

The stretched horizon and black hole complementarity

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Three postulates asserting the validity of conventional quantum theory, semiclassical general relativity, and the statistical basis for thermodynamics are introduced as a foundation for the study of black-hole evolution. We explain how these postulates may be implemented in a “stretched horizon” or membrane description of the black hole, appropriate to a distant observer. The technical analysis is illustrated in the simplified context of $(1+1)$ -dimensional dilaton gravity. Our postulates imply that the dissipative properties of the stretched horizon arise from a coarse graining of microphysical degrees of freedom that the horizon must possess. A principle of black-hole complementarity is advocated. The overall viewpoint is similar to that pioneered by ’t Hooft but the detailed implementation is different.

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I. INTRODUCTION

The formation and evaporation of a macroscopic black hole is a complex process which certainly leads to a practical loss of information and an increase of thermal entropy. The same is true of almost all macroscopic phenomena. It is exceedingly difficult to keep track of all the degrees of freedom involved when a large block of ice melts or a bomb explodes, but, *in principle*, it can be done. According to the standard rules of quantum field theory in a fixed Minkowski spacetime, the time evolution of any system from a given initial state is described unambiguously by a unitary transformation acting on that state, and in this sense there is never any loss of fundamental, *fine-grained* information.

The situation is less clear when gravitational effects are taken into account. It has been suggested [1] that fundamental information about the quantum state of matter undergoing gravitational collapse will be irretrievably lost behind the event horizon of the resulting black hole. In this view, the Hawking emission from the black hole is in the form of thermal radiation, which carries little or no information about the initial quantum state of the system. If the black hole evaporates completely, that information would be lost, in violation of the rules of quantum theory. We believe such a conclusion is unnecessary [2].

This paper is based on the assumption that black hole evolution can be reconciled with quantum theory, a viewpoint which has been most strongly advocated by ’t Hooft [3].¹ We shall introduce three postulates upon which we believe a phenomenological description of black holes should be based. These postulates extrapolate the validity of the empirically well-established principles

of quantum theory, general relativity, and statistical mechanics to phenomena involving event horizons. We argue that a phenomenological description of black holes, based on the idea of a “stretched horizon”² which can absorb, thermalize, and reemit information, is consistent with these postulates.

The postulates are the following:

Postulate 1. The process of formation and evaporation of a black hole, as viewed by a distant observer, can be described entirely within the context of standard quantum theory. In particular, there exists a unitary S -matrix which describes the evolution from infalling matter to outgoing Hawking-like radiation.

This postulate agrees with the S -matrix approach of ’t Hooft [3]. Furthermore, we assume there exists a Hamiltonian which generates the evolution for finite times.

The second postulate states the validity of semiclassical gravitation theory, including quantum corrections to the classical equations of motion, in the region outside a massive black hole. The semiclassical equations should contain enough quantum corrections to account for the outgoing Hawking flux and the evaporation of the black hole.

Postulate 2. Outside the stretched horizon of a massive black hole, physics can be described to good approximation by a set of semiclassical field equations.

No consistent formulation of such a set of equations has been achieved in four-dimensional gravity. Furthermore, the concept of a dynamical stretched horizon is quite complicated for arbitrary time-dependent black holes in four dimensions. The situation is much simpler in two-dimensional gravity and recent months have seen significant progress in constructing a semiclassical description appropriate for Postulate 2. The stretched horizon is easily defined in this simplified context. For these reasons we shall illustrate the stretched horizon

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¹This viewpoint has been recently also been put forward in the work of Schoutens, Verlinde, and Verlinde, in the context of a two-dimensional toy model [4].

²The definition of the stretched horizon will be given in Sec. III.

idea using a two-dimensional toy model. The semiclassical equations, whose nature is partly field theoretic and partly thermodynamic, describe the average energy flow and evolution of the horizon.

The third postulate is concerned with the validity of black hole thermodynamics and its connection with Postulate 1. Specifically, we assume that the origin of the thermodynamic behavior of the black hole is the coarse graining of a large, complex, ergodic, but conventionally quantum-mechanical system.

