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Covariant and gauge-invariant formulation of the Sachs-Wolfe effect

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We derive a formula relating the large-scale temperature anisotropy of the cosmic microwave background radiation with the cosmological perturbations responsible for them using the local covariant and gauge-invariant formalism developed by Ellis and Bruni. Comparisons of our covariant expression with previously derived Sachs-Wolfe formulas are given. Expanding our covariant variables in terms of Bardeen quantities, we derive a generalization of a result due to Panek.

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I. INTRODUCTION

The important discovery in 1992 by the Cosmic Background Explorer (COBE) [1,2] makes it of paramount importance to write down a formula relating the anisotropy of the cosmic microwave background radiation (CMBR) with the cosmological fluctuations responsible for them. Such a formula was first obtained by Sachs and Wolfe (SW) in 1967 [3]. Unfortunately their formula was derived using a gauge-dependent formalism which, as is well known in the literature, can lead to unphysical results due to the appearance of gauge modes, unless a complete specification or correspondence between the real perturbed Universe and the background Friedmann-Lemaître-Robertson-Walker (FLRW) model has been made.

More recently Panek [4] derived formulas for scalar, vector and tensor perturbations using the gauge-invariant variables introduced by Bardeen in his seminal paper [5]. Although this was a major step forward, there are in our view two main problems with the Bardeen variables: firstly, most of them do not have a transparent geometrical meaning, because they are defined with respect to a particular coordinate chart, and secondly, they are nonlocal due to the nonlocal nature of the Arnowitt-Deser-Misner (ADM) splitting (e.g., the splitting of a vector field is defined up to a constant [6]); indeed locality and causality are somewhat hidden in the formulas derived by Panek [7].

Improving some earlier work by Durrer [8], Magueijo [9] derived a formula using modified Bardeen variables, which are both gauge invariant and locally defined. However, the physical meaning of his variables is still unclear.

In a paper by Ellis and Bruni [10], a set of locally defined covariant and gauge-invariant variables were introduced, which are meaningful in any spacetime. Equations were derived for these variables and they were linearized about a FLRW cosmological model, recovering the standard results for a barotropic perfect fluid. Since then, this approach has been extended to multicomponent fluids and scalar fields in a number of papers [11–13]. It was also shown that a physical meaning could be found for the Bardeen variables, by expanding the covariant variables to first order with respect to the perturbed metric of Bardeen [14].

It is the goal of this paper to derive the most general formula for the large scale temperature anisotropies of the cosmic microwave background radiation in terms of locally defined covariant gauge-invariant variables without the use of any nonlocal splitting or harmonic decomposition.

In order to motivate this, consider the example of compensated topological defects studied by Magueijo [15] and Veeraraghaven and Stebbins [16]. In this case a cosmological perturbation is confined inside a compact domain Ω, which may be the causal future of the birth of a galactic seed. In general the Bardeen variables do not become trivial outside Ω and consequently the various geometrical contributions to $\delta T/T$ for a photon which has always been outside Ω may be nonvanishing. It turns out that these geometrical contributions add up to zero, but this is far from obvious. Clearly, since the underlying physics we wish to study is local, it is much more sensible to try and derive a formula that is both locally defined, gauge-invariant and has a transparent physical meaning.

In Sec. II our main formula (10) representing a covariant formulation of the SW effect is given. This formula is valid to first order in quantities that vanish in the "unperturbed" FLRW universe and it is independent of the curvature parameter ($K = 0, +1, -1$); it is also valid for arbitrary pressure and for an imperfect baryonic fluid. For adiabatic fluctuations at recombination (i.e., for $\Delta^{(\gamma b)}|_E = 0$) our result (10) is given by an integral over covariant, geometrical Ellis-Bruni-type quantities; in this case the precise choice of the integral curve (light-ray) and the last scattering surface is irrelevant to first order. If nonadiabatic fluctuations at recombination are considered we assume the idealized last scattering surface Σ_E and physical conditions on Σ_E to be given in a coordinate-independent manner (see, e.g., Panek [4] for a physical characterization of Σ_E). The main steps of the derivation of Eq. (10) are also indicated in Sec. II, whereas details of the derivation can be found in the Appendix. In Sec. III a simple formula is presented for the case of a perfect baryonic fluid and adiabatic fluctuations at recombination. Finally it is demonstrated how the classical results from Sachs and Wolfe [3] and Panek [4] can be recovered from our main formula (10). Throughout the paper we use units $8\pi G = c = 1$; Latin indices run from 0 to 3 and Greek indices from 1 to 3. The background (FLRW) metric is taken in the form

