Quantum mechanics of neutrino oscillations

J. Rich
DAPNIA-SPP, Centre d'Etudes de Saclay, 91191 Gif-sur-Yvette, France
(Received 7 June 1993)

We present a space-time approach to neutrino oscillations where neutrino emission, propagation, and absorption are treated as a single coherent quantum-mechanical process. The neutrino enters the calculation only as an unobserved intermediate state connecting initial and final states of the neutrino source and target. Equivalently, the neutrino can be considered to be an exchanged particle when the neutrino source "scatters" from the target at a macroscopic impact parameter. This approach avoids ambiguities inherent in both the standard "two-state" treatment and the "neutrino wave packet" treatment of neutrino oscillations and makes clear why and under what conditions oscillations can be observed. In simple situations, transition rates can be calculated showing how oscillations are suppressed for large neutrino mass differences.

PACS number(s): 14.60.Gh, 12.15.Ff

I. THREE WAYS TO OSCILLATE

If neutrinos have masses and nonzero mixing angles, they may "oscillate" [1]. By this we mean that a neutrino born as a given "flavor" (ν_e, ν_μ, or ν_τ) may interact later as a neutrino of a different flavor. This would result in the charged-current production of "wrong flavor" charged leptons and/or a deficit in production of "correct flavor" charged leptons. For example, neutrinos produced in pion decays in association with positive muons may oscillate from ν_μ to ν_e and produce electrons.

In the standard treatment of neutrino oscillations [2], the neutrino "flavor eigenstates" are linear combinations of "mass eigenstates." For two neutrino types one writes

$$|\nu_l\rangle = \sum_{i=1}^{2} U_{li}|\nu_i\rangle, \tag{1}$$

where $l = e, \mu$ and the matrix U is the mixing matrix $U_{11} = U_{22} = \cos\theta$ and $U_{12} = -U_{21} = \sin\theta$. To calculate the probability for a ν_e to oscillate to a ν_l one supposes the ν_e to be produced at time $t = 0$ with a definite momentum p. This state is a superposition of two states of different energies given by $E_i = \sqrt{p^2 + m_i^2} \sim p + m_i^2/2p$ where in the last form we assume $m_i \ll p$. The time evolution is then given by

$$|\nu(t)\rangle = \sum_{i=1}^{2} U_{ei}e^{-iE_it}|\nu_i\rangle = e^{-ipt}\sum_{i=1}^{2} U_{ei}e^{-im_i^2t/2p}|\nu_i\rangle \ . \tag{2}$$

(We use units where $\hbar = c = 1$.) The amplitude for the neutrino to be a ν_l at time t is

$$\langle\nu_l|\nu(t)\rangle = \sum_{i=1}^{2} U_{il}^* U_{ei}e^{-im_i^2t/2p} \ . \tag{3}$$

The time t is then replaced by x_T/c, where x_T is the distance from the production point to the neutrino target. The probability that the neutrino is a ν_l is seen to oscillate with the distance:

$$\begin{aligned} P(t = x_T, \nu_e \to \nu_l) &= |\langle\nu_l|\nu(t)\rangle|^2 = |\sum_{i=1}^{2} U_{il}^* U_{ie}e^{-im_i^2t/2p}|^2 \\ &= |U_{1l}U_{1e}|^2 + |U_{2l}U_{2e}|^2 + 2U_{1l}U_{1e}U_{2l}U_{2e}\cos(2\pi x_T/l_v), \end{aligned} \tag{4}$$

where we assume the U are real. The vacuum oscillation length is given by

$$l_v = 4\pi p/(m_1^2 - m_2^2). \tag{5}$$

The treatment is simple and elegant and gives the right answer under certain conditions. However, it immediately raises a number of conceptual questions. Why is the ν_e given a definite momentum rather than a definite energy? How can we change t into x_T/c when a neutrino with a definite momentum must have a wave function of infinite extent? Why should we set $t = x_T/c$ when the two neutrinos have different velocities neither of which is equal to c. Finally, would not the time-of-flight and momentum measurements allow us to determine if a ν_1 or ν_2 was emitted and thus destroy the interference that makes the oscillations?

Wave-packet treatments [3] eliminate some of these problems. Here the neutrino is created not as a simple two-state system but rather as a superposition of two wave packets, one for each mass eigenstate. Appeals to the uncertainty principle indicate under what conditions oscillations are observable. Unfortunately, it is not clear what determines the size of the wave packet at the moment of creation or even if it makes sense to talk of a precise time of creation. Additionally, the treatment cannot give the right answer when the source and target are separated by a distance smaller than the presumed size of the wave packet.