Postulate 3. To a distant observer, a black hole appears to be a quantum system with discrete energy levels. The dimension of the subspace of states describing a black hole of mass M is the exponential of the Bekenstein entropy $S(M)$ [5].

In particular, we assume there is no infinite additive constant in the entropy.

The above three postulates all refer to observations performed from outside the black hole. Although we shall not introduce specific postulates about observers who fall through the global event horizon, there is a widespread belief which we fully share. The belief is based on the equivalence principle and the fact that the global event horizon of a very massive black hole does not have large curvature, energy density, pressure, or any other invariant signal of its presence. For this reason, it seems certain that a freely falling observer experiences nothing out of the ordinary when crossing the horizon. It is this assumption which, upon reflection, seems to be sharply at odds with Postulate 1. Let us review the argument.

Consider a Penrose diagram for the formation and evaporation of a black hole, as in Fig. 1. Foliate the space-time with a family of spacelike Cauchy surfaces, as shown. Some of the Cauchy surfaces will lie partly within the black hole. Consider the surface Σ_P which contains the point P where the global event horizon intersects the curvature singularity. P partitions Σ_P into two

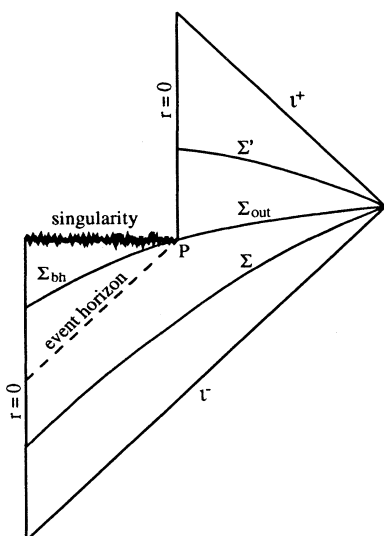


FIG. 1. Penrose diagram for black hole evolution.

disjoint surfaces Σ_{BH} and Σ_{out} which lie inside and outside the black hole, respectively.

Now assume that there exists a linear Schrödinger equation, derivable from a local quantum field theory, which describes the evolution of state vectors from one Cauchy surface to the next. An initial state $|\Psi(\Sigma)\rangle$ defined on some Cauchy surface Σ which does not intersect the black hole can be evolved without encountering any singularity until the surface Σ_P is reached. On Σ_P the Hilbert space of states \mathcal{H} can be written as a tensor product space $\mathcal{H} = \mathcal{H}_{\text{BH}} \otimes \mathcal{H}_{\text{out}}$ of functionals of the fields on Σ_{BH} and Σ_{out} , respectively.

Next, consider evolving the state further to some surface Σ' in the future, as indicated in Fig. 1. The resulting state, $|\Psi(\Sigma')\rangle$, represents the observable world long after the black hole has evaporated. According to Postulate 1, $|\Psi(\Sigma')\rangle$ must be a *pure state* which is related to the original incoming state $|\Psi(\Sigma)\rangle$ by a linear operator S , the S matrix. By assumption, $|\Psi(\Sigma')\rangle$ has evolved by the Schrödinger equation from some state $|\chi(\Sigma_{\text{out}})\rangle$ defined on Σ_{out} , which must then also be a pure state. This, in turn, implies that $|\Psi(\Sigma_P)\rangle$ must be a product state:

$$|\Psi(\Sigma_P)\rangle = |\Phi(\Sigma_{\text{BH}})\rangle \otimes |\chi(\Sigma_{\text{out}})\rangle, \quad (1.1)$$

where $|\Phi(\Sigma_{\text{BH}})\rangle \in \mathcal{H}_{\text{BH}}$ and $|\chi(\Sigma_{\text{out}})\rangle \in \mathcal{H}_{\text{out}}$. The product state is obtained by linear Schrödinger evolution from the initial state $|\Psi(\Sigma)\rangle$, but as seen above, the external factor $|\chi(\Sigma_{\text{out}})\rangle$ alone depends linearly on $|\Psi(\Sigma)\rangle$, so we arrive at the conclusion that the state inside the black hole, $|\Phi(\Sigma_{\text{BH}})\rangle$, must be independent of the initial state. In other words, all distinctions between initial states of infalling matter must be obliterated before the state crosses the global event horizon. But this is an entirely unreasonable violation of the equivalence principle. Therefore, the argument goes, the outside observer cannot see a pure state.

