$I=2$ pion scattering amplitude with Wilson fermions

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We present an exploratory calculation of the $I=2$ $\pi\pi$ scattering amplitude at threshold using Wilson fermions in the quenched approximation, including all the required contractions. We find good agreement with the predictions of chiral perturbation theory even for pions of mass 560–700 MeV. Within 10% error, we do not see the onset of the bad chiral behavior expected for Wilson fermions. We also derive rigorous inequalities that apply to two-particle correlators and as a consequence show that the interaction in the antisymmetric state of two pions has to be attractive.

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I. INTRODUCTION

In this paper we calculate the $I=2$ $\pi\pi$ scattering amplitude at threshold using Wilson fermions. The theoretical foundations of the calculation have been established in a series of papers by Lüscher where he shows how the finite volume dependence of the two-particle energy levels in a sufficiently large cubic box is related to the scattering amplitude [1,2]. In the context of lattice field theories, the leading term in an infinite volume expansion was first given in Ref. [3]. The subtleties of the lattice calculations have been described in detail in Ref. [4], where results of a calculation using staggered fermions is presented. We closely follow the notation of this reference.

The calculation of the $I=2$ $\pi\pi$ scattering amplitude at threshold has a number of simplifying features. In general, scattering amplitudes are complex, and are only related indirectly to the finite volume energy shift. At threshold, however, the amplitude is real, and is directly related to the energy shift. Also, the signal for pions is much better than for other, heavier, mesons, e.g., ρ's. Finally, one needs to calculate only quark and gluon exchanges for the $I=2$ channel, whereas, in general, there are also annihilation diagrams. The latter are both more difficult to calculate numerically, and are affected more strongly by the use of the quenched approximation.

A major motivation for this work is to test the chiral behavior of the scattering amplitude derived long ago by Weinberg [5] using PCAC (partial conservation of axial-vector current) and current algebra. We use Wilson's formulation of lattice fermions, for which there are lattice artifacts arising from the explicit breaking of chiral symmetry. We can quantify these corrections by comparing lattice results against the PCAC prediction and against results obtained using staggered fermions on the same set of lattices [4].

Our calculation is exploratory in at least three different ways. First, we use the quenched approximation. Second, we only use two different quark masses, neither very light. Thus we can only make rough extrapolations to the chiral limit. And, finally, we have only used one lattice size ($16^3\times 40$). This means we must assume the finite volume dependence predicted by Refs. [1,3], and cannot check the predictions.

In an earlier calculation, Guagnelli, Marinari, and Parisi [6] have done a partial calculation of the scattering amplitude using Wilson (and staggered) fermions. They did not include all the contractions which contribute in the $I=2$ channel, and thus one cannot extract the scattering amplitude from their results.

The article is organized as follows. A brief theoretical overview is given in Sec. II and the methodology and details of the lattices are given in Sec. III. The results are presented in Sec. IV, and in Sec. V we present a model which explains some features of the data. Section VI gives some conclusions. In the Appendix we derive rigorous inequalities regarding correlators of the type used in this study, in particular we show that the interaction in the antisymmetric $\pi\pi$ channel has to be attractive.

II. THEORETICAL BACKGROUND

Lüscher has derived the relationship between pion scattering amplitudes and the energies of the two-pion state in finite volume [1,2]. The derivation is valid as long as the box (which we take to be cubic) is large enough that its length L exceeds twice the range of interaction. In general, the relationship is complicated, involving scattering amplitudes over a range of energies and in many partial waves. The relationship simplifies, however, if one expands the energies in powers of $1/L$ and keeps only the first few terms. We consider only the lightest two pion state whose energy we denote by E. For infinite volume $E=2m_\pi$, but the energy is shifted by interactions as L is reduced:

$$\delta E = E - 2m_\pi = \frac{T}{L^3}\left[1 - c_1\frac{m_\pi T}{4\pi L} + c_2\left[\frac{m_\pi T}{4\pi L}\right]^2\right] + O(L^{-6}) ,$$
$$c_1 = -2.837\,297, \quad c_2 = 6.375\,183 . \quad (2.1)$$

To the order shown, the energy shift depends only on T, which is the (nonrelativistically normalized) scattering amplitude at threshold.

We use the nonrelativistically normalized amplitude in Eq. (2.1) since this simplifies its physical interpretation, as explained in Ref. [4]. T is related to the relativistically normalized scattering amplitude by $T^R = -(4m_\pi^2)T$, and to the S-wave scattering length by $T = -4\pi a_0/m_\pi$.

We extract T from our numerical data using Eq. (2.1). *A priori* we do not know the size of the $O(L^{-6})$ terms which we are dropping. Our numerical results suggest, however, that the truncation error is small.

