small-scale density fluctuations [nonlinear density regime (NLDR) with $\kappa \ll 1$] should have an order of magnitude consistent with Newtonian theory. The prediction from the Poisson equation

$$\nabla^2 \phi = -\frac{1}{2a^2} \nabla_i \nabla^i h_{00} = 4\pi\rho\Delta \quad (18)$$

is also that Δ should be $O(\epsilon^2/\kappa^2)$. This holds for weak density fluctuations as well, again provided that the scale is small, since Newtonian physics can be expected to apply in a sufficiently small region of space.

Orders of magnitude for pressure and velocity perturbations come from Eqs. (16) and (17):

	$O(\Pi)$	$O(v)$
LDR: $\epsilon/\kappa \ll 1$	ϵ^2	ϵ^2/κ
NLDR: $\epsilon/\kappa \gg 1$	ϵ^4/κ^2	ϵ

These agree nicely with the simple Newtonian argument that for bound systems (NLDR) the velocity should be proportional to the square root of the potential, $v \sim H^{1/2} \sim \epsilon$, and the pressure should be $\Pi \sim \Delta v^2 \sim \epsilon^4/\kappa^2$. More generally we can use the Euler equation for the hydrodynamics of a perfect fluid in a gravitational field:

$$\frac{d\tilde{\mathbf{v}}}{dt} = [\partial_t \tilde{\mathbf{v}} + (\tilde{\mathbf{v}} \cdot \nabla) \tilde{\mathbf{v}}] = -\tilde{\rho}^{-1} \nabla \tilde{P} - \nabla \phi. \quad (19)$$

In the linear density regime the partial time derivative is most important, and comparing orders of magnitude we find $v = O(\epsilon^2/\kappa)$ and $\Pi = O(\epsilon^2)$, as in the table above. In the nonlinear density regime $\tilde{\rho}$ is of order ϵ^2/κ^2 , and again the estimates agree with the results from Eqs. (16) and (17).

The tables show, not surprisingly, that in any allowed (subhorizon) regime, the pressure and velocity perturbations are much weaker than the density fluctuations. So under these conditions the metric perturbations $H(\eta, \mathbf{q})$ are determined primarily by $\Delta(\eta, \mathbf{q})$; that is, hydrodynamically the density fluctuations can be treated as the source.

We also have to consider effects on the scale factor a , since it makes an implicit contribution to any order of magnitude arguments. When nonlinear terms are kept in the Einstein tensor, their spatial average can be thought of as an energy density and used to construct an effective stress-energy tensor for the metric perturbations. (See, e.g., Isaacson [9].) Dropping for a moment the requirement that $a(\eta)$ come from the background, on physical

grounds we expect

$$a(\eta) = a_{\text{FRW}}[1 + O(\langle \epsilon^4/\kappa^2 \rangle)], \quad (20)$$

but clearly, the average over even a relatively small volume must be less than the maximum value; i.e., $O(\langle \epsilon^4/\kappa^2 \rangle) \ll \epsilon^4/\kappa^2 \ll 1$. So using the background scale factor does not alter the arguments about Δ , Π , and v , simply because the corrections are even smaller than any of the terms discussed previously.

V. GREEN FUNCTION

Having determined that the scalar metric perturbations are determined primarily by the density fluctuations, we can relate them directly by solving Eq. (14) for the mode amplitudes $H(\eta, \mathbf{q})$ and using the result in (12) to get $\phi(\eta, \mathbf{x})$. Equation (14) is of the Riccati type; the standard solution given in handbooks such as that of Gradshteyn and Ryzhik [10] is

$$H(\eta, \mathbf{q}) = H(\eta_0, \mathbf{q}) E(\eta_0, \eta, q) - \int_{\eta_0}^{\eta} du I(u, \mathbf{q}) E(u, \eta, q), \quad (21)$$

where

$$E(u, \eta, q) = \frac{a(u)}{a(\eta)} \exp[-(q^2 - 3k)C(u, \eta)], \quad (22a)$$

$$I(u, \mathbf{q}) = -\frac{4\pi}{3} \left(\frac{a^3 \rho}{a'} \right)_u \Delta(u, \mathbf{q}), \quad (22b)$$

$$C(u, \eta) = \frac{1}{3} \int_u^{\eta} dv \left(\frac{a}{a'} \right). \quad (22c)$$

