Weak-coupling expansions and an effective Lagrangian for compact U(1) lattice gauge theory in $D+1$ dimensions

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Weak-coupling perturbation series are calculated for the Hamiltonian version of compact U(1) lattice gauge theory in $D + 1$ dimensions. Expansions are obtained for the ground-state energy density and its finite-size corrections, the dispersion relation and photon velocity, and the axial string tension on a finite lattice. The finite-size scaling behavior can be simply understood on the basis of an effective Lagrangian which is that of free electromagnetic theory.

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I. INTRODUCTION

There has been renewed interest recently in the finite-size scaling properties of lattice models, aroused by the effective Lagrangian approach of Leutwyler and Hasenfratz and their collaborators [1–3]. A system with a continuous symmetry which undergoes spontaneous breakdown develops Goldstone bosons, and the massless Goldstone bosons then control the behavior of the system at low energies or temperatures, and large distances. One may write down a continuum "effective Lagrangian" for the Goldstone bosons, and hence obtain a systematic large volume expansion, which gives universal formulas for the leading finite-size corrections in the theory (as well as low-temperature and other effects) in terms of just two or three parameters. In a simple spin model, for instance, these parameters are the helicity modulus or spin-wave stiffness, the spin-wave velocity, and the spontaneous magnetization. The values of these parameters may then be determined from the amplitudes of the finite-size corrections, in much the same way that the finite-size amplitudes are related to the critical exponents at a second-order transition in two dimensions, by the theory of conformal invariance [4].

This approach was first applied in connection with chiral symmetry breaking in QCD by Gasser and Leutwyler, in a series of papers summarized in Ref. [1]. It has since been generalized to lattice spin models by Hasenfratz and Leutwyler [2] and Hasenfratz and Niedermayer [3]. A parallel development was meanwhile being carried out in connection with spin models having $O(N)$ symmetry by other authors [5], though in a less systematic way.

Here, we argue that a similar approach can be used for the compact U(1) lattice gauge model. First, we study the Hamiltonian version of this model in 2+1 and 3+1 dimensions, using weak-coupling perturbation theory. This is the analogue of spin-wave perturbation theory for a spin model, and produces a "low-temperature"

series expansion for the physical observables in powers of $x^{-1/2}$, where x is the lattice strong-coupling parameter. We extend the series obtained for the ground-state energy by Hofsäss and Horsley [6], and show that it agrees rather well with numerical calculations. The dispersion relation and photon velocity are also calculated, as well as the finite-size corrections to the ground-state energy, and the finite-size behavior of the axial string tension. These results are also found to be in reasonable agreement with numerical calculations [7], for the (3+1)-dimensional [(3+1)D] model in the weak-coupling region. The weak-coupling results imply a remarkable relationship:

$$\epsilon_0(L) - \epsilon_0(\infty) = -\frac{n_B v}{2L^{D+1}}\Big[\alpha^{(D)}_{-1/2}(1) + 2 - \frac{2}{D+1}\Big] + O(L^{-2D}) , \tag{1.1}$$

where $\epsilon_0(L)$ is the ground-state energy density on a lattice of L^D sites, $\alpha^{(D)}_{-1/2}(1)$ is a geometric "shape factor," v is the photon velocity or "speed of light," and n_B is the number of massless boson degrees of freedom (1 for $D = 2$, 2 for $D = 3$). This universal relationship is exactly the same as that holding in the $O(N)$ Heisenberg spin model [3,8]. Thus the amplitude of the correction to the ground-state energy provides a measure of the photon velocity, as advertised above.

The relationship (1.1) cries out for an explanation in terms of an effective Lagrangian theory. Indeed we show, using the techniques of Hasenfratz and Niedermayer [3], that one can obtain this result if one postulates a continuum effective Lagrangian whose leading term is simply that of free electromagnetic theory. There are only two parameters at leading order, an "effective coupling" ρ_s (denoted by analogy with the spin model), and the speed of light v; and in fact for the particular lattice model we have chosen there is an exact relationship between the two:

$$\frac{v^2}{2\rho_s} = 1 , \tag{1.2}$$

which again is precisely the same as a relationship holding in the $O(2)$ Heisenberg model [8].

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The question then arises, of course, as to whether such an effective Lagrangian approach is really valid in this context. The weak-coupling limit is not a first-order transition: there is no spontaneous breaking of a local gauge symmetry, according to Elitzur's theorem [9]. On the other hand, the treatment is at least internally consistent in that the interaction between the massless modes is weak at low energies (see Sec. III), just as in the spin model case: this is the requirement for a large-volume expansion to hold. For the (3+1)D model, it is known that there is a critical line at weak couplings where the system is in a massless Coulomb phase, and one may naturally expect the free electromagnetic Lagrangian effective theory to apply. The behavior of the fine-structure constant α in this phase has been discussed by Cardy [10] and Luck [11], who suggest that it may reach a universal maximum value at the end point of the critical line: we analyze our results along these lines in the concluding section.

For the (2+1)D model, the situation is more problematical. Göpfert and Mack [12] have shown that there is *no* massless phase at finite coupling, and that in the weak-coupling continuum limit the model reduces to a confining theory of free, massive scalar bosons, on a mass scale that decreases exponentially as the lattice spacing a goes to zero. On the other hand, Gross [13] has proved that in the "naive" continuum limit where one keeps the coupling *fixed* as the lattice spacing approaches zero, the U(1) model with Villain action *does* converge to the free electromagnetic theory, at the level of $F_{\mu\nu}$ or the Wilson loops. Our results on the finite-lattice behavior would seem to fit with this conclusion. It would thus seem that the same lattice model can give rise to either a massive or a massless continuum theory in the continuum limit, depending on the energy scale at which one approaches the limit.

II. WEAK-COUPLING EXPANSION

The Hamiltonian formulation of compact U(1) gauge theory on a lattice has been discussed by Banks *et al.* [14] and Drell *et al.* [15], following the general approach of Kogut and Susskind [16]. After choosing the temporal gauge $A^0 = 0$, the canonically conjugate vector fields **E** and **A** are defined by placing them on links of the spatial lattice. After appropriate rescalings, one finishes up with a quantum Hamiltonian

$$H = \sum_l E_l^2 - 2x \sum_p \cos\theta_p , \qquad (2.1)$$

where l denotes links and p denotes plaquettes of the D-dimensional spatial lattice. The strong-coupling parameter $x = 1/(e^4 a^{6-2D})$, where e is the electric charge and a is the lattice spacing (taken equal to unity hereafter). Periodic boundary conditions will be assumed throughout.

The relation between the plaquette angles θ_p and link angles A_l can be written

$$\theta_p(\mathbf{n}', \hat{k}) = \epsilon_{ijk}[A_l(\mathbf{n}, \hat{i}) - A_l(\mathbf{n} + \hat{j}, \hat{i}) + A_l(\mathbf{n} + \hat{i}, \hat{j}) \\ - A_l(\mathbf{n}, \hat{j})] , \qquad (2.2)$$

where $(\mathbf{n}', \hat{k})$ denotes a plaquette centered at coordinates $\mathbf{n}'$ and oriented in direction $\hat{k}$, while $(\mathbf{n}, \hat{i})$ denotes a link starting at site $\mathbf{n}$ and oriented in the $\hat{i}$ direction, with

$$\mathbf{n}' = \mathbf{n} + (\hat{i} + \hat{j})/2 . \qquad (2.3)$$

For the two-dimensional lattice, of course, the plaquettes have only a single possible orientation. The commutation relations are

$$[E_l(\mathbf{n}, \hat{\mu}), \ A_l(\mathbf{m}, \hat{\nu})] = -i\delta_{\mathbf{n},\mathbf{m}}\delta_{\hat{\mu},\hat{\nu}} . \qquad (2.4)$$

Perform Fourier transformations:

$$\theta_{\mathbf{k}}(\hat{\mu}) = \frac{1}{\sqrt{N}} \sum_{\mathbf{n}'} e^{-i\mathbf{k}\cdot\mathbf{n}'} \theta_p(\mathbf{n}', \hat{\mu}) ,$$

$$A_{\mathbf{k}}(\hat{\mu}) = \frac{1}{\sqrt{N}} \sum_{\mathbf{n}} e^{-i\mathbf{k}\cdot\mathbf{n}} A_l(\mathbf{n}, \hat{\mu}) , \qquad (2.5)$$

$$E_{\mathbf{k}}(\hat{\mu}) = \frac{1}{\sqrt{N}} \sum_{\mathbf{n}} e^{i\mathbf{k}\cdot\mathbf{n}} E_l(\mathbf{n}, \hat{\mu}) ,$$

where N is the number of sites; then

$$A_{\mathbf{k}}^\dagger(\hat{\mu}) = A_{-\mathbf{k}}(\hat{\mu}), \quad E_{\mathbf{k}}^\dagger(\hat{\mu}) = E_{-\mathbf{k}}(\hat{\mu}) . \qquad (2.6)$$

From Eq. (2.2), we find

$$\theta_{\mathbf{k}}(\hat{\mu}) = \sum_{\hat{\nu}} G_{\mathbf{k}}(\hat{\mu}, \hat{\nu}) A_{\mathbf{k}}(\hat{\nu}) , \qquad (2.7)$$

where

$$G_{\mathbf{k}}(\hat{\mu}, \hat{\nu}) = -2i\epsilon_{\mu\nu\sigma} e^{-ik_\nu/2} \sin(k_\sigma/2) . \qquad (2.8)$$

At weak coupling the cosine in Eq. (2.1) may be expanded in powers of θ_p:

$$H = \sum_l E_l^2 - ND(D-1)x + x \sum_p \theta_p^2 \\ - 2x \sum_{m=2}^\infty \frac{(-1)^m}{(2m)!} \sum_p \theta_p^{2m} . \qquad (2.9)$$

