PHYSICAL REVIEW D VOLUME 48, NUMBER 10 15 NOVEMBER 1993

Dynamical origin of the entropy of a black hole

Valery Frolov*

Theoretical Physics Institute, University of Alberta, Edmonton, Canada T6G 2J1 and P.N. Lebedev Physical Institute, Leninsky prospect 53, Moscow, 117924, Russia

Igor Novikov†

The Copenhagen University Observatory, Øster Voldgade 3, DK-1350 Copenhagen K, Denmark; Nordita, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark; and Astro Space Centre of P.N. Lebedev Physical Institute, Profsoyuznaya 84/32, Moscow, 117810, Russia

(Received 12 July 1993)

Modes of physical fields which are located inside a horizon and which cannot be observed by a distant observer are identified with the dynamical degrees of freedom of a black hole. A new invariant statistical mechanical definition of black-hole entropy is proposed. It is shown that the main contribution to the entropy is given by thermally excited "invisible" modes propagating in the close vicinity of the horizon. A calculation based on the proposed definition yields a value of the entropy which is in good agreement with the usually adopted value $A^H/(4l_{\rm Pl}^2)$, where A^H is the black-hole surface area and $l_{\rm Pl}$ is the Planck length.

PACS number(s): 97.60.Lf, 04.20.Cv, 05.30.−d

According to the thermodynamical analogy in black-hole physics, the entropy of a black hole is defined as $S^H = A^H/(4l_{\rm Pl}^2)$, where A^H is the area of the black-hole surface and $l_{\rm Pl} = (\hbar G/c^3\,)^{1/2}$ is the Planck length [1,2]. Hawking's theoretical discovery [3] of quantum black-hole evaporation proved the reality of the black-hole temperature and fixed the coefficient relating the entropy of a black hole with its surface area. The generalized second law (i.e., the statement that the sum $S = S^H + S^m$ of black-hole entropy and the entropy S^m of the outside matter cannot decrease) implies that black-hole entropy plays the same role as usual entropy and shows to what extent the energy contained in a black hole can be used to produce work [1,2,4]. Four laws of black-hole physics which form the basis in the thermodynamical analogy were formulated in [5].

Despite some promising attempts [2,6–9], the dynamical (statistical mechanical) origin of black-hole entropy has not been well understood. Bekenstein [2] who introduced the notion of black-hole entropy related it with "the measure of the inaccessibility of information (to an exterior observer) as to which a particular internal configuration of the black hole is actually realized in a given state" (i.e., for given values of black-hole parameters: mass, charge, and angular momentum). According to the "standard" interpretation these different internal states of a black hole are related with different possible initial conditions which may result in the creation of a stationary black hole with the same parameters [2]. In this approach the entropy of a black hole is considered as the logarithm of the number of distinct ways that hole might have been made [8,9].

This definition of the black-hole entropy resembles to some extent the usual definition of the entropy of matter. But there is big difference which makes the above "standard" interpretation not completely satisfactory. The entropy of matter is connected with its real internal dynamical degrees of freedom which exist at given moment and which can be affected by an external force. By getting information about the states of some of the internal degrees of freedom one can reduce the entropy of matter. (The total entropy of matter and its environment never decreases.) The loophole of the above described "standard" interpretation is that one cannot indicate dynamical degrees of freedom of a black hole which are responsible for its entropy.

Even if an observer is falling down into a black hole a long time after its formation he cannot receive more information about the initial state than the exterior observer. The information concerning the initial conditions is lost for the interior observer for the same reason as it is lost for an exterior observer [10]. The collapsing body and its structure become invisible for the late-time observer and only a few macroscopic parameters (mass, angular momentum, and charge) remain measurable. This situation reminds us of the famous grin of the Cheshire Cat remaining after the cat himself had disappeared. In this sense different possible initial states for a black hole are "Cheshire-Cat" variables [11].

This difficulty of the "standard" interpretation becomes especially vivid if one considers a recently proposed gedanken experiment in which a traversable wormhole is used to get information from a black-hole interior [12]. Namely it was shown that the area of the surface of a black hole decreases if one of the mouths of a traversable wormhole is falling into a black hole while the other mouth remains outside it. This decrease continues until the other mouth crosses the horizon or the wormhole is destroyed. After this the surface area of a black hole returns back to its initial value. There is an

*Electronic address: frolov@phys.ualberta.ca
†Electronic address: novikov@nordita.dk

evident mystery in such behavior if in accordance with the usual rules one identifies the entropy of a black hole with the loss of information about the possible origin of a black hole, because the wormhole does not get any new information about it. One can escape a contradiction with the generalized second law only if one assumes that there exist some real internal degrees of freedom hidden inside the black hole so that additional new information about the internal states of a black hole can be obtained by using a wormhole during this process.