Although this conclusion seems to follow from fairly general principles, we believe it is unwarranted. The assumption of a state $|\Psi(\Sigma_P)\rangle$ which simultaneously describes both the interior and the exterior of a black hole seems suspiciously unphysical. Such a state can describe correlations which have no operational meaning, since an observer who passes behind the event horizon can never communicate the result of any experiment performed inside the black hole to an observer outside the black hole. The above description of the state lying in the tensor product space $\mathcal{H}_{\text{BH}} \otimes \mathcal{H}_{\text{out}}$ can only be made use of by a “superobserver” outside our Universe. As long as we do not postulate such observers, we see no logical contradiction in assuming that a distant observer sees all infalling information returned in Hawking-like radiation, and that the infalling observer experiences nothing unusual before or during horizon crossing. Only when we try to give a combined description, with a standard quantum theory valid for both observers, do we encounter trouble. Of course, it may be argued that a quantum field theoretic description of gravity dictates just such a description, whether we like it or not. If this is the case, such a quantum field theory is inconsistent with our postulates; therefore, one or the other is incorrect.

Let us now consider the process of formation and evaporation of a black hole as seen by a distant observer. It is well known that the physics of a *classical*, quasistationary black hole can be described by outside observers in terms of a “stretched horizon,” which behaves in all respects like a physical membrane with certain mechanical, electrical, and thermal properties [6–9]. The description is coarse grained in character, by which we mean that it has the typical time irreversibility and dissipative properties of a system described by ordinary thermodynamics.

The membrane is very real to an outside observer. For example, if such an observer is suspended just above the stretched horizon, he or she will observe an intense flux of energetic radiation apparently emanating from the membrane. If provided with an electrical multimeter, our observer will discover that the membrane has a surface resistivity of 377 ohms. If disturbed, the stretched horizon will respond like a viscous fluid, albeit with negative bulk viscosity. And finally, the observed entropy of the massive black hole is proportional to the area of the stretched horizon. If, on the other hand, the observer attempts to determine if the membrane is real by letting go of the suspension mechanism and falling freely past the stretched horizon, the membrane will disappear. However, there is no way to report the membrane’s lack of substance to the outside world. In this sense, there is *complementarity* between observations made by infalling observers who cross the event horizon and those made by distant observers.³

We believe that Postulates 1–3 are most naturally implemented by assuming that the coarse-grained thermodynamic description of an appropriately defined stretched horizon has an underlying microphysical basis. In other words, from the point of view of an outside observer, *the stretched horizon is a boundary surface equipped with microphysical degrees of freedom that appear in the quantum Hamiltonian used to describe the observable world.* These degrees of freedom must be of sufficient complexity that they behave ergodically and lead to a coarse-grained dissipative description of the membrane.

Much of this paper is concerned with the illustration of the concept of the stretched horizon in the context of two-dimensional dilaton gravity, for which a semiclassical description has been formulated [11–26]. We review this formalism in Sec. II. In Sec. III we define the stretched horizon and study its behavior and kinematics. The definition of the stretched horizon which we find most useful differs somewhat from that used for classical black holes in Ref. [9]. Our semiclassical stretched horizon is minimally stretched, in that its area is only one Planck unit larger than the area of the global event horizon itself, whereas in Ref. [9] the areas of the two horizons differ by a macroscopic amount. The evolution of the stretched horizon can be followed throughout the en-

tire process of black hole formation and evaporation, except for the final period when the black hole is of Planckian size. In Sec. IV, we show that the stretched horizon has statistical fluctuations which cause its area to undergo Brownian motion, and to diffuse away from its classical evolution. The semiclassical theory does not provide a microphysical description, but it helps in formulating a kinematic framework for one. In Sec. V we examine consequences of the postulates.