The quadratic term

$$\sum_p \theta_p^2 = \sum_{\mathbf{k}, \hat{\mu}} \theta_{\mathbf{k}}^\dagger(\hat{\mu}) \theta_{\mathbf{k}}(\hat{\mu}) = \sum_{\mathbf{k}} \mathbf{A}_{\mathbf{k}}^\dagger R_{\mathbf{k}} \mathbf{A}_{\mathbf{k}} , \qquad (2.10)$$

where the D-dimensional vector $\mathbf{A}_{\mathbf{k}}$ is formed from the directional components of $A_{\mathbf{k}}$, and the $D \times D$ matrix $R_{\mathbf{k}}$ is

$$R_{\mathbf{k}} = G_{\mathbf{k}}^\dagger G_{\mathbf{k}} , \qquad (2.11)$$

with the components of $G_{\mathbf{k}}$ given by (2.8). The matrix $R_{\mathbf{k}}$ is Hermitian, and has eigenvalues:

$$D = 2 : \quad \lambda_{\mathbf{k}}(1) = 4 \sum_{\hat{\mu}=1}^{2} \sin^2(k_\mu/2) \equiv \Delta_{\mathbf{k}} = z(1 - \gamma_{\mathbf{k}}) , \tag{2.12a}$$

$$\lambda_{\mathbf{k}}(2) = 0;$$

$$D = 3 : \quad \lambda_{\mathbf{k}}(1) = \lambda_{\mathbf{k}}(2) = 4 \sum_{\hat{\mu}=1}^{3} \sin^2(k_{\hat{\mu}}/2) \equiv \Delta_{\mathbf{k}} = z(1 - \gamma_{\mathbf{k}}) , \tag{2.12b}$$

$$\lambda_{\mathbf{k}}(3) = 0$$

(see Appendix A), where z is the "coordination number" of the lattice (4 for a square lattice and 6 for a cubic lattice), and $\gamma_{\mathbf{k}}$ is the "structure factor" defined by

$$\gamma_{\mathbf{k}} = \frac{1}{z} \sum_{\hat{\mu}_1,\ldots,\hat{\mu}_z} \exp(i\mathbf{k} \cdot \hat{\boldsymbol{\mu}}) . \tag{2.13}$$

A unitary transformation can be found:

$$\boldsymbol{\omega} = U_{\mathbf{k}} \mathbf{A}_{\mathbf{k}} , \tag{2.14a}$$

$$\mathbf{L}_{\mathbf{k}} = U_{\mathbf{k}}^* \mathbf{E}_{\mathbf{k}} , \tag{2.14b}$$

where

$$U_{\mathbf{k}}^\dagger = U_{\mathbf{k}}^{-1} , \quad L_{\mathbf{k}}^\dagger(\hat{\mu}) = L_{-\mathbf{k}}(\hat{\mu}) , \quad \omega_{\mathbf{k}}^\dagger(\hat{\mu}) = \omega_{-\mathbf{k}}(\hat{\mu}) , \tag{2.15}$$

which diagonalizes $R_{\mathbf{k}}$, so that in terms of these new variables the Hamiltonian becomes (up to terms quadratic in the fields)

$$H = -ND(D-1)x$$
$$+ \sum_{k} \left[\sum_{\hat{\mu}=1}^{D} L_{\mathbf{k}}^\dagger(\hat{\mu}) L_{\mathbf{k}}(\hat{\mu}) \right.$$
$$\left. + x\Delta_{\mathbf{k}} \sum_{\hat{\mu}=1}^{D-1} \omega_{\mathbf{k}}^\dagger(\hat{\mu}) \omega_{\mathbf{k}}(\hat{\mu}) \right] + \cdots . \tag{2.16}$$

The commutation relations among the new variables are simply

$$[L_{\mathbf{k}}(\hat{\mu}), \omega_{\mathbf{k}'}(\hat{\nu})] = -i\delta_{\mathbf{k},\mathbf{k}'}\delta_{\hat{\mu},\hat{\nu}} . \tag{2.17}$$

We note that the component of $\boldsymbol{\omega}_{\mathbf{k}}$ corresponding to the zero eigenvalue has vanished from the Hamiltonian at this order. In fact it vanishes at all orders (see Appendix A), and is an "ignorable" or nondynamical variable. Denote this variable by $\tilde{\omega}_{\mathbf{k}}$, then it turns out that

$$\tilde{\omega}_{\mathbf{k}} \propto \sum_{\hat{\mu}=1}^{D} (1 - e^{-ik_{\hat{\mu}}}) A_{\mathbf{k}}(\hat{\mu}) , \tag{2.18}$$

so that, in position space,

$$\tilde{\omega}(\mathbf{n}) \propto \sum_{\hat{\mu}=1}^{D} [A(\mathbf{n},\hat{\mu}) - A(\mathbf{n} - \hat{\boldsymbol{\mu}},\hat{\mu})] , \tag{2.19}$$

which is just the lattice version of $\nabla \cdot \mathbf{A}$. One can always make use of the residual symmetry under time-independent gauge transformations to fix $\nabla \cdot \mathbf{A} = 0$. The

corresponding electric field component $\tilde{L}_{\mathbf{k}}$ transforms in position space to the lattice version of $\nabla \cdot \mathbf{E}$: this is identically zero by Gauss' law, since there are no charges in the pure gauge theory.

The remaining quadratic terms in Eq. (2.16) now correspond to a many-variable harmonic oscillator problem, and can be diagonalized by a "Bogoliubov transformation":

$$\alpha_{\mathbf{k}}(\hat{\mu}) = \frac{1}{\sqrt{2}} [L_{\mathbf{k}}(\hat{\mu}) \tan \phi_{\mathbf{k}} - i\omega_{\mathbf{k}}^\dagger(\hat{\mu}) \cot \phi_{\mathbf{k}}] ,$$

$$(\hat{\mu} = 1,\ldots,D-1) , \tag{2.20}$$

where

$$\cot \phi_{\mathbf{k}} = [x\Delta_{\mathbf{k}}]^{1/4} , \tag{2.21}$$

and $\phi_{-\mathbf{k}} = \phi_{\mathbf{k}}$. Then one finds that $\alpha_{\mathbf{k}}^\dagger(\hat{\mu})$, $\alpha_{\mathbf{k}}(\hat{\mu})$ obey the commutation relation of Bose creation and destruction operators:

$$[\alpha_{\mathbf{k}}(\hat{\mu}), \alpha_{\mathbf{k}'}^\dagger(\hat{\nu})] = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\hat{\mu},\hat{\nu}} . \tag{2.22}$$

After normal ordering, the Hamiltonian becomes

$$H = N[-D(D-1)x + (D-1)\sqrt{zx}C_1^{(D)}]$$
$$+ 2\sqrt{x} \sum_{\mathbf{k}} \sum_{\hat{\mu}=1}^{D-1} \Delta_{\mathbf{k}}^{1/2} n_{\mathbf{k}}(\hat{\mu}) + \text{(higher-order terms)} , \tag{2.23}$$

where

$$n_{\mathbf{k}}(\hat{\mu}) = \alpha_{\mathbf{k}}^\dagger(\hat{\mu})\alpha_{\mathbf{k}}(\hat{\mu}) \tag{2.24}$$

is the boson number operator, and

$$C_1^{(D)} = \frac{1}{N} \sum_{\mathbf{k}} (1 - \gamma_{\mathbf{k}})^{1/2} . \tag{2.25}$$

Note that the number of independent boson degrees of freedom per site has been reduced to one for $D = 2$, and two for $D = 3$, as we expect for the U(1) gauge theory.

To carry the weak-coupling expansion to higher orders, one needs to include higher-order terms in the Hamiltonian H. Inverting Eq. (2.20), one finds

$$\omega_{\mathbf{k}}(\hat{\mu}) = -\frac{i}{\sqrt{2}} [x\Delta_{\mathbf{k}}]^{-1/4} [\alpha_{\mathbf{k}}^\dagger(\hat{\mu}) - \alpha_{-\mathbf{k}}(\hat{\mu})] , \tag{2.26}$$

while the relationship between the plaquette angle and

the $\omega_{\mathbf{k}}$ is given by inverting Eqs. (2.14) and (2.7). For $D = 2$ the relationship is particularly simple, since there is only one component remaining per site for both θ_p and $\omega_{\mathbf{k}}$, and one finds

$$\theta_{\mathbf{k}} = \Delta_{\mathbf{k}}^{1/2}\omega_{\mathbf{k}} \qquad (2.27a)$$

(dropping the "direction" indices). For the $D = 3$ case,

the relation is a little more complex:

$$\theta_{\mathbf{k}}(\hat{\mu}) = \sum_{\hat{\nu}=1}^{2} B_{\mathbf{k}}(\hat{\mu}, \hat{\nu})\omega_{\mathbf{k}}(\hat{\nu}) , \qquad (2.27b)$$

where the components $B_{\mathbf{k}}(\hat{\mu}, \hat{\nu})$ are listed in Appendix A. Using (2.26) and (2.27), the higher-order terms in the Hamiltonian can be written out explicitly.

$D = 2$ square lattice:

$$H = N[-2x + 2\sqrt{x}C_1] + 2\sqrt{x}\sum_{\mathbf{k}}\Delta_{\mathbf{k}}^{1/2}n_{\mathbf{k}}$$

$$-2Nx\sum_{m=2}^{\infty}\frac{1}{(2N\sqrt{x})^m}\frac{1}{(2m)!}\sum_{\mathbf{k}_1,\dots,\mathbf{k}_{2m}}\delta\left(\sum_{i=1}^{2m}\mathbf{k}_i, 0\right)\prod_{i=1}^{2m}\{\Delta_{\mathbf{k}_i}^{1/4}(\alpha_{\mathbf{k}_i}^{\dagger} - \alpha_{-\mathbf{k}_i})\} . \qquad (2.28)$$

$D = 3$ cubic lattice:

$$H = N[-6x + 2\sqrt{6x}C_1] + 2\sqrt{x}\sum_{\mathbf{k}}\Delta_{\mathbf{k}}^{1/2}\sum_{\hat{\nu}}^{2}n_{\mathbf{k}}(\hat{\nu})$$

$$-\sum_{m=2}^{\infty}\frac{2Nx}{(2N\sqrt{x})^m}\frac{1}{(2m)!}\sum_{\mathbf{k}_1,\dots,\mathbf{k}_{2m}}\delta\left(\sum_{i=1}^{2m}\mathbf{k}_i, 0\right)\sum_{\hat{\mu}=1}^{3}\prod_{i=1}^{2m}\Delta_{\mathbf{k}_i}^{-1/4}\left\{\sum_{\hat{\nu}_i=1}^{2}[\alpha_{\mathbf{k}_i}^{\dagger}(\hat{\nu}_i) - \alpha_{-\mathbf{k}_i}(\hat{\nu}_i)]B_{\mathbf{k}_i}(\hat{\mu}, \hat{\nu}_i)\right\} . \qquad (2.29)$$