York [7] tried to solve the problem of "Cheshire-Cat" variables by proposing that quasinormal modes of a black hole can play the role of its dynamical degrees of freedom. This attempt also cannot be considered as completely satisfactory. The entropy of the quasinormal modes which are thermally excited at the given moment of time is much smaller than $A^H/(4l_{\rm Pl}^2)$ (the adopted value for the entropy of a black hole). In order to obtain the black-hole entropy York [7] makes an additional assumption that the entropy "results when we add up all the excited (normal modes) states that disappear if we allow the hole to evaporate down to a final mass zero." In other words the black-hole entropy is again related not with dynamical degrees of freedom which are really excited at the given moment of time but with a number of different possibilities to excite them during the lifetime of a black hole.

In this paper we propose a new approach to the problem of statistical mechanical calculation of the entropy of a black hole. The main idea of this approach is to identify the dynamical degrees of freedom of a black hole with those states of all physical (quantum) fields which are located inside a black hole and cannot be seen by a distant observer. An excitation of these states does not change the external parameters of the black hole. For fixed external (macroscopic) parameters there exist many microscopically different (internal) states which cannot be distinguished by observations made in the exterior region. Because of quantum effects these internal states become thermally excited and give contribution to the black-hole entropy. This makes the definition of the entropy and other thermodynamical characteristics of black holes quite similar to those adopted in the usual statistical mechanical description of matter. In the absence of mutual interactions one can consider the contribution to the entropy of each of the physical fields (including gravitational perturbations) independently. It will be shown that the resulting entropy of a black hole does not depend on the number of fields.

In order to make the definition of the black-hole entropy more concrete we assume that there exists a stationary black hole and denote by $\hat{\rho}^{\rm init}$ the density matrix describing in the Heisenberg representation the initial state of quantum fields propagating in its background. One may consider, e.g., the in-vacuum state for a black hole evaporating in the vacuum, or the Hartle-Hawking state for a black hole in equilibrium with thermal radiation. For an exterior observer the system under consideration consists of two parts: a black hole and radiation outside of it. The state of radiation outside the black hole is described by the density matrix which is obtained from $\hat{\rho}^{\rm init}$ by averaging it over the states which are located inside the black hole and are invisible in its exterior:

$$\hat{\rho}^{\rm rad} = {\rm Tr}^{\rm inv}\hat{\rho}^{\rm init}. \tag{1}$$

For an isolated black hole this density matrix $\hat{\rho}^{\rm rad}$ in particular describes its Hawking radiation at infinity.

Analogously we define the density matrix describing the state of a black hole as

$$\hat{\rho}^{H} = {\rm Tr}^{\rm vis}\hat{\rho}^{\rm init}. \tag{2}$$

The trace operators ${\rm Tr}^{\rm vis}$ and ${\rm Tr}^{\rm inv}$ in these relations mean that the trace is taken over the states located either outside ("visible") or inside ("invisible") the event horizon correspondingly. We define the entropy of a black hole as

$$S^H = -{\rm Tr}^{\rm inv}(\hat{\rho}^H \ln \hat{\rho}^H). \tag{3}$$

The proposed definition of the entropy of a black hole is invariant in the following sense. Independent changes of vacuum definitions for "visible" and "invisible" states do not change the value of S^H. Bogolubov's transformations describing an independent change of the vacuum states inside and outside the black hole can be represented by the unitary operator $\hat{U} = \hat{U}^{\rm vis}\otimes\hat{U}^{\rm inv}$ where $\hat{U}^{\rm vis}$ and $\hat{U}^{\rm inv}$ are unitary operators in the Hilbert spaces of "visible" and "invisible" particles, correspondingly. The above used trace operators are invariant under such transformations.

In order to define the states one usually uses mode expansion. The modes are characterized by a complete set of quantum numbers. Because of the symmetry properties one can choose such a subset J of quantum numbers connected with conservation laws (such as orbital and azimuthal angular momenta, helicity, and so on) that guarantees the factorization of the density matrices.

In the absence of mutual interaction of different fields the subset J necessarily includes also the parameters identifying the type of the field (e.g., mass, spin, and charge).