It is the purpose of this paper to emphasize that it is not necessary to worry about when or in what state the neutrino is created because neutrinos are neither "prepared" nor "observed." The only things that can be prepared are the unstable source particle (pion, nucleus,...) and the target particle. The only things that can be observed are the hadrons and charged leptons created in the decay or in the neutrino interaction.

This suggests that a less ambiguous treatment of neutrino oscillations could be performed by considering a system consisting initially of an unstable source particle in a definite quantum state and a target particle in a definite quantum state at some distance from the source. One can then use the Schrödinger equation to calculate the amplitude for this system to evolve into a state consisting of a set of particles associated with the source and another set associated with the target. Since only weak interactions are involved, the time evolution can be calculated easily with second-order time-dependent perturbation theory of standard quantum mechanics [4]. The amplitude is second order because the Hamiltonian acts twice, once at the source and once at the target. The neutrino enters the calculation only as an unobserved intermediate state, *which is precisely what it is.* In the language of quantum field theory, the neutrinos are exchanged particles when the source particle "scatters" from the target particle [5]. The associated "diagram" is shown in Fig. 1.

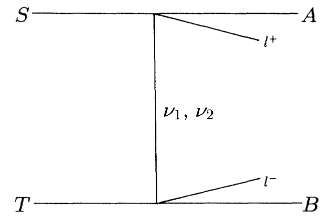

FIG. 1. The diagram for neutrino oscillations due to a neutrino source S and target T. Set A contains one charged antilepton l^+ and set B contains one charged lepton l^-.

II. A RADIOACTIVE SOURCE

To get an idea how this can work we take our initial state to consist of an unstable particle trapped in a potential well centered at $x = 0$ and a target particle trapped in another potential well centered at $x = x_T$. (To simplify things we take space to be one dimensional.) We assume that initially ($t = 0$) the source and target particles are in the ground states of their potential wells. This is a realistic model of a radioactive nucleus and target nucleus placed in two crystal lattices at the absolute zero of temperature. (Temperature effects will be discussed later.) The total energy (mass plus potential energy) of the source particle is E_S while that of the target particle is E_T. The final state is taken to consist of two sets of particles. One set (A) contains one positron and any other particles produced in the decay of S. The set has momentum p_A and energy E_A. The other set (B) contains one lepton (electron or muon) and any other particles produced in the neutrino interaction with T. The set has momentum p_B and energy E_B. The intermediate states are those with set A of particles and one neutrino of mass $m_i, i = 1, 2$ of momentum p_ν and energy $E_\nu \sim p_\nu + m_i^2/2p_\nu$.

As a specific example, we could use ^{22}Na as the source S and ^{71}Ge as the target T. The sequence of events would be $^{22}\text{Na} \to ^{22}\text{Ne}\ e^+\ \nu$ followed by $\nu\ ^{71}\text{Ga} \to ^{71}\text{Ge}\ e^-$. The set A consists of ^{22}Ne and the positron while the set B consists of ^{71}Ge and the electron. The intermediate states consists of ^{22}Ne, the positron and the neutrino.

Starting at $t = 0$ with the source and target in their ground states, the intermediate states start to be populated because the interaction Hamiltonian has matrix elements, $\langle A, \nu_i|H|S\rangle$, that connect the initial state with the intermediate states. We denote by $c_{A\nu_i} e^{-i(E_A+E_\nu)t}$ the amplitude for the system to be in the intermediate state at time t. The standard first-order perturbation result obtained by integrating the Schrödinger equation is

$$c_{A\nu_i}(t) = \int_0^t dt' \langle A, \nu_i|H|S\rangle e^{i(E_\nu+E_A-E_S)t'} c_{ST}(t=0), \tag{6}$$

where $c_{ST}(0) = 1$ is the amplitude for the system to be in the initial state. The intermediate states then feed into the final state because of the matrix elements $\langle B|H|\nu_i, T\rangle$. Denoting by $c_{AB} e^{-i(E_A+E_B)t}$ the amplitude for the system to be in the final state at time t, the standard perturbation result is

$$c_{AB}(t) = -\sum_{i=1}^{2}\sum_{p_\nu} \int_0^t dt'' \langle B|H|\nu_i, T\rangle e^{i(E_B-E_T-E_\nu)t''} c_{A\nu_i}(t'')$$

$$= -\sum_{i=1}^{2}\sum_{p_\nu} \int_0^t dt'' \langle B|H|\nu_i, T\rangle e^{i(E_B-E_T-E_\nu)t''} \int_0^{t''} dt' \langle A, \nu_i|H|S\rangle e^{i(E_\nu+E_A-E_S)t'} c_{ST}(0) \ . \tag{7}$$

The two sums are over the two neutrino types (masses) and all possible neutrino momenta.

