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Effective action of a scalar field in a curved spacetime with a small inhomogeneity

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The effective action of a quantum field in an inhomogeneous spacetime is studied. We extend the Hartle-Hu method, which was developed to investigate the model in a homogeneous spacetime with a small anisotropy, to evaluate the one-loop effective action for a scalar field in a curved spacetime with a small inhomogeneity. Through a rather lengthy calculation we obtain the final expression which is then applied to numerically investigate the quantum-field effect on the dissipation of the space inhomogeneity in the early Universe.

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The effective action is a useful tool for studying the quantum corrections to a classical theory and has been employed in investigating the theory of quantum fields in curved spacetime [1,2]. In a series of papers, Hartle and Hu developed a method to evaluate the one-loop contribution of a conformally invariant scalar field to the effective action for a homogeneous cosmological model with small anisotropy [3]. From the effective action they obtained the dynamical equations of classical geometry, which were then used to evaluate the particle pair-production spectrum and to investigate the problem of dissipation of space anisotropy near the cosmological singularity [4,5].

The Hartel-Hu method has been applied to evaluate the effective action for neutrino and photon fields in homogeneous cosmologies with small anisotropy [6]. The finite temperature correction to the effective action has also been discussed [7]. In recent papers, several authors have extended the closed-time-path formalism [8] to curved spacetimes and obtained a real and causal effective action [9–11].

In this Brief Report we will extend the Hartle-Hu method to evaluate the one-loop effective action for a scalar field in a curved spacetime with a small inhomogeneity. As the space has inhomogeneity the calculations are more lengthy than those in the Hartle-Hu paper. We then apply these results to numerically investigate the particle production and space homogenization, along the methods in Refs. [4,5]. Using the present result we will calculate the effective action in closed-time-path formalism in the near future.

The present work is one of a series of investigations about the back reaction of quantum field in inhomogeneous spacetime. We have [12] made efforts to overcome the mode-mixing behavior coming from the space homogeneity which may be *small or large,* to find the adiabatically regularized quantum stress-energy tensor. We hoped that these results will enable us to investigate the back reaction of a quantum field in the Einstein equation. However, even if such an approach may work, we could obtain the renormalized stress energy tensor only after a *rather lengthy numerical calculation.* On the other hand, using the effective action approach, which has been developed only for the modes with a *small deviation* from the homogeneous spacetime, we could have an *analytic form* of the renormalized stress energy tensor. This makes the investigation of back-reaction effects of a quantum field in the einstein equation easier. We have also studied the problems of particle creation and the Coleman-Weinberg mechanism in an inhomogeneous spacetime in recent papers [13,14].

We consider the action describing a massless scalar field (ϕ) conformally coupled to the gravitational background

$$S=\int \mathcal{L} d^4x=\int \sqrt{-g}\,\phi[\tfrac{1}{2}\Box-\tfrac{1}{6}R]\phi\, d^4x \ , \tag{1}$$

where R is the Ricci scalar and $\Box\phi=g^{\mu\nu}\nabla_\mu\nabla_\nu\phi=g^{\mu\nu}\phi_{;\mu\nu}$. For simplicity, we consider an inhomogeneous spacetime with the line element

$$ds^2=a(\eta)^2\left[-d\eta^2+\sum_i e^{2\beta_i(\eta,\mathbf{x})}dx_i^2\right], \quad \sum_i \beta_i=0 \ , \tag{2}$$

where β_i are small values. In terms of a new variable $\varphi=a\phi$, the Lagrangian defined in Eq. (1) becomes

$$\begin{aligned}\mathcal{L}=&-\tfrac{1}{2}\left[\varphi\ddot{\varphi}-\varphi\sum_i \partial_i^2\varphi\right]+\tfrac{1}{2}\varphi\sum_i(-2\beta_{i,i}\partial_i-2\beta_i\partial_i^2)\varphi\\&+\tfrac{1}{2}\varphi\sum_i\left[4\beta_i\beta_{i,i}\partial_i+2\beta_i^2\partial_i^2-\tfrac{1}{6}\left[\dot{\beta}_i^2+\sum_j\beta_{i,j}^2\right]\right]\varphi\\&+O(\beta^3)\ .\end{aligned} \tag{3}$$