After normal ordering the boson operators, the first few terms of the Hamiltonian take the following form.
$D = 2$ square lattice:

$$H = N\left(-2x + 2x^{1/2}C_1 - \frac{C_1^2}{4} + \frac{C_1^3}{24x^{1/2}} - \frac{C_1^4}{192x}\right) + \left(4x^{1/2} - C_1 + \frac{C_1^2}{4x^{1/2}} - \frac{C_1^3}{24x}\right)\sum_{\mathbf{k}}(1 - \gamma_{\mathbf{k}})^{1/2}n_{\mathbf{k}}$$

$$+\left(\frac{C_1}{2} - \frac{C_1^2}{8x^{1/2}}\right)\sum_{\mathbf{k}}(1 - \gamma_{\mathbf{k}})^{1/2}(\alpha_{\mathbf{k}}\alpha_{-\mathbf{k}} + \alpha_{\mathbf{k}}^{\dagger}\alpha_{-\mathbf{k}}^{\dagger})$$

$$+\left(-\frac{1}{12N} + \frac{C_1}{24Nx^{1/2}}\right)\sum_{\mathbf{k}_i}\delta_{1+2+3+4,0}\prod_{i=1}^{4}(1 - \gamma_{k_i})^{1/4}\Big[(\alpha_{-1}\alpha_{-2}\alpha_{-3}\alpha_{-4} + \alpha_1^{\dagger}\alpha_2^{\dagger}\alpha_3^{\dagger}\alpha_4^{\dagger})$$

$$-4(\alpha_1^{\dagger}\alpha_{-2}\alpha_{-3}\alpha_{-4} + \alpha_2^{\dagger}\alpha_3^{\dagger}\alpha_4^{\dagger}\alpha_{-1}) + 6\alpha_1^{\dagger}\alpha_2^{\dagger}\alpha_{-3}\alpha_{-4}\Big]$$

$$-\frac{1}{360N^2x^{1/2}}\sum_{k_i}\delta_{1+2+3+4+5+6,0}\prod_{i=1}^{6}(1 - \gamma_{k_i})^{1/4}\Big[(\alpha_{-1}\alpha_{-2}\alpha_{-3}\alpha_{-4}\alpha_{-5}\alpha_{-6}$$

$$+\alpha_1^{\dagger}\alpha_2^{\dagger}\alpha_3^{\dagger}\alpha_4^{\dagger}\alpha_5^{\dagger}\alpha_6^{\dagger}) - 6(\alpha_1^{\dagger}\alpha_{-2}\alpha_{-3}\alpha_{-4}\alpha_{-5}\alpha_{-6} + \alpha_2^{\dagger}\alpha_3^{\dagger}\alpha_4^{\dagger}\alpha_5^{\dagger}\alpha_6^{\dagger}\alpha_{-1})$$

$$+15(\alpha_1^{\dagger}\alpha_2^{\dagger}\alpha_{-3}\alpha_{-4}\alpha_{-5}\alpha_{-6} + \alpha_3^{\dagger}\alpha_4^{\dagger}\alpha_5^{\dagger}\alpha_6^{\dagger}\alpha_{-1}\alpha_{-2}) - 20\alpha_1^{\dagger}\alpha_2^{\dagger}\alpha_3^{\dagger}\alpha_{-4}\alpha_{-5}\alpha_{-6}\Big] + \cdots . \qquad (2.30)$$

$D = 3$ cubic lattice:

$$H = N\left[-6x + 2\sqrt{6x}C_1 - \frac{C_1^2}{2} + \frac{C_1^3}{6\sqrt{6x}}\right] + \left[2\sqrt{6x} - C_1 + \frac{C_1^2}{2\sqrt{6x}}\right]\sum_{\mathbf{k}}(1 - \gamma_{\mathbf{k}})^{1/2}\sum_{\hat{\nu}}^{2}n_{\mathbf{k}}(\hat{\nu})$$

$$-\frac{1}{12N}\sum_{k_i}\delta_{1+2+3+4,0}\Big\{V_1A_1(1) + V_2A_1(2) + V_3A_2 + V_4A_3(1)[\alpha_{-4}(2) - \alpha_4^{\dagger}(2)]$$

$$+V_5A_3(2)[\alpha_{-4}(1) - \alpha_4^{\dagger}(1)]\Big\} + \cdots , \qquad (2.31)$$

where we adopt the convention of writing $\mathbf{1}$ instead of k_1, etc., and

$$A_1(\hat{\mu}) = \alpha_{-1}(\hat{\mu})\alpha_{-2}(\hat{\mu})\alpha_{-3}(\hat{\mu})\alpha_{-4}(\hat{\mu}) + \alpha_1^{\dagger}(\hat{\mu})\alpha_2^{\dagger}(\hat{\mu})\alpha_3^{\dagger}(\hat{\mu})\alpha_4^{\dagger}(\hat{\mu})$$

$$-4[\alpha_1^{\dagger}(\hat{\mu})\alpha_{-2}(\hat{\mu})\alpha_{-3}(\hat{\mu})\alpha_{-4}(\hat{\mu}) + \alpha_2^{\dagger}(\hat{\mu})\alpha_3^{\dagger}(\hat{\mu})\alpha_4^{\dagger}(\hat{\mu})\alpha_{-1}(\hat{\mu})]$$

$$+6\alpha_1^{\dagger}(\hat{\mu})\alpha_2^{\dagger}(\hat{\mu})\alpha_{-3}(\hat{\mu})\alpha_{-4}(\hat{\mu}) - 6\delta_{1+2,0}[\alpha_{-3}(\hat{\mu})\alpha_{-4}(\hat{\mu}) + \alpha_3^{\dagger}(\hat{\mu})\alpha_4^{\dagger}(\hat{\mu})] , \qquad (2.32a)$$

$$A_2 = \alpha_{-1}(1)\alpha_{-2}(1)\alpha_{-3}(2)\alpha_{-4}(2) + \alpha_1^\dagger(1)\alpha_2^\dagger(1)\alpha_3^\dagger(2)\alpha_4^\dagger(2) + \alpha_{-1}(1)\alpha_{-2}(1)\alpha_3^\dagger(2)\alpha_4^\dagger(2)$$
$$+\alpha_1^\dagger(1)\alpha_2^\dagger(1)\alpha_{-3}(2)\alpha_{-4}(2) - 2[\alpha_{-1}(1)\alpha_{-2}(1)\alpha_3^\dagger(2)\alpha_{-4}(2) + \alpha_1^\dagger(1)\alpha_2^\dagger(1)\alpha_3^\dagger(2)\alpha_{-4}(2)$$
$$+\alpha_1^\dagger(1)\alpha_{-2}(1)\alpha_{-3}(2)\alpha_{-4}(2) + \alpha_1^\dagger(1)\alpha_{-2}(1)\alpha_3^\dagger(2)\alpha_4^\dagger(2)] + 4\alpha_1^\dagger(1)\alpha_{-2}(1)\alpha_3^\dagger(2)\alpha_{-4}(2)$$
$$-\delta_{3+4,0}[\alpha_{-1}(1)\alpha_{-2}(1) + \alpha_1^\dagger(1)\alpha_2^\dagger(1)] - \delta_{1+2,0}[\alpha_{-3}(2)\alpha_{-4}(2) + \alpha_3^\dagger(2)\alpha_4^\dagger(2)] , \tag{2.32b}$$

$$A_3(\hat\mu) = \alpha_{-1}(\hat\mu)\alpha_{-2}(\hat\mu)\alpha_{-3}(\hat\mu) - \alpha_1^\dagger(\hat\mu)\alpha_2^\dagger(\hat\mu)\alpha_3^\dagger(\hat\mu) + 3\alpha_1^\dagger(\hat\mu)\alpha_2^\dagger(\hat\mu)\alpha_{-3}(\hat\mu)$$
$$-3\alpha_3^\dagger(\hat\mu)\alpha_{-1}(\hat\mu)\alpha_{-2}(\hat\mu) + 3\delta_{1+2,0}[\alpha_3^\dagger(\hat\mu) - \alpha_{-3}(\hat\mu)] , \tag{2.32c}$$

$$V_1(1,2,3,4) = \left[\prod_{i=1}^4 c_1(i)c_2^{1/2}(i)\right]^{-1}\left[\prod_{i=1}^4 c_1^2(i) + \prod_{i=1}^4 s_x(i)s_z(i) + \prod_{i=1}^4 s_y(i)s_z(i)\right] , \tag{2.32d}$$

$$V_2(1,2,3,4) = \frac{\prod_{i=1}^4 c_2^{1/2}(i)}{\prod_{i=1}^4 c_1(i)}\left[\prod_{i=1}^4 s_x(i) + \prod_{i=1}^4 s_y(i)\right] , \tag{2.32e}$$

$$V_3(1,2,3,4) = -\frac{6s_z(1)s_z(2)\prod_{i=1}^4[c_2^{1/2}(i)]}{c_2(1)c_2(2)\prod_{i=1}^4 c_1(i)}[s_y(1)s_y(2)s_x(3)s_x(4) + s_x(1)s_x(2)s_y(3)s_y(4)] , \tag{2.32f}$$

$$V_4(1,2,3,4) = \frac{4ic_2(4)s_z(1)s_z(2)s_z(3)}{\prod_{i=1}^4 c_1(i)c_2^{1/2}(i)}[s_x(1)s_x(2)s_x(3)s_y(4) - s_x(4)s_y(1)s_y(2)s_y(3)] , \tag{2.32g}$$

$$V_5(1,2,3,4) = -\frac{4is_z(4)\prod_{i=1}^4 c_2^{1/2}(i)}{c_2(4)\prod_{i=1}^4 c_1(i)}[s_x(4)s_y(1)s_y(2)s_y(3) - s_x(1)s_x(2)s_x(3)s_y(4)] , \tag{2.32h}$$

s_x, s_y, s_z, c_1, and c_2 are defined in Appendix A.

It can be seen that the weak-coupling expansion is effectively an expansion in powers of $x^{-1/2}$. It is expected to be an asymptotic expansion, and does not reproduce (for instance) the nonperturbative effects discussed by Polyakov [17] which are responsible for "linear confinement" in 2+1 dimensions.