Equation (7) contains two matrix elements which we will now calculate. We use the standard Fermi theory so the elements are just proportional to the overlap of the relevant wave functions:

$$\langle A, \nu_i|H|S\rangle \propto U_{ei} \int dx e^{-ip_\nu x} e^{-ip_A x} \psi_S(x), \tag{8}$$

$$\langle B|H|\nu_i, T\rangle \propto U_{li}^* \int dx e^{-ip_B x} e^{ip_\nu x} \psi_T(x), \tag{9}$$

where ψ_S and ψ_T are the wave functions of the source and target particle and $l = e, \mu$ depending on whether an electron or a muon is produced in the target. The matrix U enters to give the relative amplitudes for emitting and absorbing ν_1 and ν_2. Taking the two wave functions to be Gaussians

$$\psi_S \propto e^{-x^2/2\sigma_x^2}, \tag{10}$$

$$\psi_T \propto e^{-(x-x_T)^2/2\sigma_x^2}, \tag{11}$$

the matrix elements are Gaussians in momentum space

$$\langle A, \nu_i|H|S\rangle \propto U_{ei} \exp\left[\frac{-(p_\nu + p_A)^2}{2\sigma_p^2}\right], \tag{12}$$

$$\begin{aligned}\langle B|H|\nu_i, T\rangle &\propto U_{li}^* \, e^{i(p_\nu - p_B)x_T} \exp\left[\frac{-(p_\nu - p_B)^2}{2\sigma_p^2}\right] \\ &\propto U_{li}^* \, e^{i(E_\nu - p_B)x_T} e^{-i(m_i^2/2E_\nu)x_T} \\ &\quad \times \exp\left[\frac{-(p_\nu - p_B)^2}{2\sigma_p^2}\right],\end{aligned} \tag{13}$$

where $\sigma_p = 1/\sigma_x$. In the last expression we note that appearance of the factor $e^{-i(m_i^2/2E_\nu)x_T}$ that will lead to oscillations.

As can be seen, the matrix elements serve to constrain the momentum of ingoing and outgoing particles in the decay and absorption. Since neither the source nor target has definite momentum, momentum conservation is only enforced to a precision σ_p.

Before evaluating the amplitude c_{AB} given by Eq. (7), we can anticipate that oscillations will result only if both neutrinos ($i = 1, 2$) contribute to c_{AB}. Whether or not this is possible can be determined from elementary considerations. In Eq. (7), the two time integrals of the complex exponentials rapidly oscillate with time unless the arguments of the exponentials are near zero in which case the amplitudes grow with time. This effectively enforces energy conservation in the decay of S and in the neutrino interaction with T. Thus, the sum on neutrino momentum in Eq. (7) is dominated by states where the intermediate neutrino has an energy given by

$$E_\nu = E_S - E_A = E_B - E_T. \tag{14}$$

The momentum of the dominating states depends on the neutrino mass:

$$p_\nu = E_\nu - m_i^2/2E_\nu. \tag{15}$$

As said before, the matrix elements enforce momentum conservation in decay and interaction to a precision σ_p. If momentum were exactly conserved, only one neutrino could contribute to c_{AB} since if momentum is conserved for one neutrino it cannot be for the other. However, we only require approximate momentum conservation which is possible for both neutrinos, according to Eq. (15), only if

$$|m_1^2 - m_2^2|/2E_\nu \ < \ \sigma_p. \tag{16}$$

If this condition is fulfilled, the two neutrinos will have momenta differing by less than the uncertainty of the momentum of the initial state so both can contribute to c_{AB}.

Evaluation of Eq. (7) confirms this for $t > x_T$:

$$\begin{aligned}c_{AB}(t) &\propto \delta_t(E_B + E_A - E_T - E_S) \; e^{i(E_B - E_T - p_B)x_T} \\ &\quad \times \sum_{i=1}^{2} U_{il}^* U_{ie} e^{-im_i^2 x_T/2E_\nu} \; f(m_i),\end{aligned} \tag{17}$$

where the subscript t reminds us that the δ function actually has a width $1/t$ and where

$$\begin{aligned}f(m_i) \; &= \exp\left[\frac{-(E_\nu - m_i^2/2E_\nu - p_B)^2}{2\sigma_p^2}\right] \\ &\quad \times \exp\left[\frac{-(E_\nu - m_i^2/2E_\nu + p_A)^2}{2\sigma_p^2}\right].\end{aligned} \tag{18}$$

The energy E_ν is to be understood as the function of the initial-state and final-state energies given by Eq. (14).