All of this, along with the definitions

$$H(\eta_0, \mathbf{q}) = - \int dV(\mathbf{y}) Q^*(\mathbf{y}, \mathbf{q}) \phi(\eta_0, \mathbf{y}), \quad (23a)$$

$$\Delta(u, \mathbf{q}) = \int dV(\mathbf{y}) Q^*(\mathbf{y}, \mathbf{q}) \Delta(u, \mathbf{y}), \quad (23b)$$

is inserted into Eq. (12). The integrals are re-ordered to contract the mode sums as much as possible, to give a kernel against which the initial conditions $\phi(\eta_0, \mathbf{y})$ and source $\Delta(u, \mathbf{y})$ are integrated over space. After some manipulation this gives

$$\begin{aligned} \phi(\eta, \mathbf{x}) = & \int dV(\mathbf{y}) G(\eta_0, \eta, \mathbf{x}, \mathbf{y}) \phi(\eta_0, \mathbf{y}) \\ & - \frac{4\pi}{3} \int_{\eta_0}^{\eta} du \frac{a^3 \rho}{a'} \int dV(\mathbf{y}) G(u, \eta, \mathbf{x}, \mathbf{y}) \Delta(u, \mathbf{y}) \\ & + O(\epsilon^4) \end{aligned} \quad (24)$$

where

$$G(u, \eta, \mathbf{x}, \mathbf{y}) = \frac{a(u)}{a(\eta)} e^{3kC(u, \eta)} \int d\mu(\mathbf{q}) Q(\mathbf{x}, \mathbf{q}) Q^*(\mathbf{y}, \mathbf{q}) \times e^{-q^2 C(u, \eta)}, \quad (25)$$

which is the central result of this paper: a Green function for metric perturbations due to scalar density fluctuations in a Robertson-Walker background. Equation (24) has several properties expected from an expression for a pseudo-Newtonian potential; for instance an overden-

sity $\Delta(u, \mathbf{y}) > 0$ decreases ϕ in its neighborhood. Other simple cases can be checked with the help of the following formulas for special arguments of the Green function, which come from the completeness and orthogonality relations, respectively:

$$G(u, u, \mathbf{x}, \mathbf{y}) = \delta(\mathbf{x}, \mathbf{y}), \quad (26)$$

$$\int dV G(u, \eta, \mathbf{x}, \mathbf{y}) = \frac{a(u)}{a(\eta)} e^{3kC(u, \eta)}. \quad (27)$$

The first shows that ϕ matches its initial conditions as $\eta \rightarrow \eta_0$ in (24). The second, which holds as long as $Q = \text{const}$ is an allowed mode, shows that ϕ is a function of time alone if Δ is, which means only that the background density has been shifted: $\rho \rightarrow \rho\Delta(\eta)$; of course $\Delta = 0$ is also a solution. (Any spatially homogeneous form for the density fluctuation leads to a standard Robertson-Walker metric after transformation of the scale factor and time coordinate.)

Specific forms for the Green function (25) come from choosing a particular representation (a coordinate system, in other words) for the Q 's, given k . Regardless of the representation, in closed or open spaces ($k = \pm 1$) the integrals can be very ugly, especially for $k = -1$. But for angles smaller than the curvature scale $k = 0$ is a good approximation, and under inflation it would hold generally. In this case we can represent the Q 's as plane waves and carry out the integrals explicitly. Using rectangular coordinates for \mathbf{q} , the integral in (25) is separable, and completing the square in each exponent gives

$$G_{k=0}(u, \eta, \mathbf{x}, \mathbf{y}) = \frac{a(u)}{a(\eta)} \frac{1}{[4\pi C(u, \eta)]^{3/2}} \exp \left[-\frac{|\mathbf{y} - \mathbf{x}|^2}{4C(u, \eta)} \right]. \quad (28)$$

The rest of the paper is devoted primarily to examining this formula and its use in Eq. (24) for the potential. Henceforth k is implicitly zero, unless noted otherwise.