The factorization in particular means that

$$\hat{\rho}^{\rm init} = \otimes_J \hat{\rho}_J^{\rm init}, \tag{4}$$

where $\hat{\rho}_J^{\rm init}$ is acting in the Hilbert space $\mathcal{H}_J$ of states with the chosen quantum numbers J, while the complete Hilbert space is $\mathcal{H} = \otimes_J \mathcal{H}_J$. The factorization also means that the separation into "visible" and "invisible" states can be done independently in each subset of modes with a fixed J so that

$$\hat{\rho}^H = \otimes_J \hat{\rho}_J^H, \quad \hat{\rho}_J^H = {\rm Tr}_J^{\rm vis}\hat{\rho}_J^{\rm init}, \tag{5}$$

$$S^H = \sum_J S_J^H, \qquad S_J^H = -{\rm Tr}_J^{\rm inv}(\hat{\rho}_J^H \ln \hat{\rho}_J^H), \tag{6}$$

where all the operators with subscript J are acting in the Hilbert space $\mathcal{H}_J$.

We begin by calculating S^H for a nonrotating uncharged black hole. We suppose that the black hole is contained inside a spherical cavity B of radius r_0 with a mirrorlike boundary. We choose r_0 small enough to guarantee the stable equilibrium of a black hole with thermal radiation inside the cavity. The Penrose diagram for the eternal static black hole is shown in Fig. 1. Instead of physical metric ds^2_{phys} it is convenient to use its dimensionless form $ds^2 = r_g^{-2} ds^2_{\text{phys}}$, where $r_g = 2M$ and M is the mass of the black hole. In the Kruskal coordinates (U,V) the metric ds^2 reads

$$ds^2 = -2B\,dU\,dV + x^2 d\omega^2, \qquad d\omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2,$$

$$B = 2x^{-1}\exp(1-x), \qquad UV = (1-x)\exp(x-1), \tag{7}$$

where $x = r/r_g$. The Killing vector ξ normalized to unity at infinity in the metric ds^2 is

$$\xi^\mu \partial_\mu = \tfrac{1}{2}(V\partial_V - U\partial_U). \tag{8}$$

For simplicity we consider a conformal massless scalar field $\hat{\varphi}$ obeying the equation

$$\Box\hat{\varphi} - \tfrac{1}{6}R\hat{\varphi} = 0. \tag{9}$$

The two sets f^{up} and f^{side} of classical complex solutions of this equation which we call "up" and "side" modes,

$$f^{\text{up,side}}_{\nu l m} = x^{-1} F^{\text{up,side}}_{\nu l}(U,V) Y_{lm}(\theta,\phi),$$
$$\xi^\mu\partial_\mu F^{\text{up}}_{\nu l} = -i(\nu/2)F^{\text{up}}_{\nu l}, \;\; \xi^\mu\partial_\mu F^{\text{side}}_{\nu l} = i(\nu/2)F^{\text{side}}_{\nu l}, \tag{10}$$

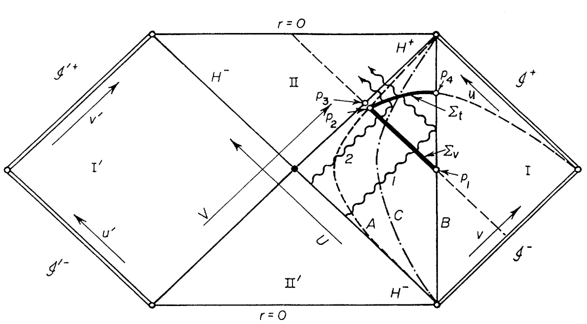

FIG. 1. The Penrose diagram for the eternal static black hole. The line B represents a spherical mirrorlike boundary of radius $r = r_0$ inside which the black hole is located. The line C indicates the position of the centrifugal barrier. Due to quantum fluctuations the horizon is spreaded. The out boundary of the spreaded horizon [lying at $r = 2M(1+\lambda)$] is schematically shown by the line A. The up-modes of high frequency ($\nu > 4\mu_l/\sqrt{27}$) penetrate the centrifugal barrier (E modes, schematically shown by the line 1). The up-modes of lower frequency ($\nu < 4\mu_l/\sqrt{27}$) are almost completely reflected by the centrifugal barrier (T modes shown by the line 2). The side-modes are radiated into the region II by the part $U > 0$ of the horizon H^-.