A. Ground-state energy

Using Rayleigh-Schrödinger perturbation theory, we can treat the off-diagonal terms in the above Hamiltonian as perturbations. Up to the order $1/x$, there are six perturbation diagrams shown in Fig. 1 contributing to the ground-state energy in 2+1 dimensions; the contribution of each diagram is

$$\Delta E_a = -\frac{C_1^3}{16x^{1/2}} + \frac{C_1^4}{64x} , \tag{2.33a}$$

$$\Delta E_b = -\frac{1}{24x^{1/2}}\left(1 - \frac{3C_1}{4x^{1/2}}\right)C_2 , \tag{2.33b}$$

$$\Delta E_c = -\frac{C_1^4}{128x} , \tag{2.33c}$$

$$\Delta E_d = -\frac{C_1^4}{128x} , \tag{2.33d}$$

$$\Delta E_e = -\frac{C_1}{24x}C_2 , \tag{2.33e}$$

$$\Delta E_f = -\frac{1}{16x}C_3 , \tag{2.33f}$$

where we denote the contribution from Fig. 1(a) as ΔE_a, etc., and C_2 and C_3 are defined by

(a)   (b)   (c)
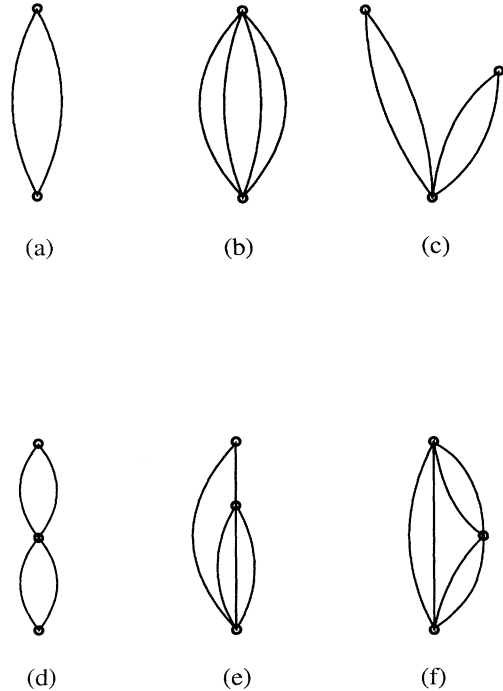

(d)   (e)   (f)

FIG. 1. The perturbation diagrams that contribute to the ground-state energy E_0/N. The points represent interaction vertices; the lines represent boson excitations in the intermediate states. To save space, we have not differentiated between $\alpha(1)$ and $\alpha(2)$ bosons, and different time orderings of the vertices in the diagrams.

$$C_2 = \frac{1}{N^3} \sum_{k_1,k_2,k_3,k_4} \delta_{1+2+3+4,0} \frac{\prod_{i=1}^4 (1-\gamma_i)^{1/2}}{\sum_{i=1}^4 (1-\gamma_i)^{1/2}} , \tag{2.34a}$$

$$C_3 = \frac{1}{N^4} \sum_{k_1,k_2,k_3,k_4,k_5,k_6} \delta_{1+2+3+4,0} \delta_{1+2+5+6,0} \frac{\prod_{i=1}^6 (1-\gamma_i)^{1/2}}{[\sum_{i=1}^4 (1-\gamma_i)^{1/2}][\sum_{k_1,k_2,k_5,k_6}(1-\gamma_i)^{1/2}]} . \tag{2.34b}$$

For 3+1 dimensions, there are two perturbation diagrams [Fig. 1(a) and Fig. 1(b)] contributing up to order $x^{-1/2}$ to the ground-state energy; the contribution of each diagram is

$$\Delta E_a = -\frac{1}{288\sqrt{6x}} \frac{1}{N^3} \sum_{k_1,k_2,k_3} \Big\{ \frac{9}{2}[V_4(1,-1,3,-3) + V_5(1,-1,-3,3)][V_4(2,-2,-3,3)$$

$$+ V_5(2,-2,3,-3)] + [6V_1(1,-1,3,-3) + V_3(3,-3,1,-1)][6V_1(2,-2,3,-3)$$

$$+ V_3(3,-3,2,-2)] + [6V_2(1,-1,3,-3) + V_3(1,-1,3,-3)][6V_2(2,-2,3,-3)$$

$$+ V_3(2,-2,3,-3)] \Big\} \Big/ (1-\gamma_3)^{1/2} , \tag{2.35a}$$

$$\Delta E_b = -\frac{1}{72\sqrt{6x}} \frac{1}{N^3} \sum_{k_i} \delta_{1+2+3+4,0} \Big[6V_1(1,2,3,4)V_1(-1,-2,-3,-4)$$

$$+ 6V_2(1,2,3,4)V_2(-1,-2,-3,-4) + V_3(1,2,3,4)V_3(-1,-2,-3,-4)$$

$$+ \frac{3}{2}V_4(1,2,3,4)V_4(-1,-2,-3,-4)$$

$$+ \frac{3}{2}V_5(1,2,3,4)V_5(-1,-2,-3,-4) \Big] \Big/ \Big[\sum_{i=1}^4 (1-\gamma_i)^{1/2} \Big] . \tag{2.35b}$$

After some calculation, we can get

$$\Delta E_a = -\frac{C_1^3}{4\sqrt{6x}} . \tag{2.36}$$

Using the value given for various lattice sums in Appendix B, one finds, for a $D = 2$ square lattice of $N = L \times L$ sites,

$$\epsilon_0(N) \equiv E_0/N = -2x + 1.916183x^{1/2} - 0.2294848 - 0.0268602x^{-1/2} - 0.009315x^{-1} + O(x^{-3/2})$$

$$- \frac{1}{L^3}[1.4376x^{1/2} - 0.34434 - 0.07533x^{-1/2} - 0.0421x^{-1} + O(x^{-3/2})] + \cdots , \tag{2.37a}$$

where the leading finite-size corrections listed here arise from corrections to the lattice sums, as discussed in Appendix B.

This prediction may be compared with the numerical results of Hamer, Oitmaa, and Zheng [19], who computed strong-coupling series approximants to the ground-state energy, and matched them at intermediate coupling to a weak-coupling form:

$$\epsilon_0(\infty) = -2x + 1.916183x^{1/2} - 0.28 ,$$

where the first two coefficients were fixed a priori, while the constant term was chosen to provide the best fit. The agreement with the prediction above is quite good. No numerical data are available to test the finite-size corrections, as far as we are aware.

For the $D = 3$ cubic lattice of $N = L^3$ sites, the weak-coupling prediction is

$$\epsilon_0(N) \equiv E_0/N = -6x + 4.77520x^{1/2} - 0.475054 - 0.04182x^{-1/2} + O(x^{-1})$$

$$- \frac{1}{L^4}[3.349x^{1/2} - 0.6664 + O(x^{-1/2})] + \cdots . \tag{2.37b}$$

This prediction may be compared with the Monte Carlo results of Hamer and Aydin [7], who matched their estimated bulk limit at intermediate coupling ($x \simeq 0.8$) to a weak-coupling form:

$$\epsilon_0(\infty) = -6x + 4.755x^{-1/2} - 0.535 , \tag{2.38}$$

where, again, the constant term was chosen to provide the best fit. It agrees quite well with (2.37b) above.

The data of Hamer and Aydin [7] can also be used

to estimate the finite-size corrections. Figure 2 graphs $\epsilon_0(N)$ against $1/L^4$ at $x = 0.7$ and $x = 0.8$, where data exist. Although these couplings are only just inside the weak-coupling region, it can be seen that the data are well fitted by a straight line, in accordance with (2.37b). The magnitude of the gradient of these lines will be considered later.

B. Dispersion relation

The energy $E(\mathbf{k})$ of a single-boson state with momentum $\mathbf{k}$ (that is, the energy gap between this state and the ground state) can be derived from the Hamiltonians (2.30) and (2.31) as follows.

$D = 2$ square lattice:

$$E(\mathbf{k}) = \sqrt{1 - \gamma_{\mathbf{k}}} \left[4x^{1/2} - C_1 - \frac{C_1^2}{8x^{1/2}} - \frac{C_1^3}{24x} - \left(1 + \frac{C_1}{4x^{1/2}}\right)\frac{C_4}{3x^{1/2}} - \frac{C_2}{48x} - \frac{C_5}{4x} - \frac{C_6}{2x} \right] . \tag{2.39}$$

$D = 3$ cubic lattice:

$$E(\mathbf{k}) = \sqrt{1 - \gamma_{\mathbf{k}}} \left[2\sqrt{6x} - C_1 - \frac{C_1^2}{4\sqrt{6x}} \right] + \frac{\Delta E^{-1/2}(\mathbf{k})}{x^{1/2}} , \tag{2.40}$$

where

$$C_4 = \frac{1}{N^2} \sum_{k_1,k_2,k_3} \delta_{1+2+3,0} \frac{\prod_{i=1}^{3}(1-\gamma_i)^{1/2}}{\sum_{i=1}^{3}(1-\gamma_i)^{1/2}} , \tag{2.41a}$$

$$C_5 = \frac{1}{N^3} \sum_{k_1,k_2,k_3,k_4,k_5} \delta_{1+2+3,0}\delta_{1+4+5,0} \frac{\prod_{i=1}^{5}(1-\gamma_i)^{1/2}}{[\sum_{i=1}^{3}(1-\gamma_i)^{1/2}][\sum_{k_1,k_4,k_5}(1-\gamma_i)^{1/2}]} , \tag{2.41b}$$

$$C_6 = \frac{1}{N^3} \sum_{k_1,k_2,k_3,k_4,k_5} \delta_{1+2+3+4,0}\delta_{1+2+5,0} \frac{\prod_{i=1}^{5}(1-\gamma_i)^{1/2}}{[\sum_{i=1}^{4}(1-\gamma_i)^{1/2}][\sum_{k_1,k_2,k_5}(1-\gamma_i)^{1/2}]} , \tag{2.41c}$$

$$\Delta E^{-1/2}(\mathbf{k}) = -\frac{2}{6^{1/2}N^2} \sum_{k_1,k_2,k_3} \delta_{1+2+3,0}[V_1(\mathbf{1,2,3,k})V_1(-\mathbf{1,-2,-3,-k})/3$$

$$+ V_5(\mathbf{1,2,3,k})V_5(-\mathbf{1,-2,-3,-k})/48 + V_3(\mathbf{1,k,2,3})V_3(-\mathbf{1,-k,-2,-3})/36$$

$$+ V_4(\mathbf{1,2,k,3})V_4(-\mathbf{1,-2,-k,-3})/16] \bigg/ \left[\sum_{i=1}^{3}(1-\gamma_i)^{1/2}\right] . \tag{2.41d}$$

Besides diagonal terms, the twenty perturbation diagrams shown in Fig. 3 for 2+1 dimensions and four perturbation diagrams shown in Fig. 3(a) for 3+1 dimensions contribute to this result.