Then, after the functional integration, the generating functional can be written as

$$Z=\int \mathcal{D}\phi e^{iS}=[\mathrm{Det}(ia\Delta a)]^{-1/2} \ , \tag{4}$$

where, to second order in β_i, the operators Δ can be expanded as

$$\Delta=\Delta_0+V_1+V_2 \ , \tag{5}$$

with

$$\Delta_0=\partial_\eta^2-\sum_i\partial_i^2 \ , \tag{6}$$

$$V_1=\sum_i(2\beta_{i,i}\partial_i+2\beta_i\partial_i^2) \ , \tag{7}$$

$$V_2=\sum_i\left[-4\beta_i\beta_{i,i}\partial_i-2\beta_i^2\partial_i^2+\tfrac{1}{6}\left[\dot{\beta}_i^2+\sum_{ij}\beta_{i,j}^2\right]\right] \ . \tag{8}$$

Defining the Green function by

$$(a\Delta a)G(x,x')=-\delta^4(x,x')\,, \tag{9}$$

the effective action we will evaluate becomes

$$\Gamma=-i\ln Z=-\frac{i}{2}\mathrm{Tr}[\ln G]\,. \tag{10}$$

Substituting the perturbative form of Δ, Eq. (5), into the equation of G, Eq. (9), we can solve the Green function. To second order in β, the solution is

$$G=G_0+G_0(a\Delta_1 a)G_0+G_0(a\Delta_2 a)G_0 +G_0(a\Delta_1 a)G_0(a\Delta_1 a)G_0\,, \tag{11}$$

where the zero-order Green function G_0 defined by Eq. (9) can be expressed as

$$G_0(x,x')=-a^{-1}(\eta)\left[\int\frac{d^nk}{(2\pi)^n}\frac{e^{ik\cdot(x-x')}}{k^2}\right]a^{-1}(\eta')\,. \tag{12}$$

Substituting the Green function solution in Eq. (11) into Eq. (10) we have the perturbative expansion of the effective action:

$$\Gamma=\Gamma_0+\Gamma_1+\Gamma_2\,, \tag{13}$$

where

$$\Gamma_0=-\frac{i}{2}\mathrm{Tr}[\ln G_0]\,, \tag{14}$$

$$\Gamma_1=-\frac{i}{2}\mathrm{Tr}[(aV_1a)G_0]\,, \tag{15}$$

$$\Gamma_2=-\frac{i}{2}\mathrm{Tr}[(aV_2a)G_0]-\frac{i}{4}\mathrm{Tr}[(aV_1a)G_0(aV_1a)G_0]\,. \tag{16}$$

Formally, these traces are divergent and a renormalization procedure is needed to make them finite. Our model requires the counteraction of the general form [15–18]

$$S_c=\frac{\mu_c^{n-4}}{n-4}\int d^nx\sqrt{-g}\left[\frac{-1}{5760\pi^2}(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}-4R_{\alpha\beta}R^{\alpha\beta}+R^2)+\frac{1}{1920\pi^2}(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}-2R_{\alpha\beta}R^{\alpha\beta}+\tfrac{1}{3}R^2)\right]\,, \tag{17}$$

where we have continued the geometry to n dimensions. The divergences in Γ will be canceled by the S_c leaving the finite physical results.

We now begin to evaluate the effective action. For the same reason as in the Hartle-Hu paper [3], in the dimensional regularization procedure the terms $\mathrm{Tr}(aV_1a)G_0$ and $\mathrm{Tr}(aV_2a)G_0$ are equal to zero. The quantum correction parts of the effective action are

$$\Gamma_q=\Gamma_A+\Gamma_B+\Gamma_C\,, \tag{18}$$

where

$$\Gamma_A=-i\int d^4x\,d^4x'a(\eta)\left[\sum_i\beta_i(x)\partial_i^2\right]a(\eta)G_0(x,x')a(\eta')\left[\sum_i\beta_i(x')\partial_i^2\right]a(\eta')G_0(x',x)\,, \tag{19}$$

$$\Gamma_B=-i\int d^4x\,d^4x'a(\eta)\left[\sum_i\beta_{i,i}(x)\partial_i\right]a(\eta)G_0(x,x')a(\eta')\left[\sum_i\beta_{i,i}(x')\partial_i\right]a(\eta')G_0(x',x)\,, \tag{20}$$

$$\Gamma_C=-2i\int d^4x\,d^4x'a(\eta)\left[\sum_i\beta_{i,i}(x)\partial_i\right]a(\eta)G_0(x,x')a(\eta')\left[\sum_i\beta_i(x')\partial_i^2\right]a(\eta')G_0(x',x)\,. \tag{21}$$