$$ds^2 = S^2(\tau)\left(-d\tau^2 + {}^3g_{\alpha\beta}dx^\alpha dx^\beta\right) . \tag{1}$$

II. A COVARIANT FORMULA FOR THE SACHS-WOLFE EFFECT

To derive a covariant and gauge-invariant Sachs-Wolfe formula we idealize the physical situation and consider a light ray being emitted from baryonic matter at point E in Fig. 1, lying on a spacelike hypersurface of last scattering Σ_E (i.e., for large-scale fluctuations details of the recombination epoch are ignored), traveling through a "perturbed" FRLW universe until it is observed by some "fundamental observer" here and now at point R. It is assumed that these fundamental observers can be represented by the word lines of baryonic matter. The derivation of our result starts from the usual expression

$$T_R = \frac{T_E}{z+1} , \tag{2}$$

where $T_R\,(T_E)$ is the temperature of the CMBR at the reception (emission) point and z is the redshift between E and R, $z = (\lambda_R - \lambda_E)/\lambda_E$. As is well known, the redshift can be obtained from

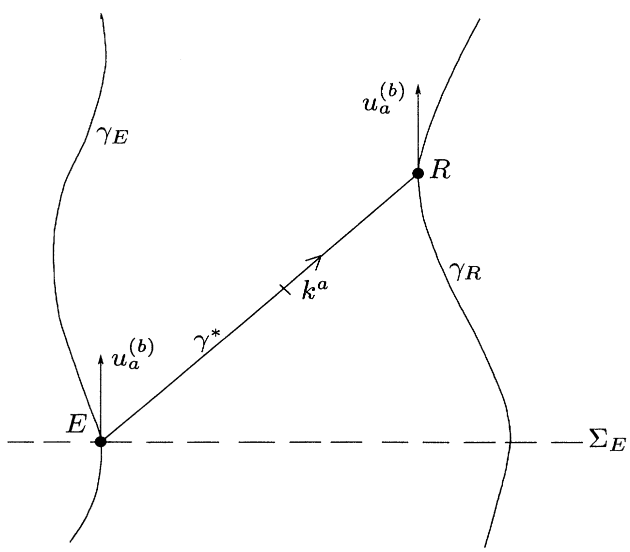

FIG. 1. Geometry in the Sachs-Wolfe effect.

$$z+1 = \frac{\left(k^a u_a^{(b)}\right)_E}{\left(k^a u_a^{(b)}\right)_R} , \tag{3}$$

where k^a is the null tangent vector to the light ray γ^* and $u_a^{(b)}$ is the four-velocity of baryons. Our formula is derived by a first order variation of Eq. (2) (see, e.g., Panek [4], Traschen [17]):

$$\left(\frac{\delta T}{T}\right)_R = \left(\frac{\delta T}{T}\right)_E - \frac{\delta z}{z+1} . \tag{4}$$