Squaring the amplitude we get the transition probability

$$\begin{aligned}|c_{AB}(t)|^2 \; &\propto t\delta(E_B + E_A - E_T - E_S) \\ &\quad \times \left|\sum_{i=1}^{2} U_{il}^* U_{ie} e^{-im_i^2 x_T/2E_\nu} \; f(m_i)\right|^2,\end{aligned} \tag{19}$$

where we use the relation $[\delta_t(E)]^2 = t\delta(E)$ in the limit $t \to \infty$. The only final states that can be populated are those that conserve energy and those that have total momentum (sets A and B) near zero.

Apart from the factor $f(m_i)$, the sum in Eq. (19) is just the standard result of Eq. (4). (The momentum p has also been replaced by E_ν.) The normal neutrino oscillation formula, is obtained only if $f(m_1) \sim f(m_2)$ so that the factor $f(m_i)$ can be taken out of the sum. Inspection of Eq. (18) shows that this is possible only if the condition defined by inequality (16) is respected.

Condition (16) is also equivalent to the requirement that the size of the source and target wave functions be smaller than the oscillation length given by Eq. (5). This means that the uncertainty in the distance between source and target must be less than one oscillation length to see the interference. This intuitively appealing conclusion was reached in Ref. [3] by using the uncertainty principle.

Condition (16) means that oscillations cannot be observed for large neutrino mass differences. For a radioactive nucleus contained in an atomic lattice, the nuclear position is uncertain to somewhat less than one interatomic distance, $\sigma_x \sim 1$ Å. In this case, neutrino oscillations cannot be observed if $|m_1^2 - m_2^2|$ is greater than 4×10^9 eV$^2(E_\nu/1$ MeV). If the lighter neutrino has negligible mass, this corresponds to the heavier neutrino having a mass of about 60 keV $\sqrt{E_\nu/1 \text{ MeV}}$. This is one reason why it is more difficult to observe oscillations of charged leptons than oscillations of neutrinos. (Charged lepton oscillations would exchange the roles of the charged leptons and neutrinos in Fig. 1. The other reason that such oscillations would be difficult to observe is that charged particles may leave tracks that betray their identity and destroy the interference.)

Summarizing, neutrino oscillations occur because there are two intermediate states that can lead to the same final state. In the case that we have considered, this is possible because the momentum of the initial state is uncertain. Only energy conservation in decay and absorption is required and this can be done with either of the two neutrinos.

It is interesting to ask what happens if we let our "final states" continue to develop in time according to the Schrödinger equation. The question is important since it can be argued that it is not the final-state particles that are observed but rather the "tracks" that they create as they pass through the medium containing the source and target. These tracks would consist of atomic excitations, lattice defects, bubbles, and the like. In a calculation [6] very similar to the one we performed above, it can be shown that each of our final momentum eigenstates will generate states of the medium containing tracks that are parallel to the momentum and that pass through the positions of the source or target. This correspondence between our final states and the subsequent states of the medium "justifies" stopping the time developement of the state vector with the momentum eigenstates. Of course, the states of the medium may also correlate well with other particle final states, e.g., wave packets of approximately defined momenta. The use of such final states is convenient if the states of the media or detector are metastable allowing one to measure the time of the interaction.

III. THREE WAYS TO NOT OSCILLATE

Before applying the method of the previous section to other situations, we discuss three ways of destroying the oscillation pattern. The first two concern averaging over final states. Each final state has its own oscillation length determined by the energy of the neutrino that dominates the sum over intermediate states. Therefore, the experimentally necessary procedure of averaging over a group of final states may average to zero the oscillating term of Eq. (4). If one averages over an interval of E_B (or E_A) equal to ΔE, it is simple to show that the oscillation pattern starts to be washed out after $E_\nu/\Delta E$ oscillations ($E_\nu = E_B - E_T$). In the case of a "monochromatic" neutrino beam where set A consists of only one particle, the energy spread of the whole neutrino spectrum will be $\Delta E = \sigma_p$. For $\sigma_p = 1/1$ Å$=0.002$ MeV, the total cross section for a 1 MeV neutrino beam will start to damp after about 500 periods.