VI. ANALOGY WITH DIFFUSION

The most striking thing about Eq. (28) is that it looks very much like the Green function for diffusion in a uniform medium, and more generally, the displacement probability distribution for an isotropic random walk [11]. For instance, we can compare the solution to the equation of heat conduction:

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T + c_p^{-1} \dot{q}. \quad (29)$$

(In this section, κ and \dot{q} represent thermal diffusivity and the rate of heat production per unit mass, not the reduced wavelength and mode numbers used in previous sections.) In an infinite uniform medium with no sources ($\dot{q} = 0$) the solution for an arbitrary initial temperature distribution is [12]

$$T(\mathbf{r}, t) = \int d^3\mathbf{r}' G(\mathbf{r} - \mathbf{r}', t - t_0) T(\mathbf{r}', t_0), \quad (30a)$$

$$G(\mathbf{r} - \mathbf{r}', t - t_0) = \frac{1}{[4\pi\kappa(t - t_0)]^{3/2}} \exp \left[-\frac{|\mathbf{r} - \mathbf{r}'|^2}{4\kappa(t - t_0)} \right]. \quad (30b)$$

Since there are no sources, the gravitational analogue is the integral over initial conditions in Eq. (24) for the potential, using the Green function (28). If the integral for the potential is written with a physical volume element $a^3(\eta_0)dV$, like the temperature integral above, we compare $a^{-3}(\eta_0)G(\eta_0, \eta, \mathbf{x}, \mathbf{y})$ with Eq. (30b). Taking η_0 and η close together to minimize the kinematic effects of expansion, the analogue of the diffusivity and time factors is just

$$\kappa(t - t_0) \rightarrow a^2(\eta_0)C(\eta_0, \eta) \simeq (3H_0)^{-1}\Delta t, \quad (31)$$

where H_0 is the Hubble parameter, evaluated at η_0 in this particular instance, and Δt is the proper (rather than conformal) time interval corresponding to $(\eta - \eta_0)$. This analogue of the diffusivity is consistent with an alternate approach that comes from converting Eq. (14) for the mode amplitudes $H(\eta, \mathbf{q})$ to an equation in (η, \mathbf{x}) , to form the analogue of (29). Summed over modes and converted to physical variables (proper time, standard Laplacian, etc.) it can be written as

$$\partial_t(a\phi) = (3H_0)^{-1}\nabla^2(a\phi) - \frac{1}{2}H_0(a\Delta), \quad (32)$$

where the Hubble parameter is taken to be a function of time. This is a diffusion equation for the potential, with time-dependent diffusivity and “specific heat capacity”:

$$T \leftrightarrow -a\phi, \quad (33a)$$

$$\kappa \leftrightarrow (3H_0)^{-1}, \quad (33b)$$

$$c_p \leftrightarrow 2H_0^{-1}, \quad (33c)$$

$$\dot{q} \leftrightarrow a\Delta, \quad (33d)$$

in agreement with (31). In simple kinetic theory, the diffusivity would imply a mean free path of H_0^{-1} for particles of average speed $c = 1$. In this approximation, changes in ϕ travel at all speeds, as can be seen both from the analogy with diffusion and from the fact that Eq. (24) assigns a nonzero value to the potential everywhere, even for (spatially) localized density fluctuations. This superficial causality problem comes from dropping terms $O(\epsilon^4/\kappa^2)$ while constructing (14). Ordinary diffusion cures the causality problem by making the diffusivity and specific heat capacity temperature dependent [13]. In the gravitational case this means making $H_0 = a'a^{-2}$ a function of the metric perturbations—which it is, in an exact treatment, since the scale factor is affected by the spatial average of $(\nabla\phi)^2$ [cf. Eq. (20)]. But causality does *not* represent a problem in the application of (24), however, because far from the source, “errors” in ϕ are extremely small—below the level of approximation for the calculation.

