QCD and the relativistic flux tube with fermionic ends

M. G. Olsson and Ken Williams
Physics Department, University of Wisconsin, Madison, Wisconsin 53705
(Received 22 December 1992)

We formulate a quantized relativistic flux tube model for mesons, in which the flux tube terminates on fermionic quarks of arbitrary mass. A semirelativistic reduction shows that all relativistic corrections correspond to the expectations of QCD.

PACS number(s): 12.40.Aa, 11.10.Qr, 11.10.St, 12.38.Aw

I. INTRODUCTION

The search for a realistic model of hadronic matter has evolved rapidly in the last few years. It now seems clear that the standard potential model does not correctly account for the observed facts in heavy quarkonia [1] or in the light states [2], and that the origin of the difficultly is with the long-range Lorentz scalar potential which seems to be inconsistent with QCD [3,4]. Although a semirelativistic potential can be formulated [5] which does, through cleverly devised retardation corrections, agree with QCD, we believe that it is more desirable to look to a more fundamental approach.

In a classical analysis [4] we have demonstrated that for large angular momenta the leading relativistic QCD corrections can be interpreted as the angular momentum and angular energy of a rotating flux tube. Thus, the simplest interpretation of these relativistic corrections implies that it is crucial to consider the momentum as well as the energy of the interacting field.

The flux tube model in a nonrelativistic form was proposed by Isgur and Paton [6] as a unifying concept for quark spectroscopy, hybrid states, and glueballs. The concept is closely related to QCD stringlike models intended to bring together [7] the old QCD string [8] and the quark model.

A relativistic string and/or flux tube model was explored by Ida [9] and LaCourse and Olsson [10]. In the latter, a quantized version was developed which for massless quarks yield a spectroscopy consisting of a straight leading Regge trajectory with parallel evenly spaced daughter trajectories. Furthermore, the meson states occur in energy-degenerate groups or towers.

In this paper we consider the construction of a relativistic flux tube model in which a straight flux tube terminates on fermion quarks. Although the model is stated in a form valid for all quark velocities, we will concentrate here on heavy quarks and consider relativistic corrections. This is an important step since considerable effort has gone into the development of QCD implications for spin-dependent [11] and spin-independent relativistic corrections [3]. For the spin dependence, our result is that, as expected [12], there are no long-range spin correlations and the tube only contributes to the kinematic Thomas precession type of spin-orbit interaction. For the spin-independent corrections we obtain

$$H_{\rm SI} = \frac{a}{36r}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}+\frac{8}{m_1 m_2}\right) - \frac{a}{6r}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}-\frac{1}{m_1 m_2}\right)L^2\,, \qquad (1)$$

where a is the tube energy per unit length or the string tension. The L^2 coefficient in (1) is our previous classical result [4] due to the rotating tube. The first term in (1) arises from spin and commutation effects in the flux tube model and agrees exactly with the expectation of QCD [3, 4].

In the spirit of the Cornell model, we may augment the flux tube with a singular short-range interaction which reproduces the well-known Lorentz vector-spin and spin-independent relativistic corrections.

In Sec. II we discuss the general formulation of the relativistic flux tube model with spin-$\frac{1}{2}$ quarks at the ends. The reduction of the wave equation to exhibit the relativistic corrections is worked out in Sec. III and our conclusions are found in Sec. IV. Some details of the symmetrization of the Hamiltonian are given in the Appendix.

II. THE FLUX TUBE MODEL WITH SPIN

Encouraged by the classical results [4] for the relativistic corrections at large angular momentum, we consider a more realistic situation. In this section we formulate a quantized, relativistic straight flux tube model with spin-$\frac{1}{2}$ quarks. We do this through the Dirac Hamiltonian, adding the required tube "field" momentum and energy while preserving Lorentz covariance. With this established we go to the two-particle Salpeter equation which will be treated by standard reduction techniques in Sec. III.

Let us begin with the free Dirac equation in position space,

$$i\frac{\partial}{\partial t}\psi(\mathbf{r},t) = [\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m]\psi(\mathbf{r},t) = H\psi(\mathbf{r},t)\,. \qquad (2)$$

As we have previously emphasized, the flux tube carries both energy and momentum. We introduce the tube by the covariant transformation

$$p^\mu \rightarrow p^\mu - p_t^\mu \,, \tag{3}$$

which means that the new canonical momentum $\mathbf{p}$ now includes the momentum associated with the flux tube. The above transformation (3) is a direct analog to the standard "minimal substitution" used to introduce the electromagnetic field.