Details of our derivation can be found in the Appendix. Here we only want to indicate some important relations used in our derivation. If $u_a\,(u_a u^a = -1)$ is the four-velocity of a cosmological fluid (i.e., the baryonic one) [18], we employed the Ehlers' decomposition of the covariant derivative $u_{a;b}$ of u_a:

$$u_{a;b} = \sigma_{ab} + \omega_{ab} + \frac{1}{3}\,\Theta\,P_{ab} - a_a u_b , \tag{5}$$

where

$$P_{ab} = g_{ab} + u_a u_b \tag{6}$$

is the projection tensor into the tangent three-space orthogonal to u^a, $\sigma_{ab} = \sigma_{(ab)} = P_a{}^c P_b{}^d u_{(c;d)} - \frac{1}{3}\,\Theta\,P_{ab}$ is the shear tensor, $\omega_{ab} = \omega_{[ab]} = P_a{}^c P_b{}^d u_{[c;d]}$ is the vorticity tensor, $\Theta = u^a{}_{;a}$ is the volume expansion and $a_a = u_{a;b}u^b$ is the acceleration. The expansion rate of baryonic matter is expressed by means of the energy conservation equation. Here we assumed the baryonic fluid to be noninteracting with other fluids, i.e., $T^{(b)0a}{}_{;a} = 0$, and the baryonic energy-momentum tensor to be given by

$$T_{ab} = \mu u_a u_b + pP_{ab} + 2q_{(a}u_{b)} + \pi_{ab} , \tag{7}$$

where μ is the rest-energy density and p is the pressure, related by the equation of state

$$p = p(\mu, s) ,$$

where s is the entropy density, $q_a = -P_a{}^c T_{cd}u^d$ is the energy flux, π_{ab} is the anisotropic pressure, and p contains a possible contribution from bulk viscosity[1]: $p = p_t - \Theta\xi$. With these variables the energy conservation equation for the baryonic fluid reads

$$\mu_{;a}u^a + (\mu + p)\,\Theta + q^a_{\ ;a} = 0 . \tag{8}$$

The goal of our work is a derivation of a SW formula in terms of covariant and gauge-invariant variables. To

[1] ξ is the coefficient of bulk viscosity, p_t is the pressure in thermodynamic equilibrium.

construct such gauge-invariant quantities the Stewart-Walker lemma [19] can be used. It states that if a quantity vanishes in a FLRW model it is gauge invariant with respect to linear perturbations of that model. It follows that the shear σ_{ab}, the acceleration a_a, the coefficient of bulk viscosity ξ, and the energy flux q^a are gauge invariant. In order to describe the CMBR anisotropy we also need variables that characterize density inhomogeneities. The most important variable for our work, introduced by Ellis and Bruni [10], is the fractional gradient of the energy density $\mathcal{X}_a$, defined by

$$\mathcal{X}_a = P_a{}^b \frac{\mu_{,b}}{\mu} . \tag{9}$$

This variable is observable in the sense that it can be determined from Virial theorem estimates [20]. With these variables we derived the formula

$$\left(\frac{\delta T}{T}\right)^*_R = \left(\Delta^{(\gamma b)}\right)_E - \int_E^R f\, d\lambda \tag{10}$$

where

$$f = \left(\frac{1}{3\left(1+w^{(b)}\right)} \left(\mathcal{X}_a^{(b)} k^a - Q^{(b)} + \Psi^{(b)}\xi^{(b)}\right) + a_a^{(b)} k^a + \mathcal{S}_{ab}^{(b)} k^a k^b \right) . \tag{11}$$

Here, $w = p/\mu$, $Q = q^a{}_{;a}/\mu S$, $\Psi = \Theta^2/\mu S$, $\mathcal{S}_{ab} = S\sigma_{ab}$, and

$$\Delta^{(\gamma b)} = \Delta^{(\gamma)} - \Delta^{(b)}$$

with

$$\Delta^{(i)} = \frac{1}{3\left(1+w^{(i)}\right)} \frac{\delta\mu^{(i)}}{\mu^{(i)}} .$$

Note that although each $\Delta^{(i)}$ is not gauge invariant, the difference for two fluids, however, is. The asterisk on $(\delta T/T)_R$ corresponds to

$$\left(\frac{\delta T}{T}\right)^*_R = \left(\frac{\delta T}{T}\right)_R - \Delta_R^{(b)} .$$