will be used in our consideration. The functions $F^{\text{up}}_{\nu l}$ and $F^{\text{side}}_{\nu l}$ obey the equation

$$\partial_U\partial_V F_{\nu l} + W_l F_{\nu l} = 0, \tag{11}$$

$$W_l \equiv x^{-3}\exp(1-x)[l(l+1) + 1/x]. \tag{12}$$

The up-modes are radiated in the exterior space from the horizon and vanish at the past null infinity $\mathcal{J}^-$, while the side-modes are radiated by the inner part $(U > 0)$ of the horizon into the black-hole interior [13]. We denote by $\hat{\alpha}$ and $\hat{\alpha}^*$ with the corresponding superscripts the operators of annihilation and creation of particles in these modes. The normalized wave-packet-type solutions f_J $(J = jnlm)$ are constructed from modes $f_{\nu lm}$ as follows $(\delta > 0)$

$$f_{jnlm} = \delta^{-1/2}\int_{j\delta}^{(j+1)\delta} \exp(2\pi i n\nu/\delta) f_{\nu lm}\, d\nu. \tag{13}$$

We denote by ν_J the average frequency of the wave packet J and introduce two sets of new modes f^B_J and f^W_J by the relations

$$f^B_J = [1 - w_J^2]^{-1/2}[f^{\text{side}}_J + w_J \bar{f}^{\text{up}}_J],$$
$$f^W_J = [1 - w_J^2]^{-1/2}[f^{\text{up}}_J + w_J \bar{f}^{\text{side}}_J], \tag{14}$$

where $w_J = \exp(-\pi\nu_J)$. These modes are of positive frequencies with respect to the affine parameter along the horizon H^-. In the presence of mirrorlike boundary surrounding the black hole the vacuum state with respect to B and W modes coincides with the Hartle-Hawking state.

The density matrix corresponding to the Hartle-Hawking state takes the form (4) with

$$\hat{\rho}^{\text{init}}_J = \rho^0_J : \exp\left[-\hat{\alpha}^{*B}_J \hat{\alpha}^B_J - \hat{\alpha}^{*W}_J \hat{\alpha}^W_J\right] :, \tag{15}$$

where ρ^0_J is the normalization constant, and $:(\cdots):$ means normal ordering with respect to operators $\hat{\alpha}^*$ and $\hat{\alpha}$ which enter the expression (15). For the Hartle-Hawking state the up-modes (as well as side-modes) are thermally excited. That is why for an eternal black hole the density matrix (15) describes the equilibrium state of a black hole with thermal radiation in its exterior. If the black hole is not eternal but is formed in the process of the gravitational collapse the expression for $\hat{\rho}^{\text{init}}$ should be modified. This modification is important for the modes emitted at the time close to the moment of black-hole formation. For late-time modes the expression (15) always provides an almost exact description. That is why instead of calculating the entropy of a stationary black hole long after its formation it is possible to calculate the entropy of an eternal black hole with the same parameters and for the initial state decribed by the density matrix (15). We use this (technically more simple) approach.

Consider a null surface $V = v$. Its intersection with the horizon (denoted p_3 at Fig. 1) represents the surface of the black hole at a given moment of time. Denote by Σ_v a part of this null surface lying between the horizon and the boundary of the cavity B.

It is evident that side-modes being propagated inside the black hole never cross Σ_v and hence according to our definition they are "invisible." Those up-modes which reach Σ_v are to be considered as "visible."

We introduce new coordinates η, ξ in the exterior ($U < 0$, $V > 0$) region (the region I in Fig. 1), where up-modes are propagating:

$$\eta = -\tfrac{1}{2}\ln(-V/U), \qquad \xi = \tfrac{1}{2}q(x), \qquad q(x) \equiv x - 1 + \ln(x-1). \tag{16}$$

Then the wave equation (11) for modes $F^{\rm up}_{\nu l}$ takes the form

$$[-\partial^2_\eta + \partial^2_\xi - V_l]F^{\rm up}_{\nu l} = 0, \tag{17}$$

where $V_l = 4(x-1)x^{-3}[\mu_l^2 - 1 + x^{-1}]$, $\mu_l^2 = l(l+1)+1$. For $l \gg 1$ the potential V_l has a maximum $\approx 16\mu_l^2/27$ at $x = x_m \approx 3/2$. We call "escaping" (or briefly E-modes) the up-modes with $\nu > 4\mu_l/\sqrt{27}$ which (in the absence of the boundary B) are propagating almost freely to infinity. We call "trapped" (or briefly T-modes) the up-modes with $\nu < 4\mu_l/\sqrt{27}$ which are mainly reflected by the potential barrier and returned to the black-hole horizon.