Substituting the solution G_0, Eq. (12), into the above equations we have

$$\Gamma_A=-\frac{i}{(2\pi)^{2n}}\int d^4x\,d^4x'\sum_{ij}\beta_i(x)\beta_j(x')K_{ij}^{(A)}(x,x')\,, \tag{22}$$

$$\Gamma_B=-\frac{i}{(2\pi)^{2n}}\int d^4x\,d^4x'\sum_{ij}\beta_{i,i}(x)\beta_{j,j}(x')K_{ij}^{(B)}(x,x')\,, \tag{23}$$

$$\Gamma_C=-\frac{2i}{(2\pi)^{2n}}\int d^4x\,d^4x'\sum_{ij}\beta_i(x)\beta_{j,j}(x')K_{ij}^{(C)}(x,x')\,, \tag{24}$$

where

$$K_{ij}^{(L)}(x)=\int d^ne^{ik\cdot x}\tilde{K}_{ij}^{(L)}(k)\,,\quad L=A,B,C\,, \tag{25}$$

with

$$\tilde{K}_{ij}^{(A)}(k)=\int d^nl\frac{(k_i+l_i)^2}{(k+l)^2l}\frac{l_j^2}{2}\,, \tag{26}$$

$$\tilde{K}_{ij}^{(B)}(k)=\int d^nl\frac{(k_i+l_i)l_j}{(k+l)^2l^2}\,, \tag{27}$$

$$\tilde{K}_{ij}^{(C)}(k)=-i\int d^nl\frac{(k_i+l_i)l_j^2}{(k+l)^2l^2}\,. \tag{28}$$

Rotating both k_0 and l_0 through an angle $+\pi/2$ in the complex plane so that the denominators in the integrals $\tilde{K}_{ij}^{(A)}(k)$, $\tilde{K}_{ij}^{(B)}(k)$, and $\tilde{K}_{ij}^{(C)}(k)$ become the norm Euclidean four-vectors we can then carry out the integrations in a standard procedure [19]. Near $n=4$ the results are

$$K_{ij}^{(A)}(x)=i\pi^2\left[\tfrac{1}{240}(1+2\delta_{ij})k^4\left[\frac{-2}{n-4}-\ln k^2-\gamma+\frac{3}{2}\right]+\tfrac{1}{40}(k_i^2+k_j^2-\tfrac{8}{3}k_ik_j\delta_{ij})k^2\left[\frac{1}{n-4}+\ln k^2+\gamma-1\right]+\tfrac{1}{30}k_i^2k_j^2\left[\frac{-2}{n-4}-\ln k^2-\gamma\right]\right]\,, \tag{29}$$

$$K_{ij}^{(B)}(x)=i\pi^2\left[\tfrac{1}{6}k_ik_j\left[\frac{2}{n-4}+\ln k^2+\gamma\right]+\tfrac{1}{12}\delta_{ij}k^2\left[\frac{2}{n-4}+\ln k^2+\gamma-1\right]\right], \tag{30}$$

$$K_{ij}^{(C)}(x)=\pi^2\left[-\tfrac{1}{24}k_ik^2\left[\frac{2}{n-4}+\ln k^2+\gamma-1\right]+\tfrac{1}{12}k_ik_j^2\left[\frac{-2}{n-4}-\ln k^2-\gamma\right]\right], \tag{31}$$

where γ is the Euler constant. Notice that as the spacetime is inhomogeneous we shall preserve the terms proportional to k_i. It is these terms that render the calculations in this Brief Report more lengthy than those in the model without inhomogeneity [3].

After the lengthy calculations we find the divergent part of Γ_q:

$$\Gamma_q^{\text{div}}=-\frac{1}{1920\pi^2(n-4)}\int d^4x(-g)^{1/2}C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}, \tag{32}$$

where $C_{\mu\nu\gamma\delta}$ is the Weyl tensor in four dimensions,

$$C_{\mu\nu\gamma\delta}C^{\mu\nu\gamma\delta}=\frac{1}{a^4}\left\{\sum_i\left[2\ddot\beta_i^2+8\beta_{i,ii}^2-4\sum_j\dot\beta_{i,j}^2+\beta_{i,ij}^2+2\left[\sum_j\beta_{i,jj}\right]^2\right]+4\left[\sum_i\beta_{i,ii}\right]^2\right\}. \tag{33}$$

This divergence shall be canceled by the counteraction in Eq. (17) and the finite parts of Γ_q will come from the logarithmic and finite terms in Eqs. (29)–(31).