Also here, both $(\delta T/T)_R$ and $\Delta_R^{(b)}$ are not gauge invariant but again the difference is. Since $\Delta_R^{(b)}$ is direction independent, $(\delta T/T)^*_R$ is just a renormalization of $(\delta T/T)_R$. If the physically meaningful quantity

$$\Delta\left(\frac{\delta T}{T}\right) = \frac{\delta T}{T}(\hat{\mathbf{r}}_1) - \frac{\delta T}{T}(\hat{\mathbf{r}}_2) \tag{12}$$

representing the difference of temperature fluctuations (temperature anisotropy) for two different directions $\hat{\mathbf{r}}_1$ and $\hat{\mathbf{r}}_2$ is considered, then this renormalization becomes unimportant, i.e., $\Delta\left(\delta T/T\right)_R = \Delta\left(\delta T/T\right)^*_R$.

III. SPECIFIC CASES AND COMPARISONS WITH PREVIOUS RESULTS

Consider for simplicity a perfect fluid ($Q^{(b)} = 0, \xi^{(b)} = 0$) and assume adiabatic perturbations at the time of last scattering ($\Delta_E^{(\gamma b)} = 0$), taking also the baryonic matter to be nonrelativistic, i.e., assuming $p^{(b)} \simeq w^{(b)} \simeq 0$. Furthermore for scales much larger than the Hubble radius corresponding to large-scale anisotropies, we can neglect the sound velocity of matter, $c^2_{s(b)} \simeq 0$, and assume $a_a^{(b)} \simeq 0$. Under these assumptions our fundamental formula (10) reduces to the simple expression

$$\left(\frac{\delta T}{T}\right)^*_R = -\int_E^R \left(\frac{1}{3}\mathcal{X}_a^{(b)} k^a + \mathcal{S}_{ab}^{(b)} k^a k^b\right) d\lambda . \tag{13}$$

A. The classical Sachs-Wolfe effect

Relation (10) can be used to recover the classical SW result. We start from the perturbed metric $ds^2 = S^2(\tau)(\eta_{ab} + \bar{h}_{ab})dx^a dx^b$ for a conformally flat universe with $\eta_{ab} = \mathrm{diag}(-1,1,1,1)$, and use the SW gauge conditions $\bar{h}_{00} = \bar{h}_{0\alpha} = \delta u^a = 0$, then for a perfect (baryonic) fluid with $p = \delta p = 0$ one obtains $a_a^{(b)} = 0$ and a solution of the perturbed field equations in the form

$$\bar{h}_{\alpha\beta} = \delta_{\alpha\beta} B + \frac{\tau^2}{10} B_{,\alpha\beta} ,$$

where the SW functions $D_{\alpha\beta}, C_\alpha$, and A were chosen to vanish. This gives, for the nonvanishing components,

$$\mathcal{S}_{\alpha\beta} = -\frac{1}{6} S^2 \delta_{\alpha\beta} \dot{\bar{h}}_\gamma{}^\gamma + \frac{1}{2} S^2 \dot{\bar{h}}_{\alpha\beta} , \tag{14}$$

$$\mathcal{X}_\alpha = -\frac{\tau^2}{20} \nabla^2 B_{,\alpha} , \tag{15}$$

where the dot indicates a derivative with respect to τ and $\bar{h}_\gamma^\gamma = \delta^{\gamma\sigma}\bar{h}_{\gamma\sigma}$. Here B is a scalar potential which is a pure function of spatial coordinates, related with the growing mode of the density perturbations. Using (10) we get

$$\left(\frac{\delta T}{T}\right)_R = \left(\frac{\delta T}{T}\right)_E - \frac{1}{2}\int_E^R \dot{\bar{h}}_{\alpha\beta} R^\alpha R^\beta d\bar{\lambda} , \tag{16}$$

where $\bar{k}^a = S^2 k^a = (1, -R^\alpha)$ and $d\bar{\lambda} = S^{-2} d\lambda$. For large-angle anisotropies the primordial temperature fluctuations can be neglected and we recover the well-known SW result in the form

$$\left(\frac{\delta T}{T}\right)_R = \frac{1}{10}\left[\tau B_{,\alpha} R^\alpha + B\right]_E^R . \tag{17}$$