Equation (2.1) holds separately for $I=0$ and 2 two pion states (Bose symmetry forbids $I=1$ at threshold). We have done the calculation, however, only for the $I=2$ channel. To understand why, consider the four types of contraction that contribute to a calculation of the two pion energy, shown in Fig. 1. With present computer resources we can calculate only the first two types, which we label the direct (D) and crossed (C) diagrams, respectively. These are not sufficient to calculate the $I=0$ amplitude which gets contributions from all four diagrams. For $I=2$, however, quark-antiquark annihilation is not

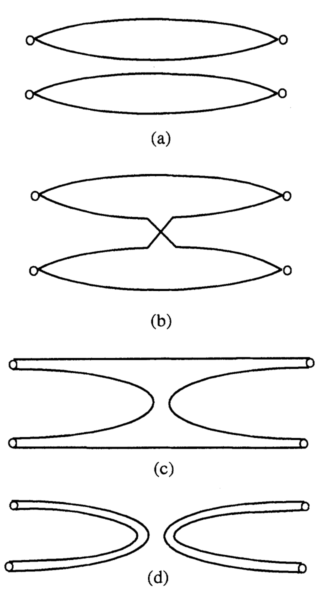

FIG. 1. The four different contractions that contribute to the two pion correlator: (a) direct, or gluon exchange, channel (D), (b) crossed, or quark exchange, channel (C), (c) single annihilation, and (d) double annihilation. The diagrams also correspond to the amplitudes that contribute to $\pi\pi$ scattering.

possible, and the required result is given by the combination $D-C$. Here we are adopting the convention of explicitly showing the minus sign from Fermi statistics, so that the lines in Fig. 1 represent c-number quark propagators.

We can also calculate the combination $D+C$. This does not project onto a definite isospin, but does select a definite representation in a theory with $N_f \geq 4$, N_f being the number of flavors. $D+C$ picks out the $\bar{q}\bar{q}qq$ representation having no traced indices, and is antisymmetric under the interchange of either quarks or antiquarks. We refer to this representation as the **A** (for $N_f=4$ it is the **20**), and to the corresponding scattering amplitude as $T(\mathbf{A})$. For arbitrary $N_f>2$, the combination $D-C$, which we call the **S**, projects onto the representation with no traced indices and is symmetric under quark exchange. The generalization of the $I=2$ amplitude T_2 is thus $T(\mathbf{S})$. In the quenched approximation the amplitudes $T(\mathbf{S})$ and $T(\mathbf{A})$ are independent of N_f, because the Wick contractions are always the same. In particular, $T_2 = T(\mathbf{S})$.

An important test of any calculation of pion scattering amplitudes is that they satisfy the constraints of chiral symmetry. In particular, the threshold amplitudes are determined, in the chiral limit, in terms of f_π (which is 93 MeV in our normalization) [5,4]:

$$4f_\pi^2 T(\mathbf{S}) = 1 + O(m_\pi^2 \ln m_\pi) ,$$
$$4f_\pi^2 T(\mathbf{A}) = -1 + O(m_\pi^2 \ln m_\pi) = -4f_\pi^2 T(\mathbf{S}) + O(m_\pi^2 \ln m_\pi) . \quad (2.2)$$

These results should apply in the quenched approximation, as discussed in Ref. [4]. Since $T(\mathbf{S})>0$, two pions in an **S** representation are repelled in the chiral limit, while in the **A** representation there is an attraction of equal strength. This equality can be understood as follows. The diagrams of Fig. 1 serve dual purpose. In addition to showing contractions contributing to E, they can also represent the contributions to a direct calculation of pion scattering amplitudes. We refer to Fig. 1(a) as the gluon exchange amplitude T_g and to Fig. 1(b) as the quark exchange amplitude T_q. Following the standard usage for amplitudes, we *include* the Fermi-statistics sign in T_q; i.e., we use the opposite convention to that for C. Thus we find that $T(\mathbf{S}) = T_g + T_q$ and $T(\mathbf{A}) = T_g - T_q$. Now, it is possible to show that T_g vanishes in the chiral limit [4], so that $T(\mathbf{S}) = T_q = -T(\mathbf{A})$.

Testing relations Eq. (2.2) is particularly important for Wilson fermions, which we use here. This is because Wilson fermions explicitly break chiral symmetry, the violation only vanishing in the continuum limit. Thus we expect the relativistically normalized amplitude not to vanish in the chiral limit but to have a constant term proportional to the lattice spacing: $T^R \propto \Lambda a + O(m_\pi^2)$. The nonrelativistically normalized amplitude will then behave as $T \propto a\Lambda/m_\pi^2$, where Λ is some nonperturbative scale. This artifact will dominate over the constant term in the chiral limit [7]. This is in contrast with staggered fermions where such artifacts are forbidden by the residual chiral symmetry [4]. With Wilson fermions one can isolate the physical result by doing the calculation at a number of values of the quark mass provided $f_\pi^2 T$ is a well-behaved function of m_π^2. This remains to be checked. We find that, for 500 MeV $< m_\pi <$ 700 MeV, the chiral-symmetry-breaking effects are smaller than the statistical errors ($\sim 10\%$).

III. CALCULATIONAL DETAILS

The energy of two pions in a finite box is obtained from the Euclidean correlator

$$C_{\pi\pi}(t) = \left\langle \sum_{\mathbf{x}_1} \mathcal{O}_1(\mathbf{x}_1,t) \sum_{\mathbf{x}_2} \mathcal{O}_2(\mathbf{x}_2,t) \mathcal{S}_3(\mathbf{x}_3,t=0) \times \mathcal{S}_4(\mathbf{x}_4,t=0) \right\rangle . \tag{3.1}$$

The sources $\mathcal{S}_i$ create the pions at $t=0$, and the operators $\mathcal{O}_i$ (which we also call the "sinks") destroy them at time t. The representation of the two pion state is determined by the flavor of the sources and sinks. For example, we can select the $I=2$ (or equivalently **S**) representation if both sources have the flavor of a π^+, and both $\mathcal{O}_1$ and $\mathcal{O}_2$ have the flavor of a π^-.