Effective averaging over momentum states also occurs if one uses time of flight to determine which of the two neutrinos was emitted and absorbed. To measure the time of emission or absorption, one must use detectors whose states correlate with wave packets of the final-state particles and not momentum eigenstates. The oscillation pattern can then be calculated with our method by taking sets A and B to consist of particles in wave packets rather than momentum eigenstates. We denote these states by $A(t_A)$ and $B(t_B)$ for packets that overlap the source or target at times t_A or t_B. The amplitude is the same as Eq. (7) except that the time integrals cover only the period when the wave packets overlap the source or target:

$$c_{A(t_A)B(t_B)}(t) = -\sum_{i=1}^{2}\sum_{p_\nu}\int_{t_B}^{t_B+\Delta t} dt'' \langle B|H|\nu_i, T\rangle e^{i(E_B-E_T-E_\nu)t''}\int_{t_A}^{t_A+\Delta t} dt' \langle A, \nu_i|H|S\rangle e^{i(E_\nu+E_A-E_S)t'} c_{ST}(0), \quad (20)$$

where Δt is the width of the wave packet. Application of the method of stationary phase shows that the $c_{A(t_A)B(t_B)}$ is negligible unless $t_B - t_A$ corresponds nearly to the time of flight for a neutrino of mass m_i:

$$t_B - t_A = x_T\left(1 + \frac{m_i^2}{2E_\nu^2}\right). \quad (21)$$

If the difference in flight time for the two neutrinos is greater than the integration interval Δt the two neutrinos cannot interfere because they produce distinct final states. Again, it is simple to show that the oscillation pattern starts to be washed out after $E_\nu \Delta t$ oscillations ($E_\nu = E_B - E_T$). Taking $\Delta t = 1$ ns gives a number of observable oscillations equal to $1.5 \times 10^{12}(E\nu/1$ MeV).

Real detectors will give information on both the momentum and the time of the event. Experimenters must then group events in "bins" of time and momentum. For practical detectors, it appears that the grouping in momentum will be most important in determining the number of observable oscillations.

A third way of destroying the oscillations is to heat the neutrino source above absolute zero. If this is done, the initial state of our source will be an incoherent thermal mixture of the quantum states of S described by a density matrix. By itself, this only increases the number of allowed final states since energy conservation will connect each initial state with final states of the same energy. However, the interactions of S with its environment give each state of S a mean lifetime τ. This fact will limit the number of observable oscillations to $E_\nu \tau$. This can be understood by using, as intermediate states, neutrino wave packets of width τ. Successive wave packets will have random phases with respect to each other because of the transitions between states of S introduce random phase changes in c_{ST}. As soon as the two neutrino wave packets emitted at the same time cease to overlap, interference will cease because of the random phase. This point was originally made by Nussinov [7] in the context of solar neutrinos. For solar neutrinos the lifetimes are very short, $\tau \sim 3 \times 10^{-17}$sec and the number of observable oscillations is equal to $\sim 4.6 \times 10^4 (E\nu/1 \text{ MeV})$.

IV. A PION BEAM

We can now apply our approach to the case where the initial state of the decaying particle (pion) is a free wave packet consisting of a superposition of pion momentum eigenstates:

$$|S(t=0)\rangle \;=\; \sum_{p_\pi} c_{p_\pi}(t=0)\; |p_\pi\rangle. \tag{22}$$

The spatial extent of the packet will be $\sigma_{x_\pi} = 1/\sigma_{p_\pi}$ where σ_{p_π} is the momentum dispersion of c_{p_π}. The wave packet can be allowed to propagate across a "decay" region of length $l_{\rm decay}$. It is natural to take $l_{\rm decay} > \sigma_{x_\pi}$. The pion will be destroyed after the decay region so it couples to the neutrino states only for a time $0 < t < l_{\rm decay}/c$. This limits the integration time in Eq. (7) so we have

$$c_{AB}(t) = -\sum_{i=1}^{2}\sum_{p_\nu}\sum_{p_\pi}\int_0^t dt''\langle B|H|\nu_i,T\rangle e^{i(E_B-E_T-E_\nu)t''}\int_0^{l_{\rm decay}} dt'\langle A,\nu_i|H|\pi\rangle e^{i(E_\nu+E_A-E_\pi)t'} c_{p_\pi}(0), \tag{23}$$

where we now sum over the neutrino states and the pion states comprising the original wave packet. The set A now consists of one muon of momentum p_μ. The matrix element connecting the pion momentum eigenstate to the neutrino is again proportional to the wave function overlap:

$$\langle A,\nu_i|H|\pi\rangle \;\propto\; U_{\mu i}\delta(p_\pi - p_\mu - p_\nu). \tag{24}$$

As before, the two time integrations enforce energy conservation in the interaction and in the decay. However, in the decay, it is only enforced to a precision $1/l_{\rm decay}$. The matrix element enforces momentum conservation in the neutrino interaction to a precision σ_p. The decay matrix element, Eq. (24), simply picks out one pion momentum component for each of the neutrinos. With all these constraints, one can again derive the conditions that both neutrinos contribute to the final-state amplitude. The first, coming from approximate momentum conservation in the neutrino interaction, is the same as condition (16), i.e., that the oscillation length be greater than the size of the target wave function. The second comes from approximate energy conservation in the decay:

$$|m_1^2 - m_2^2|/2E_\nu \;<\; 1/l_{\rm decay} \tag{25}$$

or that the oscillation length be much greater than the length of the decay region.