VII. NEWTONIAN LIMIT

The diffusion analogue is easiest to see when considering the role of the initial conditions. The Newtonian limit treats the opposite situation, where we consider compact sources (gravitationally bound systems, for instance) evolving slowly and with negligible initial conditions. If the time derivative is ignored, the gravitational diffusion equation (32) becomes the Poisson equation

tion (18) for the potential, after replacing H_0^2 by $8\pi G\rho/3$. This makes it reasonable to suppose that the Green-function expression (24) for the potential, which is essentially derived from the gravitational diffusion equation, can be reduced to a Newtonian form under appropriate conditions, providing a useful check of the formula.

We use the formulas for $k = 0$, but in fact the Newtonian limit holds in a sufficiently small region of any cosmological metric [14], and so the results can be expected to hold for $k = \pm 1$ as well. [Even when $k \neq 0$ the diffusion equation (32) still reduces to the Poisson equation when only the NLDR leading terms are kept.] Using the background equations for the scale factor and density we write the potential as

$$\phi = - \int dV \left\{ \frac{1}{2} \int_{\eta_0}^{\eta} du \frac{a'}{a} G \Delta - G_0 \phi_0 \right\} \quad (34)$$

by analogy with the Newtonian form. The contribution of the initial conditions $G_0 \phi_0$ is negligible provided that η_0 is taken far in the past and we assume that the early Universe was smooth. Then to evaluate the time integral, we can imagine that the potential is to be measured at a point outside a nearby “lump” of matter. In this situation it is physically reasonable to suppose that the Green function will peak for values of conformal time u close to η , corresponding to the time it takes for changes in the source to influence the observer. For small values of reversed conformal time $w = \eta - u$ the Green function is

$$G = \left(\frac{3}{4\pi} \frac{a'}{a} \right)^{3/2} w^{-3/2} \exp(-D_0/w) + O(w^{-1/2}), \quad (35a)$$

$$D_0 \equiv \frac{3a'}{a} \frac{|\mathbf{y} - \mathbf{x}|^2}{4}. \quad (35b)$$

This shows a peak at approximately $w_p = (2/3)D_0$, with a width of about $(2/5)D_0$. Both are small provided that the matter distribution $\Delta(\eta, \mathbf{y})$ is such that the volume integral restricts D_0 to small values. More precisely, an expansion in w is self-consistent if the width of the peak is small compared to the elapsed conformal time: $D_0 \ll (\eta - \eta_0)$. With the flat background, and using $\eta_0 = 0$, this means that (omitting overall constants of order one) we must have

$$\frac{|\mathbf{y} - \mathbf{x}|^2}{\eta^2} \ll 1 \quad (36a)$$

or

$$\kappa^2 \ll 1, \quad (36b)$$

where the last line follows from the “original” definition of κ as perturbation size divided by particle horizon length. Thus the Newtonian limit holds for density perturbations that are highly localized, as one would expect. Furthermore, if the density field changes little in the time corresponding to the width of the peak, the source and expansion terms can be evaluated at $w = 0$ and moved outside the time integral. (Strictly speaking, w_p might be a better choice, amounting to a notion of “retarded time,” but the errors due to approximating

the Green function make this academic.) Another change of variable to $z = D_0/w$ puts the time integral in nearly standard form, and we find

$$\phi \simeq - \int dV \left\{ \frac{1}{2} \left(\frac{3}{4\pi} \right)^{3/2} \left(\frac{a'}{a} \right)^{5/2} \frac{\Delta}{D_0^{1/2}} \times \int_{\delta}^{\infty} dz z^{-1/2} e^{-z} \right\}, \quad (37a)$$

$$\delta \equiv \frac{D_0}{\eta - \eta_0} = \frac{3}{4} H_0^2 \frac{a^2 |\mathbf{y} - \mathbf{x}|^2}{a(\eta - \eta_0)} \ll 1. \quad (37b)$$