Let $f^{\rm up}_J$ be a "visible" (either E- or T-) mode, i.e., a mode which crosses Σ_v. Then the operation (5) of averaging over the states corresponding to this mode gives the thermal density matrix

$$\hat\rho^H_J = Tr^{\rm vis}_J \hat\rho^{\rm init}_J = \rho^0_J \exp\left[-2\pi\nu_J \hat\alpha^{*\rm side}_J \hat\alpha^{\rm side}_J\right], \tag{18}$$

ν_J being the frequency ν corresponding to the wave packet J. The standard calculation shows that the contribution of this mode J to the entropy of a black hole is [14]

$$S_J = s(2\pi\nu_J), \quad s(z) = \frac{z}{\exp(z)-1} - \ln[1-\exp(-z)]. \tag{19}$$

It is convenient to calculate the separate contributions S^E and S^T of E and T modes and to write

$$S^H = S^E + S^T, \tag{20}$$

where

$$S^{E,T} \approx {\sum_{jn}}' \sum_{l=0}^{\infty}(2l+1)\Theta[\pm(\nu_j^2 - 16\mu_l^2/27)]s(2\pi\nu_j). \tag{21}$$

The plus sign in the step function Θ stands for E modes, the minus sign stands for T modes, and the prime in $\sum'_{jn}$ means that the summation is taken only over the modes which cross Σ_v.

In order to get S^E we change the summation over l by integration over μ_l^2 [15]:

$$\sum_{l=0}^{\infty}(2l+1)\Theta(\nu_j^2 - 16\mu_l^2/27) \approx \int_0^{27\nu_j^2/16} d\mu_l^2 = \frac{27}{16}\nu_j^2. \tag{22}$$

The expression standing under the sum for S^E does not depend explicitly on n. Denote by N^E_j the total number of different values of the index n for which a mode with a given value of j crosses Σ_v. Then we have

$$S^E \approx \frac{27}{16}\sum_j N^E_j \nu_j^2 s(2\pi\nu_j). \tag{23}$$

In DeWitt's approximation one can omit V_l in Eq.(17) and to get that $N^E_j = (u_2 - u_1)\delta/2\pi$, where $u_{1,2}$ are the values of the retarded time $u = \eta - \xi$ corresponding to the boundary points of the surface Σ_v (see Fig. 1).

The value u_2 is formally divergent. The necessary cutoff arises due to the quantum fluctuation of the horizon. Zero-point fluctuations of the horizon result in its spreading so that due to the quantum noise events happening closer than $x_\lambda = 1+\lambda$ to the horizon cannot be seen from outside. One can show that

$$\lambda = \alpha l^2_{\rm Pl}/r_g^2, \tag{24}$$

where α is a dimensionless coefficient [7,16,17]. In our context it means that we are to consider as "visible" only those particles which cross Σ^λ_v which is the part of Σ_v lying outside x_λ (between points p_1 and p_2 shown in Fig. 1).

It should be stressed that for black holes with $M \gg m_{\rm Pl}$ (which we consider here) $\lambda \ll 1$.

By using this cutoff after changing the summation over j by integration ($\delta\sum_{j=0}^{\infty} = \int_0^\infty d\nu$) we get

$$S^E \approx \frac{27}{32\pi}\Delta u \int_0^\infty d\nu\,\nu^2 s(2\pi\nu), \tag{25}$$

where $\Delta u = q(x_0) - \ln\lambda$ and the function $q(x)$ is defined in Eq. (16).

Now we turn to the calculation of the contribution S^T of T modes to the black-hole entropy. We write S^T in the form

$$S^T \approx \int_0^\infty d\nu J(\nu)s(2\pi\nu), \tag{26}$$

where

$$J(\nu_j) = \delta^{-1}\sum_{l=0}^{\infty}(2l+1)N^T_{jl}\Theta(\tfrac{16}{27}\mu_l^2 - \nu_j^2), \tag{27}$$

and N^T_{jl} is a number of T modes with given quantum numbers j and l which cross Σ^λ_v.

Note that because of the exponential decrease of $s(z)$ the frequencies $\nu \gg 1/2\pi$ do not contribute to S^T. For example, to provide the accuracy $\sim 10^{-6}$ it is sufficient to consider the frequencies $\nu < 2.2$. Denote by $x(\nu)$ a turning point of a mode of frequency ν ($V_l[x(\nu))] = \nu^2$).

For large l one has $x(\nu)-1\approx \nu^2/4\mu_l^2 \ll 1$. In the region $|x-1|\ll 1$ one can use the homogeneous-gravitational-field (HGF) approximation and put $W_l=\mu_l^2$ in Eq.(11) and $B=x=1$ in the metric (7). This approximate metric can be written in the Rindler-like form

$$ds^2=-\rho^2 d\eta^2+d\rho^2+d\omega^2, \qquad (28)$$

where $\rho=2(x-1)^{1/2}$.