Our assumptions have as consequences certain broad features of the way information is stored in the approximately thermal Hawking radiation. The information is not returned slowly in far-infrared quanta long after most of the infalling energy has been reradiated. Nor is it stored in stable light remnants. It is instead found in long-time nonthermal correlations between quanta emitted at very different times, as advocated by Page [27]. The viewpoint of this paper is essentially that of ’t Hooft [3]. However, we believe that the stretched horizon is a very complex and chaotic system. Even if the microscopic laws were known, computing an S matrix [3,4] would, according to this view, be as daunting as computing the scattering of laser light from a chunk of black coal. The validity of quantum field theory in this case is not assured by exhibiting an S matrix, but by identifying the underlying atomic structure and constructing a Schrödinger equation for the many particles composing the coal and the photon field to which it is coupled. Although the equations cannot be solved, we nevertheless think we understand the route from quantum theory to apparently thermal radiation via statistical mechanics. In the case of the stretched horizon, the underlying microphysics is not yet understood, but we hope that the semiclassical considerations in this paper will help in identifying the appropriate degrees of freedom.

II. TWO-DIMENSIONAL DILATON GRAVITY

It is very useful to have a simplified setting in which to study black hole physics. Callan, Giddings, Harvey, and Strominger (CGHS) suggested for this purpose two-dimensional dilaton gravity coupled to conformal matter [11]. Their model can be exactly solved at the classical level and has solutions which are two-dimensional analogs of black holes. Quantum corrections are much more amenable to study in this theory than in four-dimensional Einstein gravity. In this section we will review the classical theory and then show how quantum corrections can be implemented via a set of semiclassical equations which can be solved explicitly. This material is not new but it serves to fix notation and makes our discussion for the most part self-contained.

A. Classical theory

The classical CGHS model of two-dimensional dilaton gravity is defined by the action functional

³A similar view has been expressed by ’t Hooft [10].

$$S_0[f_i, \phi, g] = \frac{1}{2\pi} \int d^2x \sqrt{-g} \left[e^{-2\phi} [R + 4(\nabla\phi)^2 + 4\lambda^2] - \frac{1}{2} \sum_{i=1}^N (\nabla f_i)^2 \right]. \quad (2.1)$$

It can be viewed as an effective action for radial modes of near-extreme magnetically charged black holes in four-dimensional dilaton gravity [11–13]. The two-dimensional length scale λ is inversely related to the magnetic charge of the four-dimensional black hole. For convenience, we shall choose units in which $\lambda = 1$. In the region of the four-dimensional geometry where the two-dimensional effective description applies, the physical radius of the local transverse two-sphere is governed by the dilaton field, $r(x^0, x^1) = e^{-\phi(x^0, x^1)}$. The area calculated from this radius is proportional to the Bekenstein entropy of the four-dimensional black hole and accordingly we will refer to the function

$$\mathcal{A} = e^{-2\phi} \quad (2.2)$$

as the classical “area” function in the two-dimensional effective theory. The classical equations of motion are

$$2\nabla_\mu \nabla_\nu \phi - 2g_{\mu\nu} [\nabla^2 \phi - (\nabla\phi)^2 + 1] - e^{2\phi} T_{\mu\nu} = 0, \quad (2.3)$$

$$\frac{1}{4}R + \nabla^2 \phi - (\nabla\phi)^2 + 1 = 0, \quad (2.4)$$

$$\nabla^2 f_i = 0, \quad (2.5)$$

where $T_{\mu\nu}$ is the matter energy-momentum tensor, given by

$$T_{\mu\nu} = \frac{1}{2} \sum_{i=1}^N [\nabla_\mu f_i \nabla_\nu f_i - \frac{1}{2} g_{\mu\nu} (\nabla f_i)^2]. \quad (2.6)$$