Now, because of spatial inhomogeneity we can not perform the spatial integration. Thus, we rotate k_0 back through an angle $\pi/2$ in the complex plane. This will make the $\ln k^2$ acquire a negative imaginary part, i.e.,

$$\ln k^2=-i\pi\Theta(k_0^2-\mathbf{k}^2)+\ln|k^2| . \tag{34}$$

Using this formula and after a rather lengthy calculation we have the following expression for the one-loop approximation to the effective action:

$$\begin{aligned}\Gamma_0+\Gamma_q+S_c=\int d^4x\Bigg\{&\left[\left[\frac{a}{l}\right]^2-\lambda\left[\frac{\ddot a}{a}+\frac{\dot a^2}{a^2}\right]\right]\sum_i\dot\beta_i^2+\left[\frac{a}{l}\right]^2\sum_i\left[2\beta_{i,i}^2+\sum_j(\beta_{i,j}^2-2\beta_{i,j}\beta_{j,j})\right]\\&-\frac{2\lambda}{9}\frac{\ddot a}{a}\sum_i\left[2\beta_{i,i}^2-\sum_j\beta_{i,j}^2\right]+2\lambda\frac{\dot a}{a}\sum_i\dot\beta_i\left[2\beta_{i,ii}-\sum_j\beta_{i,jj}\right]\\&+2\lambda\left[\frac{\dot a}{a}\right]^2\sum_i\left[2\beta_{i,i}^2-\sum_j\beta_{j,i}^2\right]-\frac{\lambda}{3}\left[4\frac{\ddot a}{a}+3\left[\frac{\dot a}{a}\right]^2\right]\sum_i\left[4\beta_{i,i}^2-\sum_j\beta_{i,j}^2\right]-\frac{25\lambda}{18}\left[\sum_i\beta_{i,ii}\right]^2\\&+\lambda(\ln\mu a)\left[\sum_i\left[\tfrac{3}{2}(2\ddot\beta_i^2+\beta_{i,ii}^2)-6\sum_j(\dot\beta_{i,j}^2+\beta_{i,ij}^2)+3\left[\sum_j\beta_{i,jj}\right]^2\right]+6\left[\sum_i\beta_{i,ii}\right]^2\right]\Bigg\}\\&+\lambda\int d^4x\int d^4y\int d^4k\frac{1}{(2\pi)^4}e^{ik\cdot(x-y)}[i\pi\Theta(k_0^2-\mathbf{k}^2)-\ln|k^2|]\\&\times\Bigg\{\sum_i\Bigg[\tfrac{3}{2}\ddot\beta_i(x)\ddot\beta_i(y)+6\beta_{i,ii}(x)\beta_{i,ii}(y)-3\sum_j[\dot\beta_{i,j}(x)+\beta_{i,ij}(x)][\dot\beta_{i,j}(y)+\beta_{i,ij}(y)]\\&+\tfrac{3}{2}\left[\sum_j\beta_{i,jj}(x)\right]\left[\sum_j\beta_{i,jj}(y)\right]\Bigg]+3\left[\sum_i\beta_{i,ii}(x)\right]\left[\sum_i\beta_{i,ii}(y)\right]\Bigg\},\end{aligned} \tag{35}$$

where $l\equiv(16\pi G)^{1/2}$, G is gravitational constant, $\lambda\equiv(288\pi^2)^{-1}$, and we have combined all scales into a single regularization scale μ which appears in the term ($ln\mu a$). The above equation is the main result of this paper. Note that it can be seen that as each pole term in Eqs. (29)–(31) is always accompanied by the $\ln k^2$ term and each $\ln k^2$ will contribute an imaginary part as shown in Eq. (34), thus the total particle production probability will have the same form as that in the divergent part of Γ_q and is proportional to the square of the Weyl tensor. This result is consistent with the calculation from the S-matrix method [20].

To apply the above result to numerically investigate the quantum field effect on the inhomogeneous spacetime we need the evolution equations for $a(\eta)$ and $\beta_i(x)$, which can be obtained by varying the effective action with respect to a and β_i. However, the equations so obtained will be rather complicated and we will adopt several approximations to simplify the numerical work.