Now at small $\mathbf{k}$ (for $D + 1$ dimensions),

$$\sqrt{1 - \gamma_{\mathbf{k}}} \sim \frac{|\mathbf{k}|}{\sqrt{z}} \quad \text{as } \mathbf{k} \to 0 , \tag{2.42}$$

and using the results in Appendix B, one finds a linear dispersion relation at low momentum,

$$E(\mathbf{k}) \sim v(x)|\mathbf{k}| \quad \text{as } \mathbf{k} \to 0 , \tag{2.43}$$

corresponding to a massless boson field. The factor $v(x)$ is the boson velocity or "speed of light" in the model, which is then the following.

$D = 2$ square lattice:

$$v(x) = 2x^{1/2} - 0.479046 - 0.104783x^{-1/2}$$
$$-0.05847x^{-1} + O(x^{-3/2}) . \tag{2.44a}$$

$D = 3$ cubic lattice:

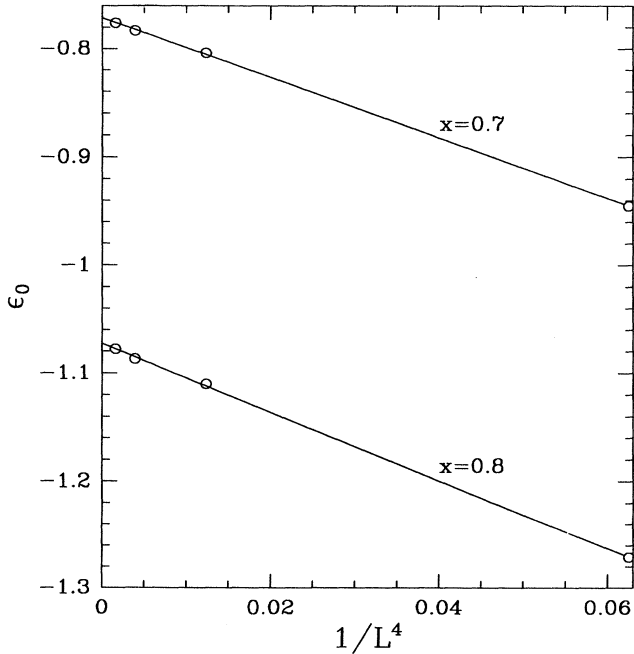$$v(x) = 2x^{1/2} - 0.3979337 - 0.0619x^{-1/2} + O(x^{-1}) . \tag{2.44b}$$

FIG. 2. Graph of $\epsilon_0(N)$ vs $1/L^4$ for the (3+1)D U(1) model, taken from the results of Hamer and Aydin [7]. Error bars for the Monte Carlo data are within the circles.
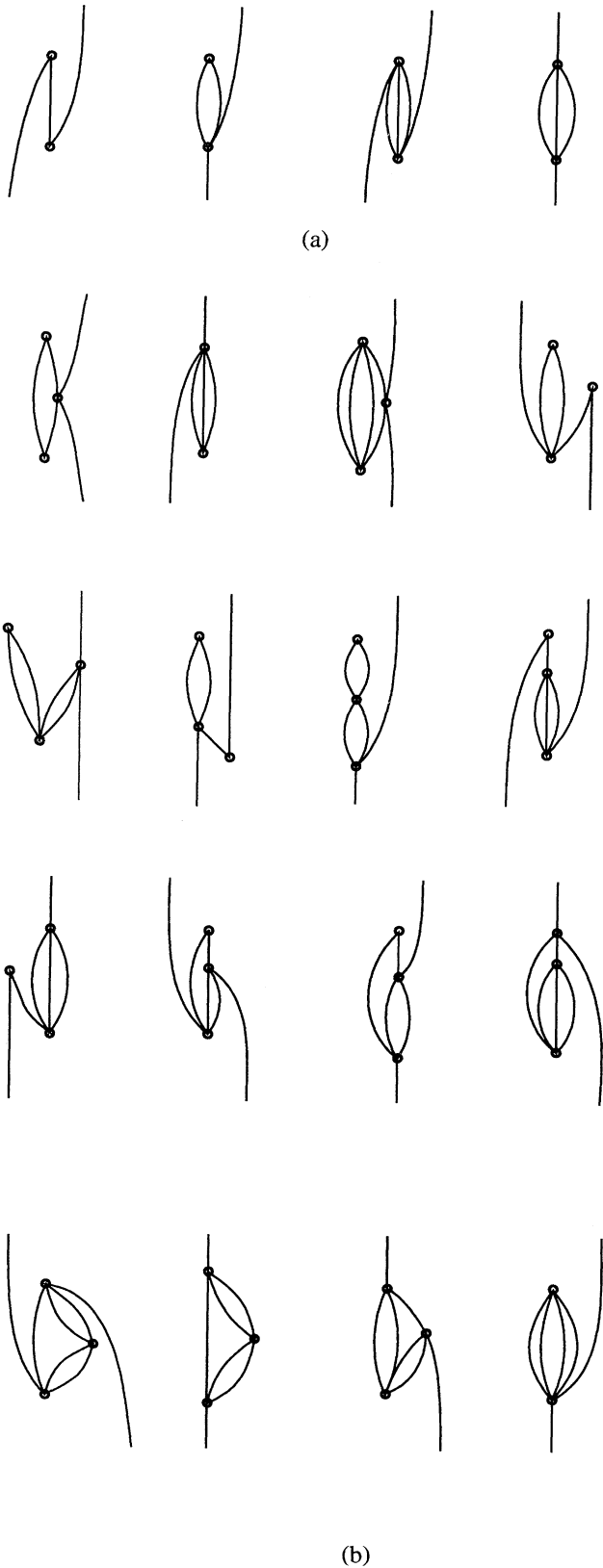
(a)

(b)

FIG. 3. The perturbation diagrams that contribute to the energy $E(k)$ of a single-boson state. Conventions as in Fig. 1.

Now upon comparing Eqs. (2.37) and (2.44), one finds a remarkable connection between the finite-size corrections to the ground-state energy density, and the boson velocity $v(x)$. It is *exactly the same relationship* as that predicted by effective Lagrangian theory [3] for the $O(N)$ lattice spin model: namely,

$$\epsilon_0(N) - \epsilon_0(\infty) = -\frac{n_B v}{2L^{D+1}}\left[\alpha^{(D)}_{-1/2}(1) + 2 - \frac{2}{D+1}\right]$$
$$+O(L^{-2D}) \,, \tag{2.45}$$

where n_B is the number of independent, massless boson fields in the effective Lagrangian, and $\alpha^{(D)}_{-1/2}(1)$ is a geometric "shape factor" given by [2]

$$\alpha^{(D)}_{-1/2}(1) + 2 - \frac{2}{D+1} = \begin{cases} \pi/3 & (D=1), \\ 1.43775 & (D=2), \\ 1.67507 & (D=3). \end{cases} \tag{2.46}$$

This universal relation is satisfied order by order in the weak-coupling expansion for the present model, as far as we have computed, with

$$D = 2: \quad n_B = 1, \quad \epsilon_0(N) - \epsilon_0(\infty) \sim -\frac{0.7188v(x)}{L^3} \,,$$
$$\tag{2.47a}$$

$$D = 3: \quad n_B = 2, \quad \epsilon_0(N) - \epsilon_0(\infty) \sim -\frac{1.675v(x)}{L^4} \,.$$
$$\tag{2.47b}$$

We expect that Eq. (2.47) could be proved to hold to all orders in the weak-coupling expansion on the basis of a diagrammatic identity, although we do not attempt such a task here.

Using (2.47b) and (2.44b) we predict that for $D = 3$ the slope of the graph of $[\epsilon_0(N) - \epsilon_0(\infty)]$ against $1/L^4$ should be -2.01 at $x = 0.7$ and -2.21 at $x = 0.8$. The measured values from Fig. 2 are -2.77 and -3.16, respectively, which are about 30% greater than the predictions. It must be recalled, however, that these x values are only just within the weak-coupling regime, and not far from the critical point $x_c = 0.675(25)$, so that higher-order perturbation terms and nonperturbative effects may be significant. In these circumstances, the agreement is probably as good as could be expected.

C. String tension

The finite-size scaling behavior of the string tension in this model is another interesting feature, to which we now turn. The "zero modes," which we have neglected hitherto, play a crucial role in this regard.

Separate out the $\mathbf{k} = 0$ terms from the Hamiltonian (2.9); then one finds *no* term involving $A_0(\hat{\mu})$: this is an "ignorable" coordinate, and its conjugate $E_0(\hat{\mu})$ is conserved. Thus

$$H_{\mathbf{k}=0} = \sum_{\hat{\mu}=1}^{D} E_0^\dagger(\hat{\mu})E_0(\hat{\mu}) \,, \tag{2.48}$$

where

$$E_0(\hat{\mu}) = \frac{1}{\sqrt{N}} \sum_{\mathbf{n}} E(\mathbf{n}, \hat{\mu}) \ . \tag{2.49}$$

Since the eigenvalues of $E(\mathbf{n}, \hat{\mu})$ are integers, it is easily seen that the eigenvalues of $E_0(\hat{\mu})$ are

$$E_0^\dagger(\hat{\mu}) = E_0(\hat{\mu}) = \frac{l(\hat{\mu})}{\sqrt{N}}, \quad l(\hat{\mu}) = \text{integer} \ , \tag{2.50}$$

and thus the zero-mode energy eigenvalues are

$$E_{\mathbf{k}=0} = \frac{1}{N} \sum_{\hat{\mu}} l^2(\hat{\mu}), \quad l(\hat{\mu}) = \text{integer} \ , \tag{2.51}$$

exact to all orders in the weak-coupling expansion.