To solve the above equations we go to conformal gauge and choose light-cone coordinates (x^+, x^-) in which the line element is $ds^2 = -e^{2\rho} dx^+ dx^-$. The equations of motion are then

$$2\partial_\pm^2 \phi - 4\partial_\pm \phi \partial_\pm \rho - e^{2\phi} T_{\pm\pm} = 0, \quad (2.7)$$

$$4\partial_+ \phi \partial_- \phi - 2\partial_+ \partial_- \phi + e^{2\rho} = 0, \quad (2.8)$$

$$2\partial_+ \partial_- \rho - 4\partial_+ \partial_- \phi + 4\partial_+ \phi \partial_- \phi + e^{2\rho} = 0, \quad (2.9)$$

$$\partial_+ \partial_- f_i = 0, \quad (2.10)$$

and the nonvanishing components of the matter energy-momentum tensor are given by

$$T_{\pm\pm} = \frac{1}{2} \sum_{i=1}^N (\partial_\pm f_i)^2. \quad (2.11)$$

The action (2.1) written in conformal gauge has a global symmetry generated by the conserved current $j^\mu = \nabla^\mu(\rho - \phi)$, and thus

$$\partial_+ \partial_- (\rho - \phi) = 0. \quad (2.12)$$

This equation allows one to fix the remaining subgroup of conformal transformations by choosing coordinates in which $\rho = \phi$. We will denote any set of light-cone coordinates in which $\rho = \phi$ as *Kruskal coordinates*.

The vacuum solution is given by

$$\begin{aligned} f_i &= 0, \\ e^{-2\phi} &= e^{-2\rho} = -y^+ y^-. \end{aligned} \quad (2.13)$$

If we define new coordinates σ^\pm by the transformation $y^\pm = \pm e^{\pm\sigma^\pm}$, we find that the spacetime can be identified as two-dimensional Minkowski space, with line elements $ds^2 = -d\sigma^+ d\sigma^-$. In these coordinates, the dilaton field is given by

$$\sigma = -\frac{1}{2}(\sigma^+ - \sigma^-) \equiv -\sigma, \quad (2.14)$$

and thus this solution is called the *linear dilaton vacuum*.

B. Classical black holes

A black hole is defined as a region of spacetime which does not lie in the causal past of future null infinity \mathcal{J}^+ , i.e., light rays which have their origin inside the black hole can never escape to \mathcal{J}^+ . The global event horizon, denoted by H_G , is the boundary of the black hole region. It is a null surface representing the last light rays which are trapped by the black hole. It is important to note that the definitions of the black hole region and global event horizon are not local. To define a black hole and its global event horizon one must have knowledge of the entire spacetime manifold—in particular, one must be able to find the causal past of \mathcal{J}^+ . As a result, observers will not be able to tell when they pass through the global event horizon of a massive black hole.

The linear dilaton vacuum solution (2.13) can easily be generalized to a one-parameter family of static black hole solutions:

$$\begin{aligned} f_i &= 0, \\ e^{-2\phi} &= e^{-2\rho} = M_0 - y^+ y^-, \end{aligned} \quad (2.15)$$

where $M_0 > 0$ is proportional to the Arnowitt-Deser-Misner (ADM) mass of the black hole. The scalar curvature is given by

$$R = \frac{4M_0}{M_0 - y^+ y^-}, \quad (2.16)$$

which becomes infinite when $M_0 - y^+ y^- = 0$. Thus there are two curvature singularities, which asymptotically approach the null curves $y^\pm = 0$. The Penrose diagram for this solution is displayed in Fig. 2. One of the curvature singularities does not lie in the causal future of any point of the spacetime and is the singularity of a white hole. The other, of course, is the black hole singularity.