First, we neglect the nonlocal term in Eq. (35). This step is hard to justify, but Hartle [5] showed that this does not change the results significantly. Note that the approximation of "local truncation" has been adopted in Refs. [4,7]. Second, we assume that β_i are functions of η and x_i; i.e., β_i do not depend on x_j if $i\neq j$. The equation for β_i is then

$$[(A\kappa_i)^{\cdot}+B\kappa_i]^{\cdot}+\tfrac{1}{2}[(A\dot{L}_{ii})^{\cdot}+CL_{ii}]_{,i}=0 \;, \tag{36}$$

where

$$A\equiv-\ln(\mu b)-i\tfrac{1}{2}\pi \;,$$

$$B\equiv\frac{2b^2}{\lambda}-\frac{1}{3}\left[\frac{\ddot{b}}{b}+\frac{\dot{b}^2}{b^2}\right] , \tag{37}$$

$$C\equiv\frac{4b^2}{\lambda}-\frac{2}{3}\left[\frac{17}{3}\frac{\ddot{b}}{b}-\frac{\dot{b}^2}{b^2}\right] ,$$

and

$$\kappa_i\equiv d\beta_i/d\eta \;,\quad L_{ii}\equiv d\beta_i/dx_i \;. \tag{38}$$

We have followed the Hartle-Hu prescription [4] which used the scale-invariant variables defined by

$$a(\eta)\equiv l\bar{\rho}_r^{1/4}b(\chi),\quad \eta\equiv\chi 6^{1/2}/\bar{\rho}_r^{1/4} \;, \qquad x_i\equiv y_i 6^{1/2}/\bar{\rho}_r^{1/4} \;, \tag{39}$$

where the value $\bar{\rho}_r$ is the constant giving the density of classical radiation according to $\rho_r=a^{-4}\bar{\rho}_r$ in the Friedmann universe. Note that the derivatives in Eq. (36) are defined by $\dot{F}\equiv dF/d\chi$ and $F_{,i}\equiv dF/dy_i$. To obtain the evolution equation (36) we have neglected the terms involving higher derivatives with respect to x_i; i.e., we assume that the spatial inhomogeneity is varying smoothly in the space. This is the third assumption.

Despite the adoption of several approximations it is still difficult to solve Eq. (36). We thus consider the case of $L_{ii}\gg\kappa_i$ (the case of $L_{ii}=0$ is that investigated by Hartle and Hu [4] for the mode in a homogeneous spacetime). Neglecting the κ_i terms in the above equation and integrating the equation we have

$$(A\dot{L}_{ii})^{\cdot}+CL_{ii}=f(\chi) \;, \tag{40}$$

where $f(\chi)$ is an integration "constant" which does not depend upon x_i. The nonhomogeneous solution of the above equation can be chosen to be spatially independent and what is really interesting to us is the solutions to the homogeneous equation corresponding to (40). We thus can write the inhomogeneous function L_{ii} as

$$L_{ii}(\chi,x_i)=c_i(x_i)L(\chi) \;. \tag{41}$$

The remaining problem is to see whether the function $L(\chi)$ will approach zero within a short time. If this happens then we may conclude that the quantum field can quickly smooth out the space inhomogeneity in the early Universe. Using the fact that for the model with a small inhomogeneity the metric $b(\chi)$ implicitly appearing in Eq. (40) will not be affected by space inhomogeneity (just like that in the homogeneous anisotropic mode discussed in Ref. [4]), thus we can easily find the homogeneous solution. We will investigate the model in the classical geometry which is the Friedmann universe.

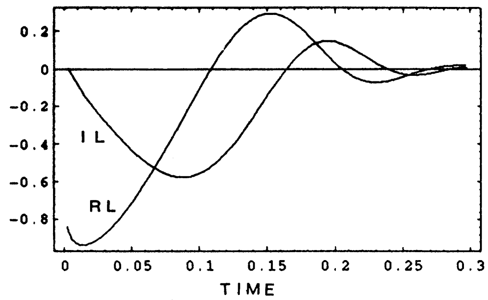

FIG. 1. Time evolution of real part RL and imaginary part IL of the inhomogeneous function L_{ii}.

Under the above discussions, a typical numerical solution of L_{ii} is shown in Fig. 1 (we use $\mu=1$). Note that the bottom scale in Fig. 1 measures the cosmic proper time in the units of Planck time. We thus see that both the imaginary part IL, which is a consequence of the particle creation, and real part RL, which represents the space inhomogeneity, approach zero within a short cosmic time. Therefore we have seen that quantum field does quickly smooth out the space inhomogeneity in the early Universe.

In conclusion, we have in this paper extended the Hartle-Hu method, which was developed to investigate the model in a homogeneous spacetime with small anisotropy, to evaluate the one-loop effective action for a scalar field in a curved spacetime with small inhomogeneity. We see that as the space has inhomogeneity the calculations are more lengthy than those in the Hartle-Hu paper. We apply our result to investigate the effect of a quantum field in inhomogeneous spacetime and find that spatial inhomogeneity in the early Universe could be quickly smoothed out.

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