At large $|t|$ the correlator will fall as

$$C_{\pi\pi}(t) = Z_{\pi\pi} \exp(-E|t|) + \cdots , \tag{3.2}$$

where E is the energy of the lightest two pion state. The ellipsis indicates contributions from excited states that are suppressed exponentially. This is similar to the behavior of the two point function used to calculate m_π:

$$C_\pi(t) = \left\langle \sum_{\mathbf{x}_1} \mathcal{O}(\mathbf{x}_1,t) \mathcal{S}(\mathbf{x}_4,t=0) \right\rangle = Z_\pi \exp(-m_\pi |t|) + \cdots . \tag{3.3}$$

We take all quarks to be degenerate, so the flavor of the pion source $\mathcal{S}$ in this equation is unimportant; all that matters is that $\mathcal{O}$ has the conjugate flavor to $\mathcal{S}$. It is useful in practice to combine Eqs. (3.2) and (3.3):

$$\mathcal{R}(t) \equiv \frac{C_{\pi\pi}(t)}{C_\pi(t)^2} = \frac{Z_{\pi\pi}}{Z_\pi^2} \exp(-\delta E|t|) + \cdots , \tag{3.4}$$

and directly extract the energy shift δE.

The contractions which can contribute to $C_{\pi\pi}$ are shown in Fig. 1. The combination which is needed depends on the flavor of the two pion state. It is easy to see that only the direct and crossed contractions contribute for two pions in **S** or **A** representations [4]. We adopt the notation that $D(t)$ is a ratio as in Eq. (3.4), with the numerator being the direct contraction. Similarly, $C(t)$

TABLE I. Summary of results from Ref. [8] needed in this calculation. f_π is normalized such that the experimental value is $f_\pi = 93$ MeV, and is obtained using the mean-field improved value 0.77 for the axial-vector current renormalization constant.

β	κ	Lattice	N_{conf}	m_π	f_π
6.0	0.154	$16^3 \times 80$	35	0.364(6)	0.057(3)
6.0	0.155	$16^3 \times 80$	35	0.297(9)	0.055(3)

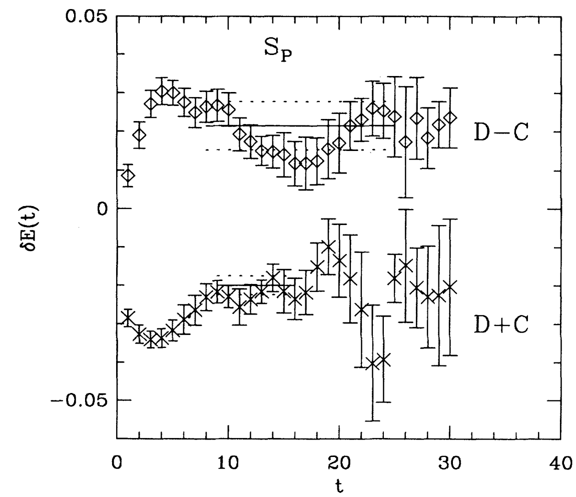

FIG. 2. The effective energy shift $\delta E(t)$ using S_P correlators at $\kappa = 0.154$, for both the $I=2$(**S**) representation ($D-C$), and the **A** representation ($D+C$).

is the ratio with the crossed contraction in the numerator. Then, as discussed above, and explained in more detail in Ref. [4], we can extract the energy shifts for the **S** and **A** representations using

$$D(t) + C(t) = Z_A e^{-\delta E_A |t|} , \tag{3.5}$$

$$D(t) - C(t) = Z_S e^{-\delta E_S |t|} . \tag{3.6}$$

The amplitudes Z_S and Z_A are shorthand for the ratios $Z_{\pi\pi}/Z_\pi^2$.

We are free to choose the form of the sources and sinks, as long as they both couple to a two pion state of the required flavor. This choice does not affect the value of the energy shift, but it does alter the signal-to-noise ra-

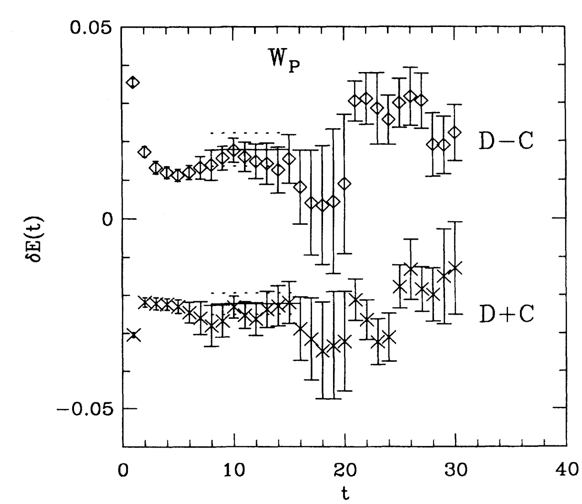

FIG. 3. As in Fig. 2 but for the W_P correlators.

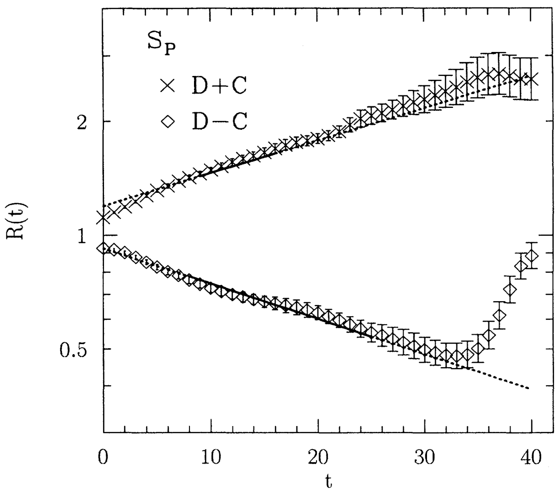

FIG. 4. $\mathcal{R}(t)$ using S_P correlators at $\kappa=0.154$.