A more physically interesting set of solutions describes black hole formation by incoming matter,

$$\begin{aligned} f_i &= f_i^+(y^+), \\ e^{-2\phi} &= e^{-2\rho} = M(y^+) - y^+ [y^- + P_+(y^+) - P_\infty], \end{aligned} \quad (2.17)$$

where $M(y^+)$ and $P_+(y^+)$ are the following functions of the infalling matter:

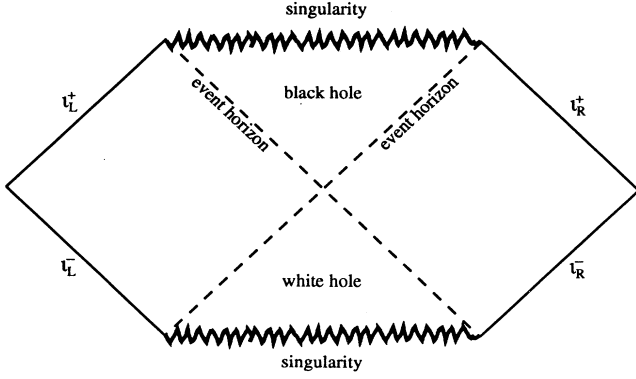


FIG. 2. Penrose diagram for the eternal black hole solution.

$$\begin{aligned} M(y^+) &= \int_0^{y^+} du u T_{++}(u), \\ P_+(y^+) &= \int_0^{y^+} du T_{++}(u), \end{aligned} \quad (2.18)$$

and $P_\infty = P_+(y^+ = \infty)$. The scalar curvature is

$$R = \frac{4M(y^+)}{M(y^+) - y^+[y^- + P_+(y^+) - P_\infty]}. \quad (2.19)$$

The functions f_i^+ are taken to be nonvanishing only on the interval $[y_1^+, y_2^+]$; i.e., the matter flux is switched on for a finite time interval. For $y^+ < y_1^+$, the solution reduces to the linear dilaton vacuum (2.13) with y^- shifted by P_∞ and, for $y^+ > y_2^+$, the solution is an eternal black-hole solution described by (2.15) with M_0 replaced by $M_\infty = M(y^+ = \infty)$. The Penrose diagram is shown in Fig. 3. The global event horizon H_G is the curve $y^- = 0$.

Kruskal coordinates are not convenient for the description of processes by an external observer. One would like to find a coordinate system which covers only the region exterior to the black hole, and reduces to Minkowski coordinates far from the black hole so that physical quan-

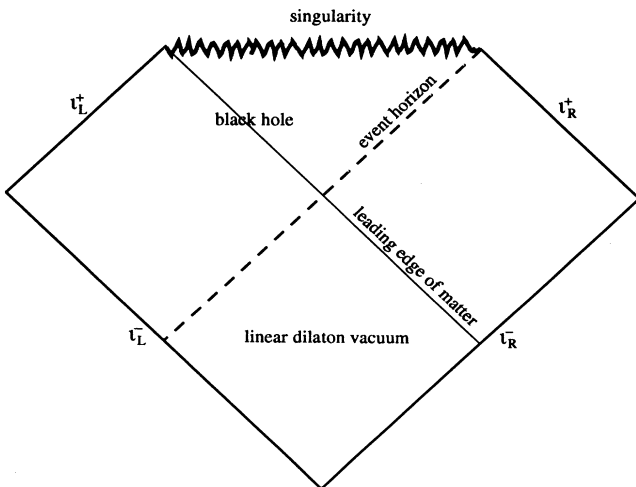


FIG. 3. Penrose diagram for the infall solution.

ties can be defined unambiguously. We define *tortoise coordinates* (t, σ) as

$$\begin{aligned} t &= \frac{1}{2} \ln(-y^+/y^-), \\ \sigma &= \frac{1}{2} \ln(-y^+y^-). \end{aligned} \quad (2.20)$$

The line element of the gravitational collapse solution (2.17) takes the form

$$ds^2 = \Lambda(t, \sigma)(-dt^2 + d\sigma^2), \quad (2.21)$$

where

$$\Lambda(t, \sigma) = \{1 + M(t, \sigma)e^{-2\phi} - [P_+(t, \sigma) - P_\infty]e^{(t-\sigma)}\}^{-1}. \quad (2.22)$$