Since $l_{\rm decay}$ is generally a macroscopic length, the second condition is the most stringent. However, it exists only because we have taken momentum eigenstates as our final states. This effectively averages over decay positions, something that is necessary experimentally if the muon is not detected. However, if atoms are placed near the decay region, they can be excited by the passage of the muon and effectively locate the decay point [6]. The condition for oscillations, Eq. (25), is then relaxed accordingly. The diagram of Fig. 1 should be modified to include legs attached to the muon that represent the initial and final states of the atoms.

We note that Eq. (25) does not concern the width of the pion wave packet (as long as $\sigma_{x_\pi} < l_{\rm decay}$). This is not surprising since the different momentum components of the pion wave packet lead to different final states and cannot interfere. It is therefore not possible to distinguish the case of a beam of pions where all particles have wave functions of mean momentum p_π and dispersion σ_{p_π} from the case of a beam of pions, each with a well-defined momentum taken randomly from a distribution of the same mean momentum and of dispersion σ_{p_π}. This difficulty in distinguishing coherent wave packets from incoherent mixtures has been noted many times [8].

While the above results seem reasonable, the use of pion wave packets is unsatisfying because the size of the packet is not determined by the calculation. (This was the problem that we wanted to avoid with the neutrino.) Indeed, it is never strictly correct to say that the state of a pion is a wave packet since, being created in a momentum conserving interaction, its state is correlated with the other particles created in the interaction.

These two problems can be addressed by going upstream one step and considering an initial proton inci-

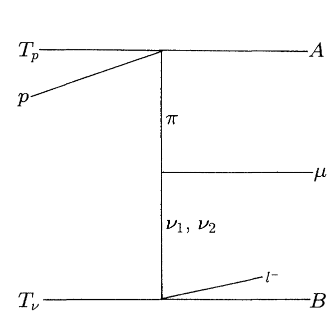

FIG. 2. The diagram for neutrino oscillations due to a proton, a proton target T_P, and a neutrino target T_ν. Set B contains one charged lepton l^-.

dent on a target nucleus in a potential well centered at $x = 0$. The pion is created in the proton-target interaction and then decays to a muon and a neutrino that subsequently interacts. The diagram for this process is shown in Fig. 2. The initial state consists of the two targets, T_p and T_ν, and the proton. The final state consists of sets A and B of particles and one muon (from the pion decay). Since the pion is charged and will interact with the medium containing the target nucleus, the definition of the state A must include the state of the medium. The intermediate states to be summed over include the pion and then the neutrino. Since the initial state has a momentum uncertainty associated with the proton target, the momentum width of the proton wave packet is irrelevant. (Different momentum components will lead to different final states.) We can then take the proton to be of definite momentum and then average over the beam spectrum later on.

Now the amplitude for the system to evolve to the final state is third order. There are two possibilities for treating the intermediate state pions. If one wishes to give the pions a finite decay region of length $l_{\rm decay}$, the pion states can be taken to be wave packets that couple to the neutrino states only as long as the pion is in the decay region. This analysis leads to the same conclusion as before, that oscillations are observable if $l_{\rm decay}$ and the sizes of the two target wave functions are all smaller than the oscillation length.

A more amusing possibility is to let the decay region be defined by the pion lifetime. This situation can be treated in the same way as resonance fluorescence [9] with the unstable pion replacing the unstable excited state of an atom. The amplitude for the system to be in the final state is

$$c_{AB\mu}(t) = -\sum_{i=1}^{2}\sum_{p_\nu}\sum_{p_\pi}\int_0^t dt''\langle B|H|\nu_i,T_\nu\rangle e^{i(E_B-E_{T_\nu}-E_\nu)t''}\int_0^{t''} dt'\langle \mu,\nu_i|H|\pi\rangle e^{i(E_\nu+E_\mu-E_\pi)t'}c_{A\pi}(t'). \tag{26}$$

Because of the pion lifetime, the amplitude $c_{A\pi}(t')$ cannot be computed in perturbation theory [as in Eq. (6)]. We can take the lifetime into account by solving the (phenomenological) Schrödinger equation [9]

$$i\dot{c}_{A\pi} = \langle A,\pi|H|p,T_p\rangle e^{i(E_A+E_\pi-E_{T_p}-E_p)t}c_{T_pp}(t) - \frac{i\Gamma}{2}c_{A\pi}, \tag{27}$$

where Γ is the pion decay rate in the rest frame of the target. The first term takes into account transitions from the initial state into the intermediate state while the second term takes into account the decay of the pion. The steady-state solution to this equation is

$$c_{A\pi}(t) = -\frac{\langle A,\pi|H|p,T_p\rangle e^{i(E_A+E_\pi-E_{T_p}-E_p)t}}{E_A+E_\pi-E_{T_p}-E_p-i\Gamma/2}. \tag{28}$$

The Breit-Wigner function means that energy will be conserved in the proton interaction only to a precision Γ. The expression for $c_{A\pi}(t)$ can now be substituted into Eq. (26).