The Dirac equation (2) becomes

$$\begin{aligned} i\frac{\partial\psi}{\partial t} &= [\boldsymbol{\alpha}\cdot(-i\boldsymbol{\nabla} - \mathbf{p}_t) + \beta m + H_t]\,\psi \,, \\ &= [\boldsymbol{\alpha}\cdot(-i\boldsymbol{\nabla}) + \beta m]\,\psi + A(\mathbf{r})\psi \,, \end{aligned} \tag{4}$$

where

$$\begin{aligned} A &= -\boldsymbol{\alpha}\cdot\mathbf{p}_t + H_t \,, \\ p_t^\mu &= (H_t, \mathbf{p}_t) \,. \end{aligned} \tag{5}$$

The Fourier transform of both sides of (4) leads to the Hamiltonian for the Dirac particle and flux tube,

$$\begin{aligned} H(\mathbf{p})\phi(\mathbf{p}) &= [\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m]\phi(\mathbf{p}) + \int d\mathbf{k}\,\tilde{A}(\mathbf{k})\phi(\mathbf{p}-\mathbf{k}) \\ &= H_0\phi(\mathbf{p}) + \int d\mathbf{k}\,\tilde{A}(-\mathbf{k})\phi(\mathbf{p}+\mathbf{k}) \,, \end{aligned} \tag{6}$$

where $\tilde{A}(\mathbf{p})$ is the Fourier transform of $A(\mathbf{r})$.

We can now use the above Hamiltonian to describe a two-fermion system. We go to the two-particle Salpeter equation [13, 14], although we may well have chosen the two-body Dirac equation [15], for example. The Salpeter equation is

$$\begin{aligned} &[M - H_{01}(\mathbf{p}_1) - H_{02}(\mathbf{p}_2)]\chi(\mathbf{p}_1, \mathbf{p}_2) \\ &\qquad = \int d\mathbf{k}\, G(\mathbf{k})\phi(\mathbf{p}_1 - \mathbf{k}, \mathbf{p}_2 + \mathbf{k}) \,. \end{aligned} \tag{7}$$

Here $G(\mathbf{k})$ is an interaction kernel for a potential type interaction. For example, $G(\mathbf{k}) \sim k^{-2}$ for a Coulomb-type interaction. In the above Salpeter equation (7) $\phi(\mathbf{p}_1, \mathbf{p}_2)$ is the Fourier transformation of $\psi(\mathbf{r}_1, \mathbf{r}_2)$ and

$$\begin{aligned} H_{0i} &= m_i\beta + \mathbf{p}_i\cdot\boldsymbol{\alpha}, \qquad i = 1, 2 \,, \\ \phi(\mathbf{p}_1, \mathbf{p}_2) &= \left[\Lambda_+^{(1)}\Lambda_+^{(2)} - \Lambda_-^{(1)}\Lambda_-^{(2)}\right]\chi(\mathbf{p}_1, \mathbf{p}_2) \,, \\ \Lambda_\pm^{(i)} &= (E_{0i} \pm H_{0i})/(2E_{0i}) \,, \\ E_{0i} &= +\sqrt{m_i^2 + p_i^2} \,, \end{aligned} \tag{8}$$

and M is the state mass.

Introducing the flux tube by covariant transformation as in (6), the Salpeter equation (7) becomes

$$[M - H_{01}(\mathbf{p}_1) - H_{02}(\mathbf{p}_2)]\,\chi(\mathbf{p}_1, \mathbf{p}_2) \int d\mathbf{k} [G(\mathbf{k})\phi(\mathbf{p}_1 - \mathbf{k}, \mathbf{p}_2 + \mathbf{k}) + \tilde{A}_1(-\mathbf{k})\chi(\mathbf{p}_1 + \mathbf{k}, \mathbf{p}_2) + \tilde{A}_2(-\mathbf{k})\chi(\mathbf{p}_1, \mathbf{p}_2 + \mathbf{k})], \tag{9}$$

with $\tilde{A}_i$ the Fourier transform of $A(\mathbf{r}_i)$.