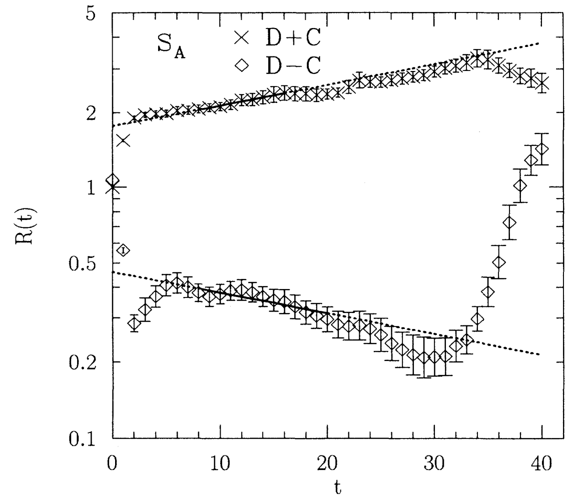

FIG. 6. $\mathcal{R}(t)$ using S_A correlators at $\kappa=0.154$.

tio. In order to improve this ratio, we should use sources with a large overlap with the lightest two pion state. This state will not consist of two independent pions. Nevertheless, as indicated in Eq. (3.1), we use sources and sinks consisting of a product of two independent single pion operators. For the single pion sources [the $\mathcal{S}_i$ in Eq. (3.1)], we use Wall and Wuppertal quark sources, which we have shown to be reasonably effective in producing single-particle correlators [8,9]. In addition, we use both pseudoscalar ($\mathcal{P}=\bar{\psi}\gamma_5\psi$) and axial-vector ($\mathcal{A}_4=\bar{\psi}\gamma_4\gamma_5\psi$) operators for each of the sources. There are thus four sources in all, which we label W_P (Wall with pseudoscalar), W_A (Wall with axial vector), S_P (Wuppertal with pseudoscalar), and S_A (Wuppertal with axial vector). For the sinks [the $\mathcal{O}_i$ in Eq. (3.1)] we use local operators, with the Dirac structure either P or A. Of the various possible combinations of sources and sinks we consider only those in which both sources are of the same type, and both

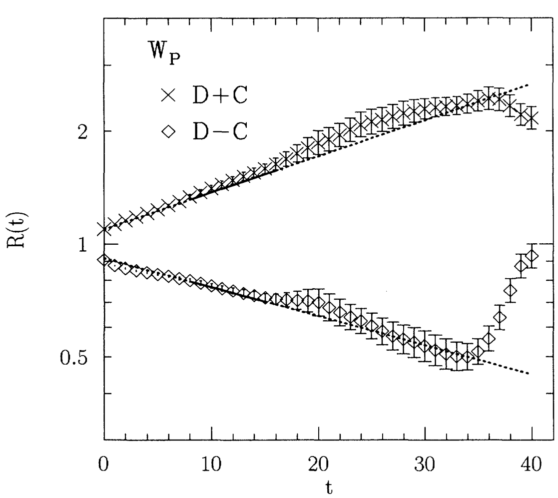

FIG. 5. $\mathcal{R}(t)$ using W_P correlators at $\kappa=0.154$.

sinks have the same Dirac structure as the sources. Thus we can label the ratios $\mathcal{R}(t)$ according to the choice of source, i.e., as W_A, W_P, etc. Finally, we always define the ratio $\mathcal{R}(t)$ with the same sources and operators in both the numerator and denominator. This is of no consequence for the energy shifts, but does affect the amplitudes $Z_{A,S}$, which we discuss further in Sec. V.

We use 35 pure gauge configurations of size $16^3\times40$ generated at $\beta=6.0$. We use Wilson fermions at two different quark masses, $\kappa=0.154$ and 0.155. The corresponding pions have masses of about 700 and 560 MeV, respectively. The quark propagators are calculated on lattices doubled in time (i.e., of size $16^3\times80$), with periodic boundary conditions in all directions. We have previously used these lattices and propagators to study the spectrum and matrix elements [8,9], and we list the relevant results in Table I. The value of f_π given in Table I is different from that quoted in Ref. [8] for two reasons. First, we use the normalization such that the experimental value is $f_\pi=93$ MeV, and second we now use the mean-field improved value 0.77 for the axial-vector current renormalization constant [10], rather than the previous estimate 0.86. The statistical errors in individual data points shown in Figs. 2–6 are calculated using the single elimination jackknife procedure. The "forward" and "backward" propagators on the underlying $16^3\times40$ lattices give us two results for $R(t)$ on each lattice. Since these are correlated, we average them and treat them as a single result.

IV. RESULTS

To display our results, we use the quantity

$$\delta E_{\text{eff}}(t)=\ln[\mathcal{R}(t)/\mathcal{R}(t+1)] \ . \tag{4.1}$$

This "effective energy shift" should reach a plateau of height δE when t is large enough that the lightest state dominates. Figures 2 and 3 show $\delta E_{\text{eff}}(t)$ for the pseudoscalar operator $\mathcal{P}$ at $\kappa=0.154$, using Wuppertal and Wall sources, respectively. The results using the axial operator $\mathcal{A}_4$ are of poorer quality, and are not shown. There is a clear signal of a nonvanishing energy shift. We extract δE by fitting $\mathcal{R}(t)$ to a single exponential, selecting the range of time slices separately for each channel based on the extent of the plateau in the effective energy shift. The solid lines in the figures indicate the fit value over the range of the fit, while the dashed lines show the 1σ jackknife errors.