In the HGF-approximation normalized solutions $F_{\nu l}^{\rm up}$ of Eq. (11) which take the value $(4\pi\nu)^{-1/2}\exp(-i\nu u)$ at the horizon H^- are

$$F_{\nu l}=A_{\nu l}\pi^{-1}\sinh^{1/2}(\pi\nu)\exp(-i\nu\eta)K_{i\nu}(\mu_l\rho), \qquad (29)$$

where

$$A_{\nu l}=-i\exp\{-i[\nu\ln(\mu_l/2)-\phi_\nu]\}\ ,$$
$$\exp(2i\phi_\nu)=\Gamma(1+i\nu)/\Gamma(1-i\nu), \qquad (30)$$

and $K_p(z)$ is a Macdonald function.

In order to find N_{jl}^T we note that each of the modes which crosses Σ_v^λ crosses also a spacelike surface Σ_t^λ located between p_2 and mirrorlike boundary B and described by the equation η =const (see Fig. 1). It is possible to show that

$$N_{jl}^T\approx 2\pi^{-2}\nu_j\delta\sinh(\pi\nu_j)\int_{2\lambda^{1/2}}^\infty d\rho\rho^{-1}K_{i\nu}^2(\mu_l\rho). \qquad (31)$$

After substituting this relation into Eq.(27) and changing summation over l by integration over μ_l^2 we can rewrite this expression in the form

$$J(\nu)\equiv J_a(\nu)=\frac{\nu\sinh(\pi\nu)}{\pi^2\lambda}\int_1^\infty\frac{dy}{y}\int_a^\infty dzzK_{i\nu}^2(zy), \qquad (32)$$

where $a=(27\lambda)^{1/2}\nu/2$. For the frequencies ν which contribute to the black-hole entropy the parameter a is extremely small $a\ll 1$. We write

$$J(\nu)=J_{a=0}(\nu)+\Delta J(\nu),\quad \Delta J(\nu)=J_a(\nu)-J_{a=0}(\nu). \qquad (33)$$

The integrals which enter $J_{a=0}(\nu)$ can be taken exactly and we get

$$J_{a=0}(\nu)=\frac{\nu^2}{4\pi\lambda}, \qquad (34)$$

while for $\Delta J(\nu)$ one can obtain the approximate expression

$$\Delta J(\nu)\approx\frac{27\nu^2}{32\pi}[\ln\lambda+\ln(27\nu^2/16)-1$$
$$-\psi(1+i\nu)-\psi(1-i\nu)]. \qquad (35)$$

Here $\psi(z)$ is the logarithmic derivative of the Γ function.

By adding this expression with the expression (25) for S^E we finally get

$$S^H=S_0^H+\Delta S, \qquad (36)$$

where

$$S_0^H=\frac{1}{4\pi\lambda}\int_0^\infty d\nu\nu^2 s(2\pi\nu)=\frac{1}{360\lambda}, \qquad (37)$$

$$\Delta S=\frac{27}{32\pi}\int_0^\infty d\nu\nu^2 Q(\nu)s(2\pi\nu), \qquad (38)$$

and $Q(\nu)=x_0-2+\ln(27\nu^2/16)-\psi(1+i\nu)-\psi(1-i\nu)$. Numerical calculations give $\Delta S\approx(9x_0-23)\times 10^{-3}$.

One cannot expect to determine the entropy with accuracy higher than it is allowed by uncertainties related with its thermodynamical fluctuations. That is why the term ΔS in Eq. (36) which is much smaller than unity can be neglected. Thus one can identify the black-hole entropy with S_0^H. By using Eq.(24) we get

$$S_0^H=\gamma\frac{A_H}{4l_{\rm Pl}^2},\qquad \gamma=\frac{1}{360\pi\alpha}, \qquad (39)$$

where $A_H=4\pi r_g^2$ is the surface area of a black hole. It is important to stress that S_0^H does not depend on r_0 (the raduis of the mirrorlike boundary). It also does not depend on the particular choice of the surface Σ_v we introduced in order to define the separation into "visible" and "invisible" modes. For a stationary black hole the obtained result is evidently invariant under the shift of the advanced time parameter v.

Instead of a null surface one can also use (without changing the result) any spacelike surface crossing the horizon. The result (39) reproduces the standard expression for the black-hole entropy $A_H/4l_{\rm Pl^2}$ for the value of the parameter $\alpha=(360\pi)^{-1}$.