Finally, we may project out the positive energy solutions [14] to avoid the Brown-Ravenhall problem [16],

$$\begin{aligned} [M - H_{01} - H_{02}]\,\phi_{++}(\mathbf{p}_1, \mathbf{p}_2) &= \int d\mathbf{k}\, G(\mathbf{k})\phi_{++}(\mathbf{p}_1 - \mathbf{k}, \mathbf{p}_2 + \mathbf{k}) \\ &\quad + \Lambda_+^{(1)}\Lambda_+^{(2)} \int d\mathbf{k} \left[\tilde{A}_1(-\mathbf{k})\chi(\mathbf{p}_1 + \mathbf{k}, \mathbf{p}_2) + \tilde{A}_2(-\mathbf{k})\chi(\mathbf{p}_1, \mathbf{p}_2 + \mathbf{k})\right]. \end{aligned} \tag{10}$$

In the above $G(\mathbf{k})$ might represent the short-range, Lorentz vector, interaction whose semirelativistic reduction is the well-known [14] Breit-Fermi terms. Since the flux tube only contributes to $\tilde{A}$, we will suppress the potential interaction $G(\mathbf{k})$ in the next section where we consider the reduction of (10).

III. RELATIVISTIC CORRECTIONS OF THE FLUX TUBE

By the standard Pauli reduction approximation [14] the Salpeter equation becomes

$$\begin{aligned} H &= H_0 + H_1 + H_2 + H_3 + H_4 \,, \\ H_0 &= m_1 + m_2 + \frac{1}{2}\left(\frac{p_1^2}{m_1} + \frac{p_2^2}{m_2}\right) + H_t(r_1) + H_t(r_2) \,, \\ H_1 &= -\frac{1}{8}\left(\frac{p_1^4}{m_1^3} + \frac{p_2^4}{m_2^3}\right) \,, \\ H_2 &= -\frac{1}{2}\left[\frac{\boldsymbol{\epsilon}_1\times\mathbf{p}_1}{m_1^2}\cdot\mathbf{S}_1 + \frac{\boldsymbol{\epsilon}_2\times\mathbf{p}_2}{m_2^2}\cdot\mathbf{S}_2\right] \,, \\ H_3 &= -\frac{i}{4}\left[\frac{\mathbf{p}_1\cdot\boldsymbol{\epsilon}_1}{m_1^2} + \frac{\mathbf{p}_2\cdot\boldsymbol{\epsilon}_2}{m_2^2}\right] \\ H_4 &= -\frac{\mathbf{h}_1\cdot\mathbf{S}_1}{m_1} - \frac{\mathbf{h}_2\cdot\mathbf{S}_2}{m_2} - \frac{\mathbf{p}_{t1}\cdot\mathbf{p}_1}{m_1} - \frac{\mathbf{p}_{t2}\cdot\mathbf{p}_2}{m_2} \,, \end{aligned} \tag{11}$$

where we define

$$\epsilon_i = -\boldsymbol{\nabla}_i H_t(r_i) \,, \qquad \mathbf{h}_i = \boldsymbol{\nabla}_i \times \mathbf{p}_t(r_i) \,. \tag{12}$$

In previous work [4, 17] the energy and momentum of the ith flux tube segment has been calculated as

$$H_t(r_i) = a r_i \frac{\arcsin v_{\perp i}}{v_{\perp i}} \,, \qquad \mathbf{p}_t(r_i) = \frac{\mathbf{L}_i}{r_i} = \frac{a r_i}{2 v_{\perp i}} \left(\frac{\arcsin v_{\perp i}}{v_{\perp i}} - \frac{1}{\gamma_{\perp i}} \right) \hat{\mathbf{v}}_{\perp i} \,. \tag{13}$$

Expanding in $v_{\perp i} \ll 1$ we obtain

$$H_t(r_i) \simeq a r_i + \frac{1}{6} a r_i v_{\perp i}^2 + \cdots \,, \qquad \mathbf{p}_t(r_i) \simeq \frac{1}{3} a r_i v_{\perp i} \hat{\mathbf{v}}_{\perp i} + \cdots \,, \tag{14}$$

where the $+\cdots$ signifies higher-order terms not required for the present work. In the heavy-quark limit most of the momentum is carried by the quark from which it follows that to leading order the total angular momentum of the ith tube-quark segment is

$$\mathbf{L}_i \simeq m_i r_i v_{\perp i} \hat{\mathbf{z}} \,, \tag{15}$$

and from (15) we then find

$$H_t(r_i) \simeq a r_i + \frac{a L_i^2}{6 m_i^2 r_i} + \cdots \,, \tag{16}$$

$$\mathbf{p}_t(r_i) \simeq \frac{a}{3 m_i r_i} \mathbf{L}_i \times \mathbf{r}_i + \cdots \,. \tag{17}$$