The z integral is simply $\pi^{1/2} \text{erfc}(\delta^{1/2})$ or, to a good approximation when δ is small, just $\pi^{1/2}$. This along with the definition of D_0 and the ever-present background equations for the scale factor reduce (37a) to the familiar Newtonian form

$$\begin{aligned} \phi &\simeq - \int dV \left\{ \frac{3}{8\pi} \left(\frac{a'}{a} \right)^2 \frac{\Delta}{|\mathbf{y} - \mathbf{x}|} \right\} \\ &\simeq - \int a^3 dV \frac{\rho \Delta}{a |\mathbf{y} - \mathbf{x}|}. \end{aligned} \quad (38)$$

To summarize, this formula holds when the initial conditions can be neglected, and the spatial dependence of the density contrast $\Delta(u, \mathbf{y})$ limits δ to small values in Eq. (37a), while the time dependence has a scale much larger than (any value of) D_0 . In fact most gravitationally bound systems satisfy these criteria quite well, and are also quite uninteresting from a cosmological standpoint. More interesting are situations where the time evolution of the density fluctuations makes a significant contribution to the metric. A theoretical description requires extending the calculation just given to post-Newtonian order by expanding the Green function and source terms in a time series. We hope to show in a forthcoming paper [15] how this is done, and under what conditions Eq. (24) for the potential predicts significant deviations from the Newtonian (and LDR) approximations.

VIII. CONCLUSION

In most situations of observational interest, the Green function expression (24) for the pseudo-Newtonian potential offers a simple and relativistically correct way of calculating the metric perturbations, taking into account effects such as multiple (perhaps closely spaced) sources, deviations from the “thin lens” approximation, nonlinear density evolution, and the cosmological expansion. The results can be applied in the calculation of observational effects such as lensing, redshift, and time delay. We hope that this will stimulate exploration of situations that are difficult to treat with current techniques. An upcoming paper will examine the post-Newtonian limit of the Green-function expression, with attention to those situations which predict significant observable effects.

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APPENDIX

This appendix gives selected references to the literature on harmonic functions. Most calculations do not require an explicit representation; in spherical coordinates a schematic representation is

$$\begin{aligned} Q(\mathbf{x}, \mathbf{q}) &= \Pi_l^{(k)}(n, \alpha) Y_{lm}(\theta, \phi), \\ \mathbf{x} &= (\alpha, \theta, \phi), \\ \mathbf{q} &= (n, l, m), \\ q^2 &= n^2 - k, \end{aligned}$$

with the eigenvalue spectra in Eqs. (3). The angular functions are ordinary spherical harmonics; good overviews of the properties of the radial functions and/or the Q 's as a whole are in Harrison [5], Bardeen [8], Kodama and Sasaki [2], and Birrell and Davies [16]. The last authors write the orthogonality and completeness relations as

$$\begin{aligned} \int dV Q^*(\mathbf{y}, \mathbf{q}) Q(\mathbf{y}, \mathbf{p}) &= \delta(\mathbf{q}, \mathbf{p}), \\ \int d\mu Q^*(\mathbf{x}, \mathbf{q}) Q(\mathbf{y}, \mathbf{q}) &= \delta(\mathbf{x} - \mathbf{y}), \end{aligned}$$

where $dV \equiv (\gamma_i^i)^{1/2} d^3y$ is the proper volume element in the static background three-space and $d\mu$ is the measure associated with the eigenvalue spectrum. More detailed information, including explicit representations of the radial eigenfunctions and proofs of orthogonality and completeness, can be found in (see also [5]) Parker and Fulling [17], and Abbott and Schaeffer [18].

Finally, note that the literature aimed at problems in quantum field theory uses only scalar harmonics, while in general relativity an arbitrary tensor function may be composed of scalar, vector, and tensor harmonics. The early sections of Kodama and Sasaki [2] give a good explanation of the distinction.

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