We have three matrix elements to evaluate. As before, the weak matrix elements are proportional to the overlap of the wave functions:

$$\langle \mu,\nu_i|H|\pi\rangle \propto U_{\mu i}\,\delta(p_\nu+p_\mu-p_\pi), \tag{29}$$

$$\langle B|H|\nu_i,T_\nu\rangle \propto U_{li}^*\,e^{i(E_\nu-p_B)x_T}e^{-i(m_i^2/2E_\nu)x_T}\times\exp\left[\frac{-(p_\nu-p_B)^2}{2\sigma_p^2}\right]. \tag{30}$$

The matrix element for the proton interaction will be more complicated but we assume that it will enforce momentum conservation to a precision no better than $1/\Delta x$ where Δx is the size of the target. We write symbolically

$$\langle A,\pi|H|T_p,p\rangle \propto F((p_\pi+p_A-p_p)\Delta x), \tag{31}$$

where F is a function peaked at zero and of width greater than unity.

Substituting these expressions into Eq. (26) we perform the various sums and integrations. As before, the time integrals fix the neutrino energy:

$$E_\nu = E_B - E_{T_\nu}. \tag{32}$$

The pion decay matrix element fixes the pion momentum (and energy):

$$p_\pi = p_\nu + p_\mu = E_B - E_{T_\nu} - \frac{m_i^2}{2E_\nu} + p_\mu. \tag{33}$$

We are left for the expression for the final-state amplitude:

$$c_{AB\mu}(t) \propto \delta_t(E_B+E_\mu+E_A-E_{T_p}-E_p-E_{T_\nu})\times\sum_{i=1}^{2}U_{il}^*U_{ie}e^{-im_i^2x_T/2E_\nu}\,g(m_i), \tag{34}$$

where

$$g(m_i) = \frac{\exp\left[\frac{-(E_\nu-m_i^2/2E_\nu-p_B)^2}{2\sigma_p^2}\right]F((p_\pi+p_A-p_p)\Delta x)}{E_A+E_\pi-E_{T_p}-E_p-i\Gamma/2} \tag{35}$$

Once again, E_ν, p_π, and E_π are to be considered the functions of m_i and the final-state energies and momenta given by Eqs. (32) and (33).

The normal oscillation formula is found only if the function $g(m_i)$ can be taken out of the sum. Approximate momentum conservation at the neutrino target gives the same condition as before:

$$|m_1^2-m_2^2|/2E_\nu < \sigma_p \tag{36}$$

or that the oscillation length be much greater than the size of the target wave function. Similarly, approximate momentum conservation at the proton target requires that the oscillation length be much greater than the size of the proton target. The final condition is due to approximate energy conservation in the proton interaction and comes from the denominator of Eq. (35). It depends on the energy and decay rate of the pion. For relativistic pions we have

$$E_\pi = E_B - E_{T_\nu} - \frac{m_i^2}{2E_\nu} + \frac{m_\pi^2}{2E_\pi} + p_\mu. \tag{37}$$

In this case the denominator can be considered to be independent of neutrino species if

$$|m_1^2 - m_2^2|/2E_\nu < \Gamma , \tag{38}$$

i.e., that the oscillation length be greater than the decay length, c/Γ. This confirms our expectation.

If the pions that dominate the sum are nonrelativistic one finds basically the same condition as (38). In this case the condition requires that the oscillation time, l_v/c, must be much greater than the pion lifetime or that the difference in momenta of the two neutrinos be much less than the pion width.

The third-order diagram of Fig. 2 can be generalized to include any number of interactions of the intermediate-state pion with atoms in the medium containing the target or in the decay volume. This would allow a more realistic quantum treatment of the process. Of course this quickly becomes prohibitively complicated and would not be expected to change drastically the conclusions about the observability of oscillations.

V. SOLAR NEUTRINOS

Solar neutrinos do not fit well into either of the two categories discussed above. In the Sun, the state of a radioactive nucleus like ^{8}B approximates a thermal mixture of free particle states subjected to a slowly varying gravitational potential. Apart from the requirement that the free particle states be confined in the Sun, we have a certain freedom to choose the base states. Since there is a temperature gradient, it is convenient to use localized states so that thermal equilibrium can be imposed on the density matrix. The problem is further complicated by the high rate of particle scatterings that leads to effective lifetimes of order 10^{-18} to 10^{-17} sec. Some of the difficulaties involved in treating this problem were discussed by Loeb [10] from an essentially classical point of view.