B. The Panek formula

To compare with the results that were obtained by Panek [4] we expand the covariant variables to first order in terms of Bardeen variables [5] and the usual harmonic functions for scalar, vector, and tensor perturbations ($Q^{(0)} \equiv Q, Q^{(1)}_\alpha, Q^{(2)}_{\alpha\beta}$; $Q^{(0)}_\alpha = -k^{-1}Q_{|\alpha}$ etc.; for conventions see Panek). Using the results of Bruni *et al.* [14] we have

$$\mathcal{X}_\alpha = -k\epsilon_m Q^{(0)}_\alpha - 3\frac{\dot{S}}{S}(1+w)v_c Q^{(1)}_\alpha \ , \tag{18}$$

$$a_\alpha = \left(\frac{\dot{S}}{S}v_s^{(0)} + \dot{v}_s^{(0)} - k\Phi_A\right) Q^{(0)}_\alpha + \left(\frac{\dot{S}}{S}v_c + \dot{v}_c\right) Q^{(1)}_\alpha \ , \tag{19}$$

$$\mathcal{S}_{\alpha\beta} = -S^2 k v_s^{(0)} Q^{(0)}_{\alpha\beta} - S^2 k v_s^{(1)} Q^{(1)}_{\alpha\beta} + S^2 \dot{H}_T^{(2)} Q^{(2)}_{\alpha\beta} \ , \tag{20}$$

$$p - p_t = p\eta Q^{(0)} \ , \qquad \frac{\delta\mu}{\mu} = Q^{(0)}\delta \ . \tag{21}$$

Since the Bardeen variables are defined in the energy frame [5,18] we can set the energy flux to zero. For tensor and vector perturbations we then immediately recover the results found by Panek:

$$\left(\frac{\delta T}{T}\right)^{(2)}_R = -\int_E^R \dot{H}_T^{(2)} Q^{(2)}_{\alpha\beta} R^\alpha R^\beta d\bar{\lambda} \tag{22}$$

and

$$\left(\frac{\delta T}{T}\right)^{(1)}_R = \int_E^R \left(\dot{v}_c Q^{(1)}_\alpha R^\alpha + k v_s^{(1)} Q^{(1)}_{\alpha\beta} R^\alpha R^\beta\right) d\bar{\lambda} \ . \tag{23}$$

For scalar perturbations our result agrees with that given by Ellis *et al.* [21]. It represents a generalization of Panek's Eq. (41):

$$\left(\frac{\delta T}{T}\right)^{(0)\dagger}_R = \left(\frac{1}{4}\delta_\gamma Q\right)_E + \left(\frac{\dot{S}}{kS}(v_b - B)Q\right)_E + \int_E^R \left[\frac{1}{3(1+w_b)}\dot{\epsilon}_{mb} Q + \frac{\dot{S}}{S}\left(\frac{c_{sb}^2 \epsilon_{mb} - w_b \epsilon_{mb} + w_b \eta_b}{1+w_b}\right) Q \right.$$
$$\left. + \left(\frac{\dot{S}}{S}v_{sb}^{(0)} + \dot{v}_{sb}^{(0)} - k\Phi_A\right) Q^{(0)}_\alpha R^\alpha + k v_{sb}^{(0)} Q^{(0)}_{\alpha\beta} R^\alpha R^\beta\right] d\bar{\lambda} \ , \tag{24}$$

where the dagger indicates that $(\delta T/T)_R$ is renormalized with $[(v_b - B)Q\dot{S}/kS]_R$. We would like to remark that the results (22)–(24) are also valid for nonvanishing curvature parameter.

In conclusion, we have succeeded in deriving a covariant formulation of the Sachs-Wolfe effect. With respect to previous formulation ours has the advantage of being independent of gauge conditions, nonlocal splittings of spacetime, and related Fourier decompositions of perturbations around a FLRW metric. Such a formulation might be of great value for the interpretation of data for the large-scale fluctuations of the CMBR. We have demonstrated how several well-known SW formulas can be derived from ours with additional assumptions.