The calculation of the coefficient α which enters the expression (24) for the quantum fluctuations of the horizon is a delicate problem which requires quantization of gravity. The following arguments allow one to estimate its value. By direct measurements of black-hole mass M during time interval τ one expects a measurement uncertainty $\Delta M\approx\xi/\tau$ where $\xi\geq 1/2$ [17]. Spontaneous quantum emission (or absorption) of particles by a black hole results in the jumps $\sim (8\pi M)^{-1}$ of the mass M. These jumps do not allow one to take the interval τ as long as one wishes without generating new additional uncertainties. The best accuracy in a single measurement can be obtained if τ coincides with the time interval t_1 between two subsequent events of emission or absorption of quanta. The accuracy of defining M can be improved if instead of a single measurement one makes a sequence of measurements. Let N be a number of independent measurements and t_i $(i=1,\ldots,N)$ be the time intervals of these measurements (i.e., the time intervals between the emission or absorption of quanta). The probability of emission of a next quantum at the time interval $(t,t+dt)$ after the last previous emission is $p(t)dt=\exp(-\bar{n}t)\bar{n}dt$, where $\bar{n}$ is the average number of quanta radiated in a unit of time. An accuracy of the black-hole mass definition after the large number N of the above described independent measurements is [18]

$$\Delta_N M\approx N^{-1/2}(\overline{t^2})^{-1/2}=(2N)^{-1/2}\bar{n}, \qquad (40)$$

where $\overline{(t^2)} = N^{-1}\sum_{i=1}^{N} t_i^2$.

In the above consideration it was inexplicitly assumed that the only origin of the uncertainty of the measurement of the mass M was connected with the measurement procedure itself. Now we note that the quantum fluctuations of the horizon do not allow us to determine the mass M of a black hole absolutely exactly even if there were no uncertainties in the measurement procedure itself. The sequence of N exact measurements of M (in case if they were possible) would give an accuracy $\bar{\Delta}_N M = N^{-1/2}\delta M$ where δM is the characteristic value of fluctuation of M (its dispersion) connected with quantum fluctuation of the horizon. It is natural to assume that both accuracies [one $(\Delta_N M)$ connected with the quantum uncertainty of the measurement procedure and the other $(\bar{\Delta}_N M)$ connected with quantum fluctuations of the horizon] are of the same order of magnitude $\Delta_N M \approx \bar{\Delta}_N M$. Hence one can write

$$\delta M \approx 2^{-1/2}\xi\bar{n}. \tag{41}$$

The value of the average particles number rate of emission $\bar{n}$ for a scalar massless field was numerically calculated by Simkins [19] who found

$$\bar{n} = 6.644 \times 10^{-4} M^{-1}. \tag{42}$$

[DeWitt's approximation gives the very close result $\bar{n}_{\text{DeWitt}} = 3^3\zeta(3)/(2^9\pi^4 M) \approx 6.5\times10^{-4}M^{-1}$.] The value of the parameter α $(\delta M = \alpha/4M)$ corresponding to the expression (42) is

$$\alpha \approx 1.88 \times 10^{-3}\xi. \tag{43}$$

This result is quite close to York's estimation of the quantum fluctuations of the event horizon based on the deflection of the apparent horizon from the event horizon for an evaporating black hole [16].

By using Eq. (43) one gets for the coefficient γ which enters the expression (39), $\gamma \approx 0.47\xi^{-1}$. For the oft-quoted minimal value $\xi = 1/2$ one has $\gamma \approx 0.94$. This estimation is in a good accordance with the exact thermodynamical value of the black-hole entropy for which $\gamma = 1$.

We make now some general remarks concerning the obtained result. The entropy of a black hole is of pure quantum nature. The gravitational field of a black hole continuously creates pairs of particles. For a lone static uncharged black hole one of the components of a created pair is always located inside the horizon while the other can escape to infinity and contribute to the Hawking radiation. For low frequencies the probability of escape is exponentially small so that almost all of the components of such pairs created outside the horizon are reflected by the potential barrier and finally fall down into the black hole. The existence of "invisible" (hidden inside the horizon) modes results in the entropy of the black hole. Equation (2) shows that only those "invisible" components of pairs contribute to the entropy of the black hole for which the other component is "visible." The lifetime of "visible" "trapped" modes is of order $\tau_{\nu l} \sim \kappa^{-1}\ln(l^2/\nu^2)$. For modes with large $l \sim \lambda^{-1/2}$ which give the main contribution to the entropy this "lifetime" is $\sim r_g \ln(r_g/\lambda_{\text{Pl}})$. The main contribution to the entropy of a black hole is given by "invisible" modes which are propagated in the narrow $\sim \lambda_{\text{Pl}}$ shell region near the horizon.