In terms of quark c.m. coordinates

$$\begin{aligned} \mathbf{p} &\equiv \mathbf{p}_1 = -\mathbf{p}_2 \,, \\ \mathbf{r} &= \mathbf{r}_1 - \mathbf{r}_2 \,, \\ m_i r_i &= \mu r + \cdots \,, \\ \mu &= \frac{m_1 m_2}{m_1 + m_2} \,, \\ \frac{\boldsymbol{\nabla}_i \cdot \hat{\mathbf{r}}_i}{m_i} &= \frac{\boldsymbol{\nabla}_{\mathbf{r}} \cdot \hat{\mathbf{r}}}{\mu} + \cdots \,, \end{aligned} \tag{18}$$

the quantities appearing in (11) can be evaluated [18] as

$$\begin{aligned} \epsilon_i \times \mathbf{p}_i &\simeq -\frac{a\mathbf{L}}{r} \,, \\ \mathbf{h}_i &\simeq \frac{a\mathbf{L}}{m_i r} \,, \\ \mathbf{p}_t(r_i) \cdot \mathbf{p}_i &\simeq \frac{\mu a L^2}{3 m_i^2 r} \,, \\ \mathbf{p}_i \cdot \epsilon_i &\simeq \frac{ia}{r} \,, \\ H_t(r_1) + H_t(r_2) &\simeq ar + \frac{aL^2}{6r}\left(\frac{1}{m_1^2} + \frac{1}{m_2^2} - \frac{1}{m_1 m_2} \right) \,. \end{aligned} \tag{19}$$

Substitution of (19) into (11) then gives to lowest order

$$\begin{aligned} H_0 &= m_1 + m_2 + \frac{p^2}{2\mu} + ar + \frac{aL^2}{6r}\left(\frac{1}{m_1^2} + \frac{1}{m_2^2} - \frac{1}{m_1 m_2} \right) \,, \\ H_1 &= -\frac{p^4}{8}\left(\frac{1}{m_1^3} + \frac{1}{m_2^3} \right) \,, \\ H_2 &= \frac{a}{2r}\left(\frac{\mathbf{L}\cdot\mathbf{S}_1}{m_1^2} + \frac{\mathbf{L}\cdot\mathbf{S}_2}{m_2^2} \right) \,, \\ H_3 &= \frac{a}{4r}\left(\frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \,, \\ H_4 &= -\frac{a}{r}\left(\frac{\mathbf{L}\cdot\mathbf{S}_1}{m_1^2} + \frac{\mathbf{L}\cdot\mathbf{S}_2}{m_2^2} \right) \\ &\quad -\frac{aL^2}{3r}\left(\frac{1}{m_1^2} + \frac{1}{m_2^2} - \frac{1}{m_1 m_2} \right) \,. \end{aligned} \tag{20}$$

Adding the above together then gives

$$\begin{aligned} H = {} & m_1 + m_2 + \frac{p^2}{2\mu} - \frac{p^4}{8}\left(\frac{1}{m_1^3} + \frac{1}{m_2^3} \right) \\ & -\frac{a}{2r}\left(\frac{\mathbf{L}\cdot\mathbf{S}_1}{m_1^2} + \frac{\mathbf{L}\cdot\mathbf{S}_2}{m_2^2} \right) + ar + \frac{a}{4r}\left(\frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \\ & -\frac{aL^2}{6r}\left(\frac{1}{m_1^2} + \frac{1}{m_2^2} - \frac{1}{m_1 m^2} \right) \,. \end{aligned} \tag{21}$$