The tortoise coordinates are asymptotically flat, and the line element is conformal to that of Minkowski space. The global event horizon is at $t = \infty, \sigma = -\infty$, and, for the eternal black-hole solution, $(\partial/\partial t)$ is a timelike Killing vector. The light-cone coordinates $\sigma^\pm = t \pm \sigma$ exactly cover \mathcal{J}_R^- and \mathcal{J}_R^+ , respectively, so we see that tortoise coordinates are the coordinates appropriate for the description of processes as seen by asymptotic inertial observers. They provide a time variable which covers the entire region accessible to an outside observer and we assume the existence of a Hamiltonian, which generates translations of this time variable.

C. Semiclassical theory

Our second postulate assumes that a semiclassical approximation to gravitation theory can be developed systematically. In the simplified world of two-dimensional dilaton gravity this can be achieved by the addition of certain quantum corrections to the classical equations of motion, as first described in the ground-breaking work of Callan *et al.* [11]. These corrections arise from the conformal anomaly of the matter fields in the theory, which takes the form

$$\langle T_\mu{}^\mu \rangle = \frac{N}{24} R. \quad (2.23)$$

The semiclassical CGHS model is obtained by adding to the classical action (2.1) the associated Liouville term:

$$\begin{aligned} S_L &= -\frac{N}{96\pi} \int d^2x \sqrt{-g(x)} \int d^2x' \sqrt{-g(x')} R(x) \\ &\quad \times G(x; x') R(x'), \end{aligned} \quad (2.24)$$

where G is a Green's function for the operator ∇^2 . This incorporates the leading-order quantum back reaction on the geometry due to the matter fields. The original CGHS equations have not been solved in closed form (see Ref. [26] for results of numerical studies) but subsequent work led to a set of semiclassical equations which can be solved exactly [20–23]. In the following we will use the model introduced by Russo, Susskind, and Thorlacius (RST) and give a summary of the results of [23,24]. This model is obtained by including in the effective action a local covariant counterterm,

$$-\frac{N}{48\pi} \int d^2x \sqrt{-g} \phi R, \quad (2.25)$$

in addition to the nonlocal Liouville term. This turns out to simplify the analysis and physical interpretation of the semiclassical solutions.

We work in conformal gauge and use light-cone coordinates (y^+, y^-) . The effective action becomes

$$S_{\text{eff}} = \frac{1}{\pi} \int d^2y \left[e^{-2\phi} (2\partial_+ \partial_- \rho - 4\partial_+ \phi \partial_- \phi + e^{2\rho}) + \frac{1}{2} \sum_{i=1}^N \partial_+ f_i \partial_- f_i - \kappa (\partial_+ \rho \partial_- \rho + \phi \partial_+ \partial_- \rho) \right], \quad (2.26)$$

where⁴ $\kappa = N/12$. The constraint equations, which follow from varying $g_{\pm\pm}$, are

$$(e^{-2\phi} + \kappa/4)(2\partial_{\pm}^2 \phi - 4\partial_{\pm} \rho \partial_{\pm} \phi) - \kappa [\partial_{\pm}^2 \rho - (\partial_{\pm} \rho)^2 - t_{\pm}] - T_{\pm\pm} = 0. \quad (2.27)$$

Here $T_{\pm\pm}$ is the physical, observable flux of energy momentum. There are subtleties involved in the regularization of the composite operator. We define $T_{\pm\pm}$ to be normal ordered with respect to the asymptotically Minkowskian tortoise coordinates (2.20).

The functions $t_{\pm}(y^{\pm})$ reflect both the nonlocal nature of the anomaly and the choice of boundary conditions satisfied by the Green's function G . They are fixed by physical boundary conditions on the semiclassical solutions.