We estimated the contribution to entropy of a nonrotating uncharged black hole by a conformal massless field. The generalization to the case of stationary (rotating and charged) black holes and different physical fields is straightforward. By using the above arguments one can expect that the number of thermally excited trapped modes which contribute to the entropy will be always proportional to the surface area of a black hole. If there are present more than one field each of them contributes to the entropy additively. On the other hand the average rate $\bar{n}$ of the emission of the particles grows as the number of fields N. That is why the parameter α which enters the expression (39) for the entropy of a black hole and which characterizes the quantum fluctuations of the horizon also grow as N. For a large number N of fields these two effects cancel each other so that the entropy does not depend on the number of fields.

In conclusion we return to the gedanken experiment with a wormhole [12] discussed at the beginning. The entropy of a black hole (identified with its surface area) decreases when one of the wormhole's mouths is falling inside a black hole and returns to its initial value after the other mouth crosses the horizon or the wormhole is destroyed. The proposed interpretation of "invisible" modes as dynamical degrees of freedom of a black hole allows one to understand this "mysterious" behavior of entropy. In principle by using a wormhole one can obtain additional information concerning the quantum states inside the black hole. The total number of the originally "invisible" modes propagating near the gravitational radius which become "visible" via the wormhole is proportional to the change of the black hole surface area. A decrease in the number of "invisible" states results in the decrease of the entropy of the black hole. In principle by using a wormhole one can change the states of some of the originally "invisible" modes without changing the external parameters of the black hole [22]. This possibility is lost after the second mouth of the wormhole crosses the horizon or the wormhole is destroyed. The "visible" components of such excited modes continue living outside the horizon only a short time compared with their "lifetime." After this they fall down into the black hole and the corresponding pair does not contribute to the entropy of the black hole. The system "forgets" an intervention (if only it did not change the black-hole parameters) and the entropy of the black hole returns to its initial value.

Note added. After this paper was submitted for publication a paper by Srednicki [20] appeared. For the massless scalar field in a flat spacetime he showed that the entropy arising after tracing the degrees of freedom of the field in the vacuum state residing inside a sphere is proportional to the area of this sphere. The analogous result was obtained earlier in Ref. [21]. In the present paper we have shown that the main contribution to the black-hole entropy is given by modes, propagating near its surface, and hence the entropy can also be considered as the "surface effect." Despite the formal similarity of the results of the flat spacetime and of the black-hole calculations there is a big difference between them. The black-hole horizon is a null surface. Its geometry differs from the geometry of a timelike surface in flat spacetime. The physical meaning and the mathematical description of modes which contribute to the entropy are quite different in both cases. That is why it is not clear how far the interesting arguments of Refs. [20,21] based on flat spacetime calculations can be directly applied to black holes.

The authors are grateful to Werner Israel for stimulating discussions. They also thank Don Page for helpful comments and his help in finding Ref. [19]. This work was supported in part by the Danish Natural Sciences Research Council, grants N11-9640-1 and 11-9524-1SE and also by the Natural Sciences and Engineering Research of Canada and by Deutsche Forschungsgemeinschaft (V.F.).

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[14] Strictly speaking $s(z) = (1+n)\ln(1+n) - n\ln(n)$, where $n = \Gamma[\exp(z)-1]^{-1}$, and Γ is the probability for the wave packet to reach Σ_v. For $\Gamma = 0$, $s(z) = 0$, while for $\Gamma = 1$ one gets the expression (19). The approximation we use is analogous to the one proposed by B.S. DeWitt [Phys. Rep. **19C**, 295 (1975)]. We put $\Gamma = 1$ for the modes which reach Σ_v and $\Gamma = 0$ for modes which do not reach it. We shall see that the main contribution to the entropy of a black hole is given by the modes propagating in the close vicinity of the horizon. Because of the very high blueshift effect for these modes one can use the geometric optics approximation and verify that the approximation we use does give the correct result.
[15] According to the Euler-MacLaurin formula $\sum_{l=0}^{\infty} F(l) \approx \frac{1}{2}F(0) - \frac{1}{12}F'(0) + \int_0^{\infty} F(x)dx$. The omitted terms $F(0)$ and $F'(0)$ are important for the calculation of the contribution of the scalar massless field into quantum radiation at infinity. (The authors are grateful to Andrei Zelnikov for this remark.) It will be shown later that the main contribution to the black-hole entropy is provided by the modes which are propagating near the surface of the black hole and have large values of l. That is why the omitted terms are not important for our consideration.
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