A final and critical step must be taken. The Hamiltonian should be totally symmetrized in order that products of coordinates and momenta will be Hermitian. This is relvant only for the last term of (21). As demonstrated in the Appendix, total symmetrization yields

$$\left[\frac{L^2}{r} \right]_{\text{sym}} = \frac{L^2}{r} + \frac{4}{3r} \,, \tag{22}$$

where the L^2 on the right side may now be regarded as a constant when operating on an angular momentum eigenstate. Inserting (22) into (21) then provides our final result:

$$\begin{aligned} H = {} & m_1 + m_2 + \frac{p^2}{2\mu} - \frac{p^4}{8}\left(\frac{1}{m_1^3} + \frac{1}{m_2^3} \right) \\ & + ar - \frac{a}{2r}\left(\frac{\mathbf{L}\cdot\mathbf{S}_1}{m_1^2} + \frac{\mathbf{L}\cdot\mathbf{S}_2}{m_2^2} \right) \\ & + \frac{a}{r}\left[\frac{1}{36}\left(\frac{1}{m_1^2} + \frac{1}{m_2^2} + \frac{8}{m_1 m^2} \right) \right. \\ & \left. \qquad - \frac{1}{6}\left(\frac{1}{m_1^2} + \frac{1}{m_2^2} - \frac{1}{m_1 m^2} \right) L^2 \right] \,. \end{aligned} \tag{23}$$

To the above we may add a short-range Lorentz vector part as previously discussed [4].

IV. CONCLUSIONS

Comparison of our spin-independent long-range relativistic corrections in (23) with the Wilson loop expansion results [3, 4] shows that both terms are correctly accounted for. The L^2 term follows from the classical rotational energy of the flux tube, while the other term has two sources. The L^2 independent term is the sum of H_3, a Darwin-like term originating from the fermionic nature of the quarks, and a commutator term from the symmetrization (22) of the L^2 term. We also note that the effective Hamiltonian (23) contains no hyperfine or tensor interaction and hence no long-range spin-spin correlations. This is an inescapable result of the original flux tube substitution (3) so that only linear spin operators will occur. The resulting spin-orbit interaction of the Thomas type follows directly from the expansion (16) and (17). We note that in our formulation the flux tube has no relation to the so-called "scalar confining potential" other than the spin dependence turns out to be the same. In fact, the substitution (3) treats the long-range interaction as if it were a Lorentz vector.

Led by the attractive features of the spinless flux tube model [4, 10] and the result of the present paper, we have some confidence that a full numerical solution of (10) will lead to an improved description of much of meson spectroscopy.

ACKNOWLEDGMENTS

This work was supported in part by the U.S. Department of Energy under Contract No. DE-AC02-76ER00881 and in part by the University of Wisconsin Research Committee with funds granted by the Wisconsin Alumni Research Foundation.

APPENDIX

We establish here the total symmetrization of the operator $\frac{L^2}{r}$ as stated in (22). To accomplish this we begin by expressing this quantity in terms of coordinates and canonical momenta and symmetrizing the result

$$\begin{aligned}\frac{L^2}{r} &= \mathbf{r}\times\mathbf{p}\,\frac{1}{r}\cdot\mathbf{r}\times\mathbf{p} = \epsilon_{ijk}\epsilon_{imn}\,r_j p_k \frac{1}{r}\, r_m p_n \\ &= \epsilon_{ijk}\epsilon_{imn}\left(p_k \frac{r_j r_m}{r}\, p_n + i\delta_{jk}\frac{r_m}{r}\, p_n\right) , \qquad \text{(A1)} \\ \frac{L^2}{r} &= \epsilon_{ijk}\epsilon_{imn}\left(p_k \frac{r_j r_m}{r}\, p_n\right) ,\end{aligned}$$

where the δ_{jk} term vanishes due to the antisymmetry of ϵ_{ijk}. In the above we have used the commutation relation

$$[p_a, r_b] = -i\,\delta_{ab}\ . \qquad \text{(A2)}$$

Other useful commutation identities following from (A2) are

$$\left[p_a,\ \frac{r_b}{r}\right] = -\frac{i}{r}\left(\delta_{ab} - \frac{r_a r_b}{r^2}\right) , \qquad \text{(A3)}$$

$$\left[p_a,\ \frac{r_b r_c}{r^2}\right] = -\frac{i}{r^2}\left(\delta_{ab} r_c + \delta_{ac} r_b - \frac{2 r_a r_b r_c}{r^2}\right) . \qquad \text{(A4)}$$

From (A3) we see that (A1) is unique in that $\frac{1}{r}L^2 = \frac{L^2}{r} = L^2\frac{1}{r}$.