While this manuscript was in preparation, we became aware that a related formulation of the Sachs-Wolfe effect was under study [23].

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APPENDIX: DERIVATION OF FORMULA (9)

In the following second order terms will be neglected without exception. We then get, from Eq. (3),

$$\frac{\delta z}{z+1} = -\left[\frac{\delta\left(k^a u_a^{(b)}\right)}{\left(k^a u_a^{(b)}\right)}\right]_E^R$$
$$= -\int_E^R \frac{d}{d\lambda}\left(\frac{\delta\left(k^a u_a^{(b)}\right)}{\left(k^a u_a^{(b)}\right)}\right) d\lambda$$
$$= \int_E^R \left[\left(\frac{d}{d\lambda}S\right)\delta\left(k^a u_a^{(b)}\right) + S\frac{d}{d\lambda}\delta\left(k^a u_a^{(b)}\right)\right] d\lambda \ ,$$

since $k^a u_a^{(b)} = -S^{-1}$ to lowest (i.e., 0th) order. Using

$$\frac{d}{d\lambda}\delta = \delta\frac{d}{d\lambda}$$

and

$$\frac{d}{d\lambda} = k^a\frac{\partial}{\partial x^a}$$

we get

$$\frac{\delta z}{z+1} = \int_E^R \left[S_{;a}k^a\,\delta\left(k^a u_a^{(b)}\right) + S\delta\left(\left(k^a u_a^{(b)}\right)_{;b} k^b\right)\right] d\lambda \ .$$

Using the geodesic equation $k^a_{;b}k^b = 0$ and remembering that S is a pure function of τ and $k^0 = S^{-2}$ to lowest order we get

$$\frac{\delta z}{z+1} = \int_E^R \left(\frac{\dot{S}}{S^2}\delta\left(k^a u_a^{(b)}\right) + S\delta\left(u^{(b)}_{a;b}k^a k^b\right)\right) d\lambda \ .$$

Using the Ehlers' decomposition (5) and $\omega_{ab}k^a k^b = 0 = k_a k^a$ we obtain

$$\frac{\delta z}{z+1}=\int_E^R\left(\frac{\dot{S}}{S^2}\delta\left(k^a u_a^{(b)}\right)+\mathcal{S}_{ab}^{(b)}k^a k^b+a_a^{(b)}k^a+\frac{1}{3}S\delta\left(\Theta u_a^{(b)}k^a\right)u_b^{(b)}k^b+\frac{1}{3}S\Theta^{(b)}u_b^{(b)}k^b\delta\left(u_a^{(b)}k^a\right)\right)d\lambda.$$

Since σ_{ab} and a_a vanish in the background they are already of first order and the δ sign can be dropped here. Since to lowest order $\Theta^{(b)}=3\dot{S}/S^2$ the last term in the integral cancels with the first one. We use the energy-conservation equation (8) to substitute Θ:

$$\frac{\delta z}{z+1}=\int_E^R\left[-\frac{1}{3}\delta\left(\frac{\mu_{;a}^{(b)}}{\mu^{(b)}+p^{(b)}}k^a\right)+\frac{1}{3\left(1+w^{(b)}\right)}\left(\mathcal{X}_a^{(b)}k^a-Q^{(b)}\right)+a_a^{(b)}k^a+\mathcal{S}_{ab}^{(b)}k^a k^b\right]d\lambda\ ,$$

where we wrote $\delta Q=Q=q^a{}_{;a}/(\mu S)$ and $\delta\mathcal{X}_a=\mathcal{X}_a$. Furthermore, it can be shown that

$$\delta\left(\frac{d\mu/d\lambda}{\mu+p}\right)=\frac{d}{d\lambda}\left(\frac{\delta\mu}{\mu+p}\right)+\frac{\Theta w}{S(1+w)}\left(\frac{\delta p}{p}-\frac{c_s^2}{w}\frac{\delta\mu}{\mu}\right)\ ,$$