It is apparent from Figs. 2 and 3 that the signal for Wall sources has smaller statistical errors than that for the Wuppertal sources. We can partly understand this as follows. The Wall source produces pions with $\mathbf{p}=0$, while the Wuppertal source couples to pions having all possible momenta. Consequently the Wuppertal source correlators have an additional unwanted contribution at small t from an excited state of two pions of equal and opposite momenta, each of magnitude $p=2\pi/L$. For large volumes this state approaches the lightest state consisting of two pions both having $\mathbf{p}=0$, and provides the largest contamination to $\mathcal{R}(t)$ rather than the states made up of radially excited pions. It may therefore become necessary to use Wall sources for calculations on larger lattices.

In Figs. 4–7 we show the ratios $\mathcal{R}(t)$ at $\kappa=0.154$, for both $\mathcal{P}$ and $\mathcal{A}_4$ operators. These plots show that there is a contamination from "wraparound" effects starting at $t\sim30$. One of the two pions can propagate $N_t-t=80-t$ time steps backwards, because of the periodic boundary conditions. This results in a contribution which is independent of t, but suppressed by roughly

$$\exp(-m_\pi N_t)/\exp(-2m_\pi t)$$

compared to the forward propagation of the two pion state. In practice, we always fit to time ranges satisfying $t_{\max}\leq 26$, for which we can ignore this contamination.

The results of our fits, together with the time ranges used, are given in Table II. For both S_P and W_P correlators we fit using the full covariance matrix over the range of the plateau. We are unable to do this for the S_A and W_A channels because some of the jackknife samples are too noisy. Our results for these channels are obtained

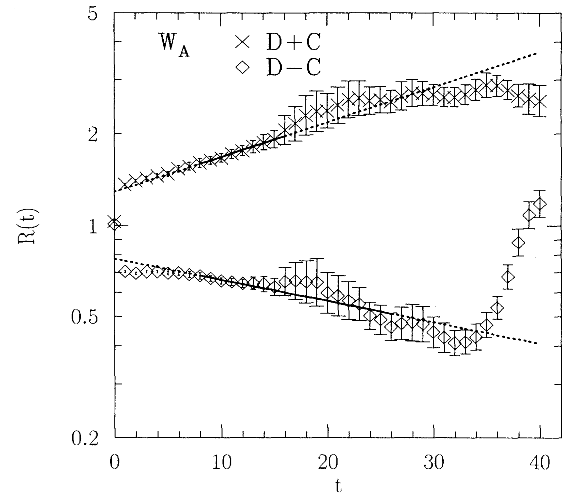

FIG. 7. $\mathcal{R}(t)$ using W_A correlators at $\kappa=0.154$.

keeping only the diagonal elements of the covariance matrix, i.e. neglecting correlations in $\mathcal{R}(t)$ between different time slices. Because of this, we use only the S_P and W_P results to extract scattering amplitudes. The data show that the interaction in the $D+C$ channel is attractive, consistent with the result derived in the Appendix.

Our results for scattering amplitudes are presented in Table III. We illustrate our procedure for obtaining these results using $T(S)$ as an example. For each jackknife sample we first average the fit value for $\delta E(S)$ from the S_P and W_P sources, and then solve the cubic polynomical given in Eq. (2.1) for $T(S)$. The central value and the error are given by the jackknife procedure, regarding the 35 data points as statistically independent. Within the same jackknife procedure we also calculate $T(A)$, and extract $T_q=[T(S)-T(A)]/2$ and $T_g=[T(S)+T(A)]/2$.

When solving Eq. (2.1) for $T(S)$ or $T(A)$ we monitor the effect of the $1/L^4$ and $1/L^5$ terms (using the values of m_π given in Table I). These turn out to be, respectively, $\sim31\%$ and 8% of the leading term. This suggests that

TABLE II. Results for the amplitude and energy shifts obtained from fits to $\mathcal{R}(t)$ of correlators for the $D\pm C$ channels. The four kinds of correlators S_P, W_P, S_A, and W_A are described in the text. We also give the range of time slices over which the fit is made.

		$D+C$			$D-C$	
Correlator	Fit	Z_A	$-E_A$	Fit	Z_S	E_S
			$\kappa=0.154$			
S_P	8–16	1.19(4)	0.020(2)	8–25	0.93(7)	0.022(6)
W^P	8–16	1.10(2)	0.022(3)	8–15	0.92(2)	0.018(4)
S_A	8–16	1.77(10)	0.019(7)	8–20	0.46(7)	0.019(10)
W_A	8–15	1.30(5)	0.025(6)	8–26	0.78(7)	0.016(9)
			$\kappa=0.155$			
S_P	8–17	1.23(6)	0.022(6)	8–18	0.87(6)	0.022(8)
W_P	8–18	1.14(3)	0.027(3)	8–15	0.92(3)	0.024(5)
S_A	8–15	2.07(17)	0.020(9)	8–20	0.27(11)	0.023(25)
W_A	8–15	1.49(9)	0.023(5)	7–20	0.67(8)	0.017(13)

TABLE III. Final results for the scattering amplitudes.