VI. CONCLUSIONS

We have presented a systematic method of treating neutrino oscillations with standard quantum mechanical techniques. It treats the whole process of emission, propagation, and absorption in a consistent way. The method can also be applied to phenomena like $K_S - K_L$ interference. In certain simple situations where transition rates can be easily calculated, it is worth the extra effort involved in treating the whole process.

Having done this we can ask how this treatment relates to the two-state and wave-packet treatments. These approaches single out two sets of intermediate states from the sum of Eq. (7). If these two sets include the states dominating the sum, the traditional treatments will then give the right answer when calculating neutrino absorption rates. This may or may not be the case depending on the initial and final states of the source and target. In the cases treated here, the dominating neutrinos have well-defined energy so the two-state treatment with neutrinos of well-defined momentum will not give the right answer if it is taken one step further to calculate the absorption probability. The wave-packet treatment takes a larger set of states so it is obviously more robust. However, it does not give the amplitude for the emission or absorption of the neutrinos in question so the criterion for oscillations must remain rather vague.

The approach presented here is much "safer" since it goes upstream in time to states of the decaying particle that can be prepared experimentally. One simply has to turn the "Schrödinger crank" until states that can be correlated with detector responses are reached.

Calculations based on this approach show clearly that oscillation phenomena occur because the finite sizes of the initial-state wave functions impose momentum conservation only approximately. This allows intermediate states for both neutrinos that would not be allowed if both energy and momentum were conserved. The arbitrariness of the wave-packet approach can be eliminated if the size of the initial bound-state wave functions is determined by atomic physics.

ACKNOWLEDGMENTS

I thank Jacques Bouchez, Gabriel Chardin, and Michel Spiro for many conversations concerning quantum mechanics and particle oscillations. I also thank Boris Kayser and Georg Raffelt for some very stimulating discussions, and Alain Milsztajn, Serguey Petcov, and Joe Weneser for helpful comments on the manuscript. After completion of this work I learned of a paper [11] where a calculation similar to the ones here was performed within the framework of quantum field theory rather than elementary quantum mechanics.

[1] B. Pontecorvo, Zh. Eksp. Teor. Fiz. **53**, 1717 (1967) [Sov. Phys. JETP **26**, 984 (1968)].

[2] For a review of oscillation phenomenology and recent experimental results, see L. Oberauer and F. von Feilitzsch, Rep. Prog. Phys. **55**, 1093 (1992).

[3] Boris Kayser, Phys. Rev. D **24**, 110 (1981).

[4] See, for example, J. J. Sakurai, *Advanced Quantum Mechanics* (Addison-Wesley, New York, 1967), pp. 39 and

40.

[5] That the neutrino can be considered a virtual particle has been observed by J. Rich, D. Lloyd Owen, and M. Spiro, Phys. Rep. **151**, 267 (1987); Boris Kayser, in *Neutrino 90*, Proceedings of the 14th International Conference on Neutrino Physics and Astrophysics, Geneva, Switzerland, 1990, edited by J. Panam and K. Winter [Nucl. Phys. B (Proc. Suppl.) **19**, 177 (1991)]; C. Giunti, C. W. Kim, and U. W. Lee, Phys. Rev. D **45**, 2414 (1992). The same can also be said about photons, see Sakurai [4], p. 256.

[6] Neville F. Mott, Proc. R. Soc. London **A126**, 79 (1929); reprinted in *Quantum Theory and Measurement*, edited by J. A. Wheeler and W. H. Zurek (Princeton University Press, Princeton, New Jersey, 1983); L. I. Schiff, *Quantum Mechanics*, 3rd ed. (McGraw-Hill, New York, 1968), pp. 335–339.

[7] S. Nussinov, Phys. Lett. **63B**, 201 (1976).

[8] See, for example, R. H. Dicke and J. P. Witke, *Introduction to Quantum Mechanics* (Addison-Wesley, New York, 1960), p. 336; G. Comsa, Phys. Rev. Lett. **51**, 1105 (1983); H. Kaiser, S. A. Werner, and E. A. George, *ibid.* **51**, 1106 (1983).

[9] Sakurai [4], pp. 53 and 54.

[10] Abraham Loeb, Phys. Rev. D **39**, 1009 (1989).

[11] C. Giunti, C. W. Kim, J. A. Lee, and U. W. Lee, preceding paper, Phys. Rev. D **48**, 4310 (1993).