where $c_s^2=\dot{p}/\dot{\mu}$ is the square of the sound speed. We define $p=p_0+\delta p$ and $p_t=p_0\left(1+\frac{c_s^2}{w}\frac{\delta\mu}{\mu}\right)$ (see [22]). We can now integrate the first term. Using Eq. (4) we get

$$\left(\frac{\delta T}{T}\right)_R=\left(\frac{\delta T}{T}\right)_E+\frac{1}{3}\left[\frac{\delta\mu^{(b)}}{\mu^{(b)}+p^{(b)}}\right]_E^R-\int_E^R f\,d\lambda\ ,$$

where f is given in Eq. (11). Our result (10) is then obtained from the Stefan-Boltzmann law ($\mu^{(\gamma)}=\sigma T^4$) in the form

$$\frac{\delta T}{T}=\frac{1}{4}\left(\frac{\delta\mu^{(\gamma)}}{\mu^{(\gamma)}}\right)=\Delta^{(\gamma)}\ ,$$

where $p^{(\gamma)}=\mu^{(\gamma)}/3$.

[1] G. Smoot *et al.*, Astrophys. J. **396**, L1 (1992).
[2] E. L. Wright *et al.*, Astrophys. J. **396**, L13 (1992).
[3] R. K. Sachs and A. M. Wolfe, Astrophys. J. **147**, 73 (1967).
[4] M. Panek, Phys. Rev. D **34**, 416 (1986).
[5] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).
[6] J. M. Stewart, Class. Quantum Grav. **7**, 1169 (1990).
[7] We of course admit that such nonlocal formulas have been successfully applied to a variety of problems, e.g., special cases with topological defects; see, e.g., R. Durrer, in *Proceedings of the 10th Seminar on Relativistic Astrophysics and Gravitation*, edited by S. Gottlöber (World Scientific, Singapore, 1992).
[8] R. Durrer, Phys. Rev. D **42**, 2533 (1990).
[9] J. C. R. Magueijo, Phys. Rev. D **47**, R353 (1993).
[10] G. F. R. Ellis and M. Bruni, Phys. Rev. D **40**, 1804 (1989).
[11] P. K. S. Dunsby, Class. Quantum Grav. **8**, 1785 (1991).
[12] P. K. S. Dunsby, M. Bruni, and G. F. R. Ellis, Astrophys. J. **395**, 54 (1992).
[13] M. Bruni, G. F. R. Ellis, and P. K. S. Dunsby, Class. Quantum Grav. **9**, 921 (1992).
[14] M. Bruni, P. K. S. Dunsby, and G. F. R. Ellis, Astrophys. J. **395**, 34 (1992).
[15] J. C. R. Magueijo, Phys. Rev. D **46**, 1368 (1992).
[16] S. Veeraraghaven and A. Stebbins, Astrophys. J. Lett. (to be published).
[17] J. Traschen, Phys. Rev. D **29**, 1563 (1984).
[18] For an imperfect fluid the four-velocity u^a is not uniquely defined [24], so we have to make a choice based on the underlying thermodynamics. Two physical possibilities exist: either we choose the energy frame $u^a=u_E^a$, that is, the frame where the energy flux q_a vanishes, or we take the particle frame where the particle drift vanishes (this is discussed in detail in [12]). The important point is that whatever choice we make, u^a must coincide with u_E^a in the background or equilibrium model; otherwise the gradient variables we define later will not be gauge invariant. In this paper we will take u^a to define the particle frame.
[19] J. M. Stewart and M. Walker, Proc. R. Soc. London **A341**, 49 (1974).
[20] J. Kristian and R. K. Sachs, Astrophys. J. **143**, 379 (1966).
[21] G. F. R. Ellis, C. Xu, M. Katz, and W. R. Stoeger (unpublished).
[22] R. Durrer and N. Straumann, Helv. Phys. Acta **61**, 1027 (1988).
[23] M. Bruni and D. H. Lyth (unpublished).
[24] S. R. de Groot, W. A. van Leeuwen, and Ch. G. van Weert, *Relativistic Kinematic Theory* (North-Holland, Amsterdam, 1980).