One must then take care in enumerating which states are physically allowed. This is easily done by considering the strong-coupling limit of the model, where the basis states are eigenstates of the electric field. The ground state has $l(\hat{\mu}) \equiv 0$. Most nonzero values of l are forbidden by the requirement that $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$; i.e., there are no "sources" of electric flux on the lattice. The exceptions to this rule are the "string" states, in which a string of electric flux wraps around the entire periodic lattice. The minimum eigenvalue for such a state is $l = L$; and thus the string tension, or energy per link of an axial string state, is

$$\sigma = \frac{1}{L} E_{\text{string}} \ , \tag{2.52}$$

i.e.,

$$\sigma = \frac{1}{L^{D-1}} \tag{2.53}$$

exact to all orders in the weak-coupling expansion (but not accounting, of course, for nonperturbative effects).

For the $D = 3$ case, the behavior $\sigma \sim 1/L^2$ $(x \to \infty)$ was already derived, and demonstrated numerically, by Hamer and Aydin [7]. There is a phase transition at $x^* \simeq 0.675$, and beyond that point the finite-lattice string tension drops sharply and then levels off at the value $1/L^2$. This behavior is characteristic of a critical point [7], and provides very clear evidence of the line of fixed points running from x^* to ∞, which is expected to occur in this model. For the $D = 2$ case, the behavior $\sigma \sim 1/L$ (which is again characteristic of a critical point) would only be expected to occur asymptotically as $x \to \infty$. There are no numerical data available to check this point. There have been many calculations of the bulk behavior of the (2+1)D model (for recent examples see Refs. [18–20]), but to our knowledge there are no reliable Monte Carlo calculations of the finite-size scaling behavior in the weak-coupling regime.

D. Mass gap

In the ground-state sector, the zero-mode eigenvalues $l(\hat{\mu})$ are strictly zero, as noted in the previous sec-

tion. The first translation-invariant excited state therefore consists, at leading order, of a pair of bosons with equal and opposite momenta $|\mathbf{k}| = 2\pi/L$; and hence the energy gap in the ground-state sector is

$$\Delta E \sim \frac{4\pi\sqrt{x}}{L} \ , \tag{2.54}$$

at leading order. The predicted $1/L$ dependence is again characteristic of a critical point. There are no numerical data presently available to test this prediction.

III. EFFECTIVE LAGRANGIAN THEORY

In view particularly of the relation (2.45), which appears to hold in weak-coupling perturbation theory, it is natural to ask whether the weak-coupling limit of these models can be described by an effective Lagrangian approach. Leaving aside for the moment the question as to whether such an approach is valid or not, let us proceed along the lines laid out by Hasenfratz and Niedermayer [3].

We are interested primarily in the finite-size behavior of the model. We begin by choosing units $\hbar = v = 1$, where v is the "speed of light," and defining the system in a Euclidean box $L_t \times L^D$, interpreting $L_t = T^{-1}$ where T is the temperature. To begin with, we shall assume $L_t \sim L_s$ ("cubic" geometry), but we shall eventually be interested in the $T \to 0$ or $L_t \to \infty$ limit ("cylinder" geometry).

A. Effective Lagrangian

In this method one starts by writing down the most general local Lagrangian L_{eff} which respects the symmetries of the underlying model. The different terms in the effective Lagrangian are multiplied by unknown couplings. Accordingly, we assume that an effective Lagrangian can be written in terms of a massless gauge field A_μ, and that the leading term in the Euclidean effective action takes the form

$$S = \int_0^{1/T} dt \int_0^L d^D x \frac{1}{4} \rho_s F_{\mu\nu} F_{\mu\nu} \ , \tag{3.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{3.2}$$

In other words, the leading term is assumed to be simply the same as a free photon field, with unknown coupling ρ_s. This respects the gauge symmetry of the original model, whereas pure polynomials in A_μ do not. In general, the effective Lagrangian may be expected to include other terms as well, which contain higher-order derivatives of the field. The coupling ρ_s is independent of T and L [3].

The low-energy excitations carry momenta $p \sim L^{-1} \sim T$. Then every derivative in the effective Lagrangian is counted as $\sim p$. The field A_μ should be counted as $\sim$

$p^{(D-1)/2}$, since fluctuations of this size have a Boltzmann factor of order 1: the leading term in the Lagrangian density then is

$$\frac{1}{4}\rho_s F_{\mu\nu}F_{\mu\nu} \sim p^{D+1} , \qquad (3.3)$$

which is integrated over a region $L^D/T \sim 1/p^{D+1}$. Higher-order terms in the effective Lagrangian, containing more fields and/or more derivatives, will therefore be higher order in p. This forms the basis for a systematic, low-energy or large-volume expansion, as in the case discussed by Hasenfratz and Niedermayer [3]. Only two unknown parameters, ρ_s and v (after restoring dimensions), enter up to this order.

One can argue, following Hasenfratz and Niedermayer [3], that a finite cutoff will not affect the leading-order contributions to the free energy density, once one allows for renormalization of the couplings.

B. Volume dependence of the ground-state energy

Now we consider a Euclidean cylinder $L_t \times L^D$ in the $L_t \to \infty$ limit. We use a (hyper)cubic lattice to regularize the effective theory. Put the lattice unit $a = 1$, and denote the lattice points by $\mathbf{x} = (t, \mathbf{x})$ where $t = 1, 2, \ldots, L_t$, etc. The Fourier transform is defined as

$$A(\mathbf{x}, \hat{\mu}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} A_{\mathbf{k}}(\hat{\mu}), \quad V = L^D \cdot L_t \qquad (3.4)$$

[where $A(\mathbf{x}, \hat{\mu})$ is defined on the link starting at $\mathbf{x}$ and pointing in direction $\hat{\mu}$, as in Sec. II]. The action has the regularized form

$$S(A) = \frac{1}{4}\rho_s \sum_{\mathbf{x}} [A(\mathbf{x}+\hat{\mu}, \hat{\nu}) - A(\mathbf{x}, \hat{\nu}) - A(\mathbf{x}+\hat{\nu}, \hat{\mu})$$
$$+ A(\mathbf{x}, \hat{\mu})]^2 . \qquad (3.5)$$

Once the theory has been placed on a lattice, the treatment can be carried out in very similar fashion to Sec. II. Choose the Hamiltonian gauge $A_0 = 0$, $\nabla \cdot \mathbf{A} = 0$, then upon Fourier transforming, the "timelike" plaquette terms give

$$\sum_{\mathbf{x}, \hat{i}} [A(\mathbf{x}+\hat{t}, \hat{i}) - A(\mathbf{x}, \hat{i})]^2$$
$$= \sum_{\mathbf{x}, \hat{i}} A_{\mathbf{k}}^{\dagger}(\hat{i}) A_{\mathbf{k}}(\hat{i}) 4\sin^2\left(\frac{k_0}{2}\right)$$
$$- \sum_{\mathbf{k}} \sum_{\hat{\mu}=1}^{D-1} \omega_{\mathbf{k}}^{\dagger}(\hat{\mu})\omega_{\mathbf{k}}(\hat{\mu}) 4\sin^2\left(\frac{k_0}{2}\right) , \qquad (3.6)$$

while the "spacelike" plaquette terms give

$$\sum_{\mathbf{x}, \hat{i}, \hat{j}} [A(\mathbf{x}+\hat{i}, \hat{j}) - A(\mathbf{x}, \hat{j}) - A(\mathbf{x}+\hat{j}, \hat{i}) + A(\mathbf{x}, \hat{i})]^2$$
$$= 2\sum_{\mathbf{k}} \sum_{\hat{\mu}=1}^{D-1} \omega_{\mathbf{k}}^{\dagger}(\hat{\mu})\omega_{\mathbf{k}}(\hat{\mu})\Delta_{\mathbf{k}} , \qquad (3.7)$$

where

$$\Delta_{\mathbf{k}} = 4\sum_{\hat{i}=1}^{D} \sin^2\left(\frac{k_i}{2}\right) , \qquad (3.8)$$

as before. Therefore the action can be rewritten

$$S(\omega) = \frac{1}{2}\rho_s \sum_{\mathbf{k}} \sum_{\hat{\nu}=1}^{D-1} \omega_{\mathbf{k}}^{\dagger}(\hat{\nu})\omega_{\mathbf{k}}(\hat{\nu})d(k) , \qquad (3.9)$$

where

$$d(k) = 4\sum_{\hat{\mu}=0}^{D} \sin^2\left(\frac{k_{\mu}}{2}\right) . \qquad (3.10)$$

The Euclidean partition function is

$$Z = \left(\prod_{\mathbf{k}} \prod_{\nu=1}^{D-1} \int d\omega_{\mathbf{k}}(\nu)\right) \exp[-S(\omega)] , \qquad (3.11)$$

leading to a free energy density

$$f(T, L) = \frac{(D-1)}{2V} \sum_{\mathbf{k}} \ln\left(\rho_s \frac{d(k)}{2\pi}\right) , \qquad (3.12)$$

after integrating the Gaussian form (3.9).

In the $L_t \to \infty$ limit one obtains

$$f(0, L) = -\frac{D-1}{2L^{D+1}}\left[\alpha_{-1/2}^{(D)}(1) + 2 - \frac{2}{D+1}\right] , \qquad (3.13)$$

where $\alpha_p^{(D)}$ are the "shape coefficients" referred to earlier [3]. The terms shown here are only the leading finite-size corrections, which should be universal; bulk terms which are nonuniversal have been omitted.

The ground-state energy then is simply

$$\epsilon_0(L) = \lim_{T \to 0} f(T, L) , \qquad (3.14)$$

and hence

$$\epsilon_0(L) - \epsilon_0(\infty) = -\frac{D-1}{2L^{D+1}}\left[\alpha_{-1/2}^{(D)}(1) + 2 - \frac{2}{D+1}\right] . \qquad (3.15)$$

Allowing for an "anisotropic" space-time with $v \neq 1$ (but still $\hbar = 1$), one arrives at the final result given in Eq. (2.45). Just as in the case treated by Hasenfratz and Niedermayer [3], this leading-order behavior is unaffected by higher-order terms in the effective Lagrangian, and is expected to be an exact, universal formula for the coefficient of the $O(L^{-D-1})$ finite-size correction term.