Total symmetrization of (A1) gives

$$\left[\frac{L^2}{r}\right]_{\rm sym} \equiv \frac{1}{3}\,\epsilon_{ijk}\epsilon_{imn}\left(p_k \frac{r_j r_m}{r}\, p_n + p_k p_n \frac{r_j r_m}{r} + \frac{r_j r_m}{r}\, p_k p_n\right) . \qquad \text{(A5)}$$

Using (A2)–(A4) to commute $\frac{r_j r_m}{r}$ to a position between p_k and p_n yields, by comparing with (A1),

$$\left[\frac{L^2}{r}\right]_{\rm sym} = \frac{L^2}{r} + \frac{i}{3}\,\epsilon_{ijk}\epsilon_{imn}\left(-p_k \frac{r_m}{r}\,\delta_{nj} + \frac{r_j}{r}\, p_n\,\delta_{km}\right) , \qquad \text{(A6)}$$

where again we take advantage of the antisymmetry of ϵ_{ijk}. From (A3) and by relabeling dummy indices we obtain our result

$$\begin{aligned}\left[\frac{L^2}{r}\right]_{\rm sym} &= \frac{L^2}{r} - \frac{1}{3r}\left(\delta_{jm}\delta_{kj} - \delta_{jj}\delta_{km}\right)\left(\delta_{km} - \frac{r_k r_m}{r^2}\right) \\ &= \frac{L^2}{r} + \frac{4}{3r}\ . \qquad \text{(A7)}\end{aligned}$$

[1] S. Jacobs, M. G. Olsson, and C. J. Suchyta III, Phys. Rev. D **33**, 3338 (1986).
[2] Alan Gara, Bernice Durand, and Loyal Durand, Phys. Rev. D **40**, 843 (1989).
[3] A. Barchielli, N. Brambilla, and G. M. Prosperi, Nuovo Cimento A **103**, 59 (1989); N. Brambilla and G. M. Prosperi, Phys. Lett. B **236**, 69 (1990); A. Barchielli, E. Montaldi, and G. M. Prosperi, Nucl. Phys. **B296**, 625 (1988); **B303**, 752(E) (1988).
[4] Collin Olson, M. G. Olsson, and Ken Williams, Phys. Rev. D **45**, 4307 (1992).
[5] N. Brambilla and G. M. Prosperi, Phys. Rev. D **46**, 1096 (1992).
[6] N. Isgur and J. Paton, Phys. Rev. D **31**, 2910 (1985); Phys. Lett. **124B**, 247 (1983).
[7] A. Chodos and C. B. Thomas, Nucl. Phys. **B72**, 509 (1974); I. Bars, *ibid.* **B111**, 413 (1976).
[8] For the development of the string model see, for instance, the following review articles: C. Rebbi, Phys. Rep. C **12**, 1 (1974); S. Mandelstam, *ibid.* **13**, 259 (1974); T. Goto, Prog. Theor. Phys. **46**, 1560 (1971); X. Artru, Phys. Rep. **97**, 147 (1983).

[9] M. Ida, Prog. Theor. Phys. **59**, 1661 (1978).
[10] D. LaCourse and M. G. Olsson, Phys. Rev. D **39**, 2751 (1989).
[11] E. Eichten and F. Feinberg, Phys. Rev. D **23**, 2724 (1981); D. Gromes, Z. Phys. C **22**, 265 (1984); **26**, 401 (1984).
[12] W. Buchmüller, Phys. Lett. **112B**, 479 (1982).
[13] E. E. Salpeter, Phys. Rev. **87**, 328 (1952).
[14] Hans A. Bethe and Edwin Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Springer-Verlag, Berlin, 1957).
[15] J. Sucher, Rep. Prog. Phys. **41**, 1781 (1978); Horace N. Crater and Peter Van Alstine, Phys. Rev. D **37**, 1982 (1988).
[16] G. E. Brown and G. Ravenhall, Proc. R. Soc. London **A208**, 552 (1951).
[17] For a heavy-quark system the total linear momentum is zero to first order in the quark c.m. system. The tube's canonical momenta are $\mathbf{p}_t(r_i) = 0\hat{\mathbf{r}}_i + \frac{L_t}{r}\hat{\mathbf{v}}_{\perp i}$.
[18] We follow closely the analogous calculations in Ref. [14], pp. 170–183.