	$\kappa=0.154$	$\kappa=0.155$
$T(S)$	58.7(8.7)	69.6(10.7)
$T(A)$	−61.8(6.5)	−73.0(6.6)
T_q	60.2(4.6)	71.3(6.6)
T_g	−1.6(5.0)	−1.7(6.0)
$4f_\pi^2T(S)$	0.76(14)	0.84(16)
$4f_\pi^2T(A)$	−0.80(12)	−0.88(13)
$4f_\pi^2T_q$	0.78(10)	0.86(12)
$4f_\pi^2T_g$	−0.02(7)	−0.02(7)

the error introduced by truncating the series is a few percent in the scattering amplitudes. This is smaller than the statistical errors, which are approximately 10%.

To test the current algebra predictions, we calculate the combinations $4f_\pi^2T$. These are included in Table III and shown in Fig. 8. The errors are calculated assuming that f_π and T are uncorrelated; we have checked on subsamples that the errors are similar if correlations are included. In the chiral limit the following relations should hold: $4f_\pi^2T(S)=1$, $T(S)=T_q=-T(A)$, and $T_g=0$. Our results show that, within our errors, the second relation holds even at the relatively large quark masses we have used, and $4f_\pi^2T$ lies between 0.76 and 0.88. There is a small increase in T between $\kappa=0.154$ and 0.155, but because of the size of the statistical errors we cannot conclude if this is related to the $1/m_\pi^2$ divergence expected for Wilson fermions in the chiral limit. Also, the errors are too large to extract a value for the gluon exchange amplitude T_g. All we can say is that it is much smaller than the quark exchange amplitude.

It is interesting to compare our results with those obtained using staggered fermions on the same lattices [4]. For technical reasons, we were able to calculate only the

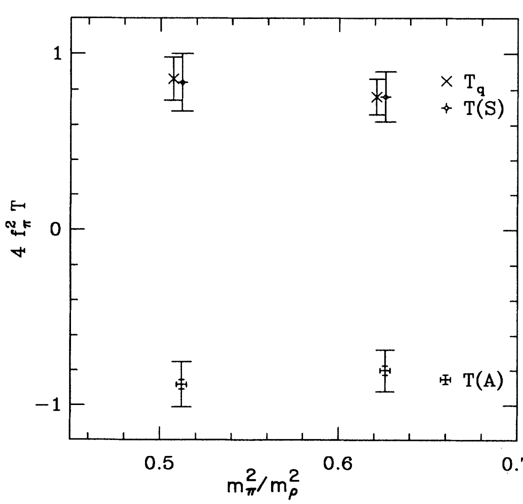

FIG. 8. $4f_\pi^2T$ plotted vs m_π^2/m_ρ^2 to test the chiral behavior. The data for T_q has been displaced by −0.05 along the x axis for clarity.

quantity $4Q=T_q-2T_qT_g(c_1m_\pi/4\pi L)$ with staggered fermions, and not T_q and T_g separately. If we use the values for T_g obtained here, however, we then find that $4Q=T_q$ to good approximation. To compare the results, we note that the pion mass and f_π match for the following parameter values: staggered $m_q=(0.02+0.03)$ with Wilson $\kappa=0.154$ and staggered $m_q=(0.01+0.02)$ with $\kappa=0.155$ [11]. The staggered results at these two masses are $T_q\approx 4Q=67(8)$ and 78(6), respectively, which agree within errors with the results obtained here. It is reassuring to find that the two formulations, each with their separate technical problems, yield mutually consistent results.

V. EXPECTED BEHAVIOR OF $\mathcal{R}(t)$ WITH WILSON FERMIONS

As shown in Table II and Figs. 4–7, different source and sink operators give considerably different values for $Z_{A,S}$. This variation is more marked for smaller quark mass. It turns out that the values for $Z_{A,S}$ can be understood semiquantitatively using a simple model, as we explain in this section. This model is similar to that used in the analysis of our staggered fermion data [4].

We begin by imagining that the source creates two pions each having $\mathbf{p}=0$. The pion operators could be local or smeared, but should have a finite extent that is much smaller than the lattice size. Also, the same source operators are used in the numerator and denominator of Eq. (3.4). In this case we expect that $Z_{A,S}\to 1$ as $L\to\infty$ because the lightest two pion state differs from two independent, zero momentum pions by terms which vanish as $L\to\infty$. Thus the Z factors in the numerator and denominator of Eq. (3.4) cancel. Assuming $Z_{A,S}=1$, and using Eqs. (3.5) and (3.6), the behavior of the direct and crossed contractions is

$$D(t)=1-\frac{E_S+E_A}{2}t+O(t^2)\,,$$
$$C(t)=-\frac{E_S-E_A}{2}t+O(t^2) \tag{5.1}$$

for t large enough that the contributions of excited states have died away. What is important here is that only $D(t)$ is nonzero when extrapolated back to $t=0$; the constant term corresponds to the two pions propagating from source to sink without interactions, which is only possible in the direct channel. In the crossed channel, at least one quark exchange interaction is required. This gives rise to the term linear in t, since the interaction can occur at any time. The term linear in t in $D(t)$ is due to the gluon exchange interaction.