C. Zero modes

Now let us look more closely at the class of collective or "zero modes" with $\mathbf{k} = (k^0, \mathbf{k} = 0)$, corresponding to gauge fields which are constant in space, and slowly varying with time.

Define the zero-mode variables

$$\Omega_{k^0}(\hat{i}) = A_{k^0, \mathbf{k}=0}(\hat{i}) ; \qquad (3.16)$$

then the term in the action associated with these variables is

$$S(\Omega) = \frac{1}{2}\rho_s \sum_{k^0, \hat{i}} \Omega_{k^0}^{\dagger}(\hat{i})\Omega_{k^0}(\hat{i}) 4\sin^2\left(\frac{k_0}{2}\right)$$

$$\sim \frac{1}{2}\rho_s \sum_{k^0, \hat{i}} k_0^2 \Omega_{k^0}^{\dagger}(\hat{i})\Omega_{k^0}(\hat{i}) \text{ as } L_t \to \infty , \qquad (3.17)$$

or, Fourier transforming,

$$S(\Omega) \sim \frac{1}{2}\rho_s \int dt \, \dot{\Omega}^2(t, \hat{i}) \text{ as } L_t \to \infty . \qquad (3.18)$$

The conjugate angular momentum density is

$$\pi(\hat{i}) = \rho_s \dot{\Omega}(t, \hat{i}) , \qquad (3.19)$$

and the term in the Minkowski space Hamiltonian density associated with the zero-mode variables is

$$\mathcal{H}(\Omega) = \frac{1}{2\rho_s} \sum_i \pi^2(\hat{i}) . \qquad (3.20)$$

The discussion from here on is exactly the same as in the weak-coupling expansion, so that for the compact U(1) theory one finds the string tension for an axial string state is predicted as

$$\sigma = \frac{1}{2\rho_s L^{D-1}} , \qquad (3.21)$$

or, allowing $v \neq 1$,

$$\sigma = \frac{v^2}{2\rho_s L^{D-1}} . \qquad (3.22)$$

This agrees with Eq. (2.53) provided

$$\frac{v^2}{2\rho_s} = 1 . \qquad (3.23)$$

An identity of exactly the same form holds for the parameters of the effective Lagrangian in the O(2) Heisenberg spin model in $D+1$ dimensions [5].

IV. SUMMARY AND CONCLUSIONS

Weak-coupling expansions have been calculated for the compact U(1) gauge theory in $D+1$ dimensions, defined by Eq. (2.1). Let us summarize the results here, for convenience.

(i) Bulk ground-state energy per site:
$D = 2$ square lattice:

$$\epsilon_0(\infty) = -2x + 1.916183x^{1/2} - 0.2294848$$
$$-0.0268602x^{-1/2} - 0.009315x^{-1} + O(x^{-3/2}) . \qquad (4.1)$$

$D = 3$ cubic lattice:

$$\epsilon_0(\infty) = -6x + 4.77520x^{1/2} - 0.475054 - 0.04182x^{-1/2}$$
$$+O(x^{-1}) . \qquad (4.2)$$

(ii) Photon velocity:
$D = 2$ square lattice:

$$v = 2x^{1/2} - 0.479046 - 0.104783x^{-1/2} - 0.05847x^{-1}$$
$$+O(x^{-3/2}) . \qquad (4.3)$$

$D = 3$ cubic lattice:

$$v = 2x^{1/2} - 0.3979337 - 0.0619x^{-1/2} + O(x^{-1}) . \qquad (4.4)$$

(iii) Finite-size corrections to the ground-state energy per site:
$D = 2$ square lattice:

$$\epsilon_0(N) - \epsilon_0(\infty) \sim -\frac{1}{L^3}[1.4376x^{1/2} - 0.34434$$
$$-0.07532x^{-1/2} - 0.0421x^{-1}$$
$$+O(x^{-3/2})] . \qquad (4.5)$$

$D = 3$ cubic lattice:

$$\epsilon_0(N) - \epsilon_0(\infty) \sim -\frac{1}{L^4}[3.349x^{1/2} - 0.6664 + O(x^{-1/2})] , \qquad (4.6)$$

for a lattice of $N = L^D$ sites.

(iv) Finite-size axial string tension:

$$\sigma = \frac{1}{L^{D-1}} , \qquad (4.7)$$

exact to all orders in the weak-coupling expansion.

(v) Energy gap in the ground-state sector:

$$\Delta E \sim \frac{1}{L}[4\pi\sqrt{x} + \text{const}] . \qquad (4.8)$$

An "effective Lagrangian" theory [3] has also been constructed to describe the long distance behavior of the model at the critical limit. Its leading term is simply the free photon Lagrangian density, with coupling parameter ρ_s (the "helicity modulus"). From this approach one can derive a remarkable universal relationship for the leading finite-size correction to the ground-state energy,

$$\epsilon_0(N) - \epsilon_0(\infty) = -\frac{(D-1)v}{2L^{D+1}}\left[\alpha_{-1/2}^{(D)} + 2 - \frac{2}{D+1}\right]$$
$$+O(L^{-2D}) , \qquad (4.9)$$

in terms of the speed of light v, which is satisfied order by order by Eqs. (4.3)–(4.6), and which is expected to be an exact relationship. One also obtains an expression for the string tension,

$$\sigma = \frac{v^2}{2\rho_s L^{D-1}} , \qquad (4.10)$$

which agrees with Eq. (4.7) provided the parameters ρ_s and v obey an identity

$$\frac{v^2}{2\rho_s} = 1 . \qquad (4.11)$$

Precisely analogous relationships to these have previously been found to apply [5] to the $O(N)$ Heisenberg spin model in $D+1$ dimensions.

We should now address the question of whether this effective Lagrangian approach is really valid in the present context. Hasenfratz, Leutwyler, and their collaborators [1–3] have hitherto only applied it to cases where spontaneous symmetry breaking occurs, and the long distance behavior is dominated by massless Goldstone bosons. Here, there is no spontaneous symmetry breaking in the conventional sense, and no local order parameter, but we have presumed that the long distance behavior is dominated by massless photon modes.

The approach seems to be internally consistent at least, in that higher-order terms in the effective Lagrangian can be shown to involve higher powers of momentum, and hence one can develop a large volume expansion in powers of $(\hbar v/\rho_s L^{D-1})$, just as in the spin model case.

The approach is almost trivially valid for the *noncompact* lattice model, which is obtained by truncating the expansion (2.9) after the quadratic term in θ_p. This is a noninteracting theory, and does indeed reduce to the theory of free photons in the continuum limit as one would naively expect. In this case, the parameters ρ_s and v are simply given by the leading-order terms

$$v = 2x^{1/2} , \qquad (4.12a)$$

$$\rho_s = 2x , \qquad (4.12b)$$

satisfying Eq. (4.11).

In fact, Kovner, Rosenstein, and Eliezer [21] have argued that in this case the photon can indeed be regarded as a Goldstone boson, arising from spontaneous breakdown of a global symmetry generated by the magnetic flux. They noted the softness of the interactions between the photons at low energy, and discussed the form of the effective "chiral" Lagrangian. Their conclusions did not extend to the case of compact QED, however.

For the compact (3+1)D model, it has been proven [22,23] that there is a confining phase at strong coupling, and a nonconfining phase at weak coupling. There is a massless, Coulomb phase extending from the weak-coupling limit to an end point at some finite coupling. The nature of the phase transition at the end point is still a matter of debate [24,25,7], and may depend on the exact form of the action chosen. Driver [26] has proved that in a certain sense the lattice model converges to the free electromagnetic theory in the weak-coupling continuum limit, and so our effective Lagrangian approach certainly seems valid here.

Cardy [10] has discussed the renormalization of the fine-structure constant α in the Coulomb phase due to magnetic monopoles in the compact theory. Confinement occurs when the monopole susceptibility diverges. He argues that the transition should occur at a universal critical value α_c, by analogy with the $O(2)$ planar spin model in two dimensions, where the critical index η reaches a universal value of 1/4 at the transition. Luck [11] has estimated the value of α_c at the transition in the Euclidean gauge model, obtaining a value

$$4\pi\alpha_c = 1.9 \pm 0.1 . \qquad (4.13)$$

In the present Hamiltonian version of the model, the fine-structure constant is related to the parameters ρ_s and v by

$$\alpha = \frac{v}{4\pi\rho_s} = \frac{1}{2\pi v} \qquad (4.14)$$

(see Appendix C). From Eq. (4.4) we can obtain a weak-coupling expansion for α, which at the transition point $x_c = 0.675 \pm 0.025$ has a value

$$4\pi\alpha_c = 1.7(2) , \qquad (4.15)$$

which agrees within errors with Luck's value.

The compact $D = 2$ case is more problematical still. Here there is no massless phase at finite x, and the weak-coupling results can only apply asymptotically as $x \to \infty$ [although the bulk ground-state energy seems to be quite well described by (2.37a) at moderate x values]. Furthermore, Göpfert and Mack [12] have shown analytically that the model is confining for all values of the coupling constant e^2, and that in the continuum limit it reduces to a theory of free, massive scalar bosons, with a mass scale that decreases exponentially as the lattice spacing a goes to zero:

$$M^2 a^2 \sim \frac{c_1}{g^2} \exp(-c_2/g^2) \quad \text{as } a \to 0 , \qquad (4.16)$$

where $g^2 = e^2 a$ is the dimensionless coupling.

The question then arises, whether the finite-size scaling behavior predicted by the weak-coupling expansion and the effective Lagrangian approach will really apply in these circumstances. We expect that it *will* apply, on the basis of the following argument: if one takes the "naive" continuum limit, keeping the coupling e^2 *fixed* as $a \to 0$, then the mass scale M goes to zero, and one indeed obtains a theory of massless bosons with a single degree of freedom per site, for which the finite-size scaling behavior (4.9), for instance, should apply. Thus the finite-size scaling predictions (4.5), (4.7), and (4.8) should apply *provided* we look at energy scales $\gg M$, i.e.,

$$1/L \gg Ma \sim \frac{c_1^{1/2}}{g} \exp\left(-\frac{c_2}{2g^2}\right) ; \qquad (4.17)$$

or, in other words, the predictions apply in an asymptotic sense at large couplings x, provided the lattice size L is not too large. It would be interesting to have this checked by a Monte Carlo calculation.

In fact, Gross [13] has proved that in the "naive" continuum limit the Villain version of the model, at least, converges to the naive free electromagnetic theory at the level of $F_{\mu\nu}$ or the Wilson loops. The effective Lagrangian which we have assumed is compatible with this, of course. It seems therefore that the free photon theory which one obtains as the "naive" continuum limit describes the system at a fixed scale of energy and distance, while the confining massive scalar theory describes the system on an exponentially smaller mass scale given by (4.16), and therefore on an exponentially larger scale of distance.

The number of degrees of freedom is the same in each case, and the ratio between these energy scales becomes infinite in the continuum limit, of course, and so these statements are not incompatible.

Perhaps the major point arising from this study is the fact that for an Abelian gauge theory the interactions between the gauge bosons are "soft," i.e., become weak at low energies [21]. Hence in an effective Lagrangian theory of this sort, the massless gauge bosons will control the low-temperature and large-volume behavior of the system, just as massless Goldstone bosons do.

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$$D = 2: \quad U_{\mathbf{k}} = \frac{i}{c_1} \begin{pmatrix} -e^{-i_x} s_y & e^{-i_y} s_x \\ e^{-i_x} s_x & e^{-i_y} s_y \end{pmatrix} , \tag{A3a}$$

$$D = 3: \quad U_{\mathbf{k}} = \begin{pmatrix} ie^{-i_x} s_y / c_1 & -ie^{-i_y} s_x / c_1 & 0 \\ -e^{-i_x} s_x s_z / c_3 & -e^{-i_y} s_y s_z / c_3 & e^{-i_z}(s_x^2 + s_y^2)/c_3 \\ ie^{-i_x} s_x / c_2 & ie^{-i_y} s_y / c_2 & ie^{-i_z} s_z / c_2 \end{pmatrix} , \tag{A3b}$$

where $c_1 = \sqrt{s_x^2 + s_y^2}$, $c_2 = \sqrt{s_x^2 + s_y^2 + s_z^2}$, $c_3 = c_1 c_2$.

The relationship between the $\theta_{\mathbf{k}}$ and the $\omega_{\mathbf{k}}$, from (2.7) and (2.14), is

$$\theta_{\mathbf{k}}(\hat{\mu}) = \sum_{\hat{\nu},\hat{\sigma}} G_{\mathbf{k}}(\hat{\mu},\hat{\nu}) U_{\mathbf{k}}^{\dagger}(\hat{\nu},\hat{\sigma}) \omega_{\mathbf{k}}(\hat{\sigma}) , \tag{A4}$$

so that, in Eq. (2.27),

$$B_{\mathbf{k}}(\hat{\mu},\hat{\nu}) = \sum_{\hat{\sigma}} G_{\mathbf{k}}(\hat{\mu},\hat{\sigma}) U_{\mathbf{k}}^{\dagger}(\hat{\sigma},\hat{\nu}) . \tag{A5}$$

For $D = 2$, $B_{\mathbf{k}}$ has only one nonzero component:

$$B_{\mathbf{k}} = (\Delta_{\mathbf{k}}^{1/2}, 0) . \tag{A6a}$$

For $D = 3$, we find

$$B_{\mathbf{k}} = 2 \begin{pmatrix} s_x s_z / c_1 & is_y c_2 / c_1 & 0 \\ s_y s_z / c_1 & -is_x c_2 / c_1 & 0 \\ -c_1 & 0 & 0 \end{pmatrix} , \tag{A6b}$$

APPENDIX A

Here we give explicit details of some of the matrices involved in the weak-coupling expansion. Let us denote

$$i_x \equiv \frac{ik_x}{2}, \quad s_x \equiv \sin\left(\frac{k_x}{2}\right) , \tag{A1}$$

and similarly for y and z components; then the matrix $R_{\mathbf{k}}$ defined by Eq. (2.11) is given by

$$R_{\mathbf{k}}(\hat{\mu},\hat{\nu}) = 4\left(\delta_{\mu,\nu} \sum_{\sigma} s_{\sigma}^2 - e^{i_{\mu} - i_{\nu}} s_{\mu} s_{\nu}\right) . \tag{A2}$$

The eigenvalues of this matrix are given by Eq. (2.12). The eigenvectors can be used to construct the matrix $U_{\mathbf{k}}$, which can be taken as

where again the components coupling to $\omega_{\mathbf{k}}(3)$, the eigenvector corresponding to $\lambda_{\mathbf{k}}(3) = 0$, are all zero.

APPENDIX B

Here we show how to calculate the bulk limits of various lattice constants C_n, etc., together with their finite-lattice corrections.

The evaluation of C_n involves a summation over momentum $\mathbf{k}$ in the first Brillouin zone. For the bulk system, the momentum $\mathbf{k}$ is continuous over the first Brillouin zone, but for the finite-lattice system, the momentum $\mathbf{k}$ is discrete. For the square lattice and cubic lattice, the structure factor $\gamma_{\mathbf{k}}$, the first Brillouin zone for a bulk system and the discrete momentum k for a finite-lattice system are the following.

$D = 2$ square lattice:

$$\gamma_{\mathbf{k}} = \frac{1}{2}[\cos(k_x a) + \cos(k_y a)],$$

momentum $\mathbf{k} : 0 < k_x a, \ k_y a < 2\pi$ (bulk system); \tag{B1}

$$k_x(i) = \frac{2\pi i}{La}, \quad k_y(i) = \frac{2\pi i}{La}, \quad i = 1, \ldots, L \quad \text{(finite-lattice system)}.$$

$D = 3$ cubic lattice:

$$\gamma_{\mathbf{k}} = \frac{1}{3}[\cos(k_x a) + \cos(k_y a) + \cos(k_z a)],$$

momentum $\mathbf{k} : 0 < k_x a, \ k_y a, \ k_z a < 2\pi$ (bulk system); $\hspace{3cm}$ (B2)

$$k_x(i) = \frac{2\pi i}{La}, \quad k_y(i) = \frac{2\pi i}{La}, \quad k_z(i) = \frac{2\pi i}{La}, \quad i = 1, \ldots, L \quad \text{(finite-lattice system)}.$$

$C_n(\infty)$ and its finite-lattice corrections can be evaluated by a least-squares fit of $C_n(L)$ to the form $C_n(\infty) + A/L^{D+1} + B/L^{D+2} + C/L^{D+3}$. The results for C_n, ΔE_b, and $\Delta E^{-1/2}(\mathbf{k})$ are summarized as follows.

$D = 2$ square lattice:

$$C_1 = 0.958091399 - 0.7188/L^3 \ , \tag{B3a}$$

$$C_2 = 0.204910823 - 0.8180/L^3 \ , \tag{B3b}$$

$$C_3 = 0.04610568 - 0.3080/L^3 \ , \tag{B3c}$$

$$C_4 = 0.2844704 \ , \tag{B3d}$$

$$C_5 = 0.08548917 \ , \tag{B3e}$$

$$C_6 = 0.06387607 \ . \tag{B3f}$$

$D = 3$ cubic lattice:

$$C_1 = 0.97473453393 - 0.6837/L^4 \ , \tag{B4a}$$

$$\Delta E_b = -0.010313x^{-1/2} \ , \tag{B4b}$$

$$\Delta E^{-1/2}(\mathbf{k}) = -0.05464(1 - \gamma_{\mathbf{k}})^{1/2} \quad \text{as} \quad \mathbf{k} \to 0 \ . \tag{B4c}$$

APPENDIX C

Here we show how the fine-structure constant is related to the parameters ρ_s and v.

Assume the continuum effective Lagrangian density (in Minkowski space) is

$$\mathcal{L} = -\frac{\rho_s}{4} F_{\mu\nu} F^{\mu\nu} - k j^\mu A_\mu \ , \tag{C1}$$

and the "speed of light" is v, where a term coupling the field A_μ to a charge current j^μ has been added in. Our first task is to establish the value of the coupling k which corresponds to our original lattice model.

From (C1), the canonical $\mathbf{E}$ field is given by

$$E^i = v \frac{\partial \mathcal{L}}{\partial \dot{A}_i} \ , \tag{C2}$$

and hence

$$\mathbf{E} = -\rho_s \left(\boldsymbol{\nabla} \Phi + \frac{1}{v} \frac{\partial \mathbf{A}}{\partial t} \right) \ . \tag{C3}$$

The canonical equation of motion (Gauss' law) reads

$$\boldsymbol{\nabla} \cdot \mathbf{E} = k j^0 \ , \tag{C4}$$

and the Hamiltonian density is

$$\mathcal{H} = \frac{1}{2\rho_s} \mathbf{E}^2 + \frac{\rho_s}{2} \mathbf{B}^2 + \mathbf{E} \cdot \boldsymbol{\nabla} \Phi + k j^\mu A_\mu \ . \tag{C5}$$

Now comparing (C5) with (2.1) and (3.23), we see that our lattice $\mathbf{E}$ field,

$$E_l \to \mathbf{E}' = \frac{1}{v} \mathbf{E} \ , \tag{C6}$$

while the lattice $\mathbf{E}$ field obeys

$$\boldsymbol{\nabla} \cdot \mathbf{E}' = j^0 \tag{C7}$$

—a unit charge produces unit flux of $\mathbf{E}'$. Hence one finds the equivalent continuum coupling

$$k = v \ . \tag{C8}$$

Now let us rescale our units and fields to bring the Lagrangian density into standard form. Let

$$\mathcal{L} \to \xi_1 \mathcal{L}, \quad t \to \frac{1}{\xi_1} t, \quad A^\mu \to \xi_2 A^\mu \ , \tag{C9}$$

where

$$\xi_1 = \frac{1}{v}, \quad \xi_2 = \sqrt{\frac{\rho_s}{v}} \ ; \tag{C10}$$

then the rescaled Lagrangian density is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - e j^\mu A_\mu \ , \tag{C11}$$

with speed of light equal to unity, and

$$e = \frac{v \xi_1}{\xi_2} = \sqrt{\frac{v}{\rho_s}} \ ; \tag{C12}$$

therefore

$$\alpha = \frac{e^2}{4\pi} = \frac{v}{4\pi \rho_s} \ . \tag{C13}$$

As a check, note that in the weak-coupling limit $x \to \infty$, $\rho_s \to 2x$, $v \to 2\sqrt{x}$, where $x = 1/e_0^4$ and e_0 is the bare lattice coupling; and so $\alpha \to e_0^2/4\pi$, as expected.

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