In practice Z_A and Z_S differ from unity, for two reasons. First, the two pion state is altered by interactions. This gives corrections proportional to $1/L^2$ [4], which we assume are small and ignore. Second, our sources are not two independent pion operators each having $\mathbf{p}=0$, but rather a single Wall or Wuppertal quark source. This gives large corrections to $Z_{A,S}$, due to the overlap, which do not vanish as $L\to\infty$, and it is these which we estimate.

In our setup a state of two quarks and two antiquarks, all in close proximity to one another, is created rather than two separate pions. We attempt to pair the quarks and antiquarks into pions by making two color singlets, each with pseudoscalar quantum numbers. But, by performing a combined color and Dirac Fierz transformation, we find that we are also creating, with nonvanishing amplitude, two pions with the opposite $\bar{q}q$ pairings. Explicitly, the Fierz transformations are

$$\begin{aligned} \mathcal{P}\otimes\mathcal{P} &\to \tfrac{1}{12}(\mathcal{P}\otimes\mathcal{P}+\mathcal{A}_4\otimes\mathcal{A}_4\cdots)\ , \\ \mathcal{A}_4\otimes\mathcal{A}_4 &\to \tfrac{1}{12}(\mathcal{P}\otimes\mathcal{P}+\mathcal{A}_4\otimes\mathcal{A}_4\cdots)\ , \end{aligned} \tag{5.2}$$

where

$$\mathcal{P}\otimes\mathcal{P}\equiv(\bar{\psi}_1\gamma_5\psi_2)(\bar{\psi}_3\gamma_5\psi_4)$$

and

$$\mathcal{A}_4\otimes\mathcal{A}_4\equiv(\bar{\psi}_1\gamma_4\gamma_5\psi_2)(\bar{\psi}_3\gamma_4\gamma_5\psi_4)\ ,$$

parentheses implying spin and color traces. The sign due to fermion exchange is not included in these Fierz identities, since we do not include the sign in our definitions of $D(t)$ and $C(t)$. The Fierz relations hold for both Wall and Wuppertal sources, in the latter case because all the products of link matrices begin at the same site. We have shown only the $\mathcal{P}\otimes\mathcal{P}$ and $\mathcal{A}_4\otimes\mathcal{A}_4$ parts of the Fierz-transformed combinations because the correlators of these operators make the dominant contribution. Other tensor structures contribute in two ways, both of which are suppressed by powers of $1/L$. For definiteness, we focus on $\mathcal{V}\otimes\mathcal{V}$. If the q and $\bar{q}$ in each of the two vector bilinears are near to one another, but the bilinears themselves are well separated, then the source will create two ρ mesons. Since with wall sources the bilinears can have any relative position, the two ρ mesons have zero three-momentum. The ρ's then scatter into two pions, leading to a factor of $1/L^3$ because of the interaction. Alternatively, when the two bilinears overlap there can be a direct coupling to two pions. This is suppressed by powers of $1/L$, because it requires all four fields in our smeared sources to be close. Note that if all four fields are distant from one another the result is suppressed both by powers of $1/L$ and by the rapid falloff of meson wave functions. Contributions of operators consisting of two color octets require all four fields to be close, and are thus also suppressed by an overlap factor. The magnitude of these neglected terms can be significant, especially for Wuppertal sources, as shown by the difference between our data and the estimates presented below.

There is no large Fierz contribution from the sinks, since these do consist of two independent pion operators, each having $\mathbf{p}=0$. The only contribution occurs when the two operators overlap and are suppressed by powers of $1/L$.

The Fierz relations mean that our crossed contractions contain a part in which the quarks have already been exchanged before we can identify the state as one with two pions, so that no subsequent quark-exchange interaction is necessary. This leads to a constant term in the crossed contraction. The Fierz contributions to the direct contraction do not, however, affect the constant term, for there must be an additional quark exchange interaction to bring the quarks and antiquarks back to their original pairings. This discussion motivates the following assumptions for the constant terms:

$$\begin{aligned} D(t=0)_{P,A} &\approx 1\ , \\ C(t=0)_P &\approx \tfrac{1}{12}(1+C_A^2/C_P^2)\ , \\ C(t=0)_A &\approx \tfrac{1}{12}(C_P^2/C_A^2+1)\ . \end{aligned} \tag{5.3}$$

The subscript indicates the type of correlator, and the constants C_A, C_P are the amplitudes for creating single pions with $\mathbf{p}=0$ using the operators $\mathcal{A}_4$ and $\mathcal{P}$, respectively. The ratio C_P/C_A is 2.5 and 3.1 for Wuppertal sources, and 1.6 and 1.9 for Wall sources, at $\kappa=0.154$ and 0.155, respectively. Using these values, we can calculate the Z's using

$$Z_S \approx D(t=0)-C(t=0),\quad Z_A \approx D(t=0)+C(t=0)\ . \tag{5.4}$$

TABLE IV. Model predictions for the intercepts of $\mathcal{R}(t)$.

Correlator	Z_A	Z_S
	$\kappa=0.154$	
S_P	1.10	0.90
W_P	1.12	0.88
S_A	1.60	0.40
W_A	1.30	0.70
	$\kappa=0.155$	
S_P	1.09	0.91
W_P	1.11	0.89
S_A	1.88	0.12
W_A	1.38	0.62

The predictions are collected in Table IV. They give a good semiquantitative representation of the data for $Z_{A,S}$ in Table II. In particular, we can understand the small value of Z_S for the S_A operators as being due to a large cancellation between the direct contraction and the Fierz contribution to the crossed contraction, the latter being enhanced by the large ratio C_P/C_A for Wuppertal sources. This cancellation is most likely the reason why the signal is so noisy in this channel.

VI. CONCLUSIONS

We find that it is straightforward to calculate the finite volume energy shift for channels not involving $\bar{q}q$ annihilation. The calculation is much less involved using Wilson fermions than the one we carried out with staggered fermions [4]. We are able to work on a lattice of modest size ($L\approx1.6$ fm) because the interactions in the channels we consider are relatively weak. From the energy shifts we extract the quark-exchange amplitude and place a bound on the gluon exchange amplitude. Our results are consistent with the predictions of current algebra; on the other hand, the quarks used in the calculation are not light enough to expose the expected artifacts due to the breaking of chiral symmetry by Wilson fermions.

It is important to extend this work to smaller quark masses, where the divergence in T due to chiral symmetry breaking should show up. Furthermore, the result should be checked on a larger volume to verify that the asymptotic form of the finite volume dependence can be used.

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APPENDIX: CORRELATION INEQUALITIES FOR SCATTERING AMPLITUDES

One can derive rigorous inequalities among correlation functions for vectorlike gauge theories such as QCD. The basis of such inequalities is the positively of the measure in the Euclidean path integral:

$$[dA_\mu][d\bar{\psi}][d\psi]\exp[-S_{\rm gauge}-S_{\rm fermion}]\geq 0 \ . \tag{A1}$$

This property has been exploited, both on the lattice [12] and in the continuum [13], to derive inequalities among two-point correlation functions. As a result it was shown that the pion is the lightest meson. Here we apply the same arguments to four-point correlation functions in finite volume to derive constraints on $\pi\pi$ scattering amplitudes. The only other work extending the derivation of inequalities to higher-order correlation functions that we are aware of is Ref. [14], where it is shown that the pion wave function is largest at $\mathbf{r}=0$ using a four-point correlation function.

We consider the four-point correlation functions corresponding to the $\pi\pi$ scattering at threshold. Let the sources for the two pion be at time $t=0$ and the two pion sinks be at time $t=T$. We take all four pion operators to be pointlike with Dirac structure $\bar{\psi}\gamma_5\psi$ and zero three-momentum. In terms of the quark propagator $G(\mathbf{x},0;\mathbf{y},T)$, the direct and crossed correlators are

$$\begin{aligned} D(0,T;\mathbf{p}=0) &= \sum_{\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{w}} \langle \mathrm{Tr}[G(\mathbf{x},0;\mathbf{y},T)G^\dagger(\mathbf{x},0;\mathbf{y},T)] \\ &\qquad \times \mathrm{Tr}[G(\mathbf{z},0;\mathbf{w},T)G^\dagger(\mathbf{z},0;\mathbf{w},T)]\rangle \ , \\ C(0,T;\mathbf{p}=0) &= \sum_{\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{w}} \langle \mathrm{Tr}[G(\mathbf{x},0;\mathbf{y},T)G^\dagger(\mathbf{z},0;\mathbf{y},T)G(\mathbf{z},0;\mathbf{w},T) \\ &\qquad \times G^\dagger(\mathbf{x},0;\mathbf{w},T)]\rangle \ , \end{aligned} \tag{A2}$$

where the trace is taken over the color and spin indices.

Using the Schwarz inequality

$$\langle ff^\dagger\rangle \geq |\langle f\rangle|^2 \ , \tag{A3}$$

we get the relation

$$D(0,T;\mathbf{p}=0)\geq [P(0,T;\mathbf{p}=0)]^2 \ , \tag{A4}$$

where the zero three-momentum pion correlator,

$$P(0,T;\mathbf{p}=0)=\sum_{\mathbf{x},\mathbf{y}} \langle \mathrm{Tr}[G(\mathbf{x},0;\mathbf{y},T)G^\dagger(\mathbf{x},0;\mathbf{y},T)]\rangle \ , \tag{A5}$$

is by itself positive definite. This inequality implies that the contribution of this diagram to the two pion interaction is attractive.

These results can be generalized to other Dirac structures. In fact, the interaction between any two mesons, e.g., two ρ's, is attractive in the direct channel. Note that the derivation of this inequality did not depend on the volume of the system. This is analogous to the zero temperature mass inequalities of Refs. [12,13] which can be used at finite temperature to give relations between hadronic screening lengths.

The crossed correlator given in Eq. (A2) can be rewritten as

$$C(0,t;\mathbf{p}=0)=\sum_{\mathbf{x},\mathbf{z}}\left\langle \mathrm{Tr}\left[\left[\sum_{\mathbf{y}} G(\mathbf{x},0;\mathbf{y},T)G^\dagger(\mathbf{z},0;\mathbf{y},T)\right]\left[\sum_{\mathbf{w}} G(\mathbf{x},0;\mathbf{w},T)G^\dagger(\mathbf{z},0;\mathbf{w},T)\right]^\dagger\right]\right\rangle \ , \tag{A6}$$

and is therefore also positive. Combining this fact with the inequality in Eq. (A4) shows that the scattering amplitude in the flavor antisymmetric channel, corresponding to the combination $D+C$, is

$$D(0,T;\mathbf{p}=0)+C(0,T;\mathbf{p}=0)\geq [P(0,T;\mathbf{p}=0)]^2 \ . \tag{A7}$$

This inequality on the correlators implies that the exponential falloff with time in this channel is slower than that for two noninteracting pions. It follows that the interaction energy δE in the antisymmetric channel is negative, i.e., the scattering length is positive.

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