If we define the two-component vector

$$\Phi = \begin{pmatrix} \phi \\ \rho \end{pmatrix}, \quad (2.28)$$

then the kinetic terms in the action (2.26) may be written $(\partial_+ \Phi) M (\partial_- \Phi)$, and one finds that

$$\left[-\frac{\det(M)}{4} \right]^{-1/4} = \left[e^{-2\phi} - \frac{\kappa}{4} \right]^{-1/2} \quad (2.29)$$

plays the role of the gravitational coupling constant for the f_i fields. This coupling becomes infinite on a curve γ_{cr} on which the classical area function (2.2) takes on the value

$$\mathcal{A}_{\text{cr}} = \kappa/4. \quad (2.30)$$

The curve γ_{cr} has been interpreted to be a boundary of the semiclassical spacetime [17,24], which plays the same

role as the surface $r=0$ in the Schwarzschild solution of four-dimensional Einstein gravity. Accordingly, we have to impose a boundary condition on γ_{cr} when it is timelike. Following Ref. [24], the boundary condition we use is to require the scalar curvature to be finite on γ_{cr} . This boundary condition implements a weak form of the cosmic censorship hypothesis, in that curvature singularities on γ_{cr} will necessarily be spacelike and cloaked by a global event horizon, except possibly for isolated points.

We next define the fields⁵

$$\begin{aligned} \Omega &= e^{-2\phi} + \frac{\kappa}{2} \phi, \\ \chi &= e^{-2\phi} + \kappa(\rho - \frac{1}{2}\phi), \end{aligned} \quad (2.31)$$

for which the effective action takes the simple form

$$S_{\text{eff}} = \frac{1}{\pi} \int d^2y \left[\frac{1}{\kappa} (-\partial_+ \chi \partial_- \chi + \partial_+ \Omega \partial_- \Omega) + e^{2(\chi - \Omega)/\kappa} + \frac{1}{2} \sum_{i=1}^N \partial_+ f_i \partial_- f_i \right]. \quad (2.32)$$

The resulting equations of motion and constraint equations are

$$\partial_+ \partial_- \chi = \partial_+ \partial_- \Omega = -e^{2(\chi - \Omega)/\kappa}, \quad (2.33)$$

$$\frac{1}{\kappa} [(\partial_{\pm} \Omega)^2 - (\partial_{\pm} \chi)^2] + \partial_{\pm}^2 \chi + T_{\pm\pm} - \kappa t_{\pm} = 0. \quad (2.34)$$

The field Ω can be viewed as a quantum corrected area function. At the horizon of a massive black hole it agrees to leading order with the classical area function (2.2). More specifically, we will define the semiclassical area function as

$$\mathcal{A} = \Omega - \Omega_{\text{cr}}, \quad (2.35)$$

where

$$\Omega_{\text{cr}} = \Omega(\gamma_{\text{cr}}) = \frac{\kappa}{4} \left[1 - \ln \frac{\kappa}{4} \right].$$

With this definition the area vanishes at the boundary curve.

The effective action (2.32) has a symmetry generated by the same conserved current as we had in the classical theory:

$$j^{\mu} = \nabla^{\mu}(\rho - \phi) = \frac{1}{\kappa} \nabla^{\mu}(\chi - \Omega).$$

We can therefore again choose Kruskal coordinates, in which $\chi = \Omega$, and the general solution of (2.33) takes the form

$$\begin{aligned} \chi(y^+, y^-) &= \Omega(y^+, y^-) \\ &= \alpha_+(y^+) + \alpha_-(y^-) - y^+(y^- - P_{\infty}), \end{aligned} \quad (2.36)$$

⁴We will not go into the technical issues involving reparameterization ghosts, etc., which are involved in the determination of the value of κ . Our goal here is limited to obtaining exactly solvable equations, which incorporate the leading semiclassical corrections and exhibit reasonable physical behavior, such as having a rate of Hawking radiation proportional to the number of matter fields.

⁵Note that the normalizations of the fields Ω and χ defined here differ by a factor of $\sqrt{\kappa}$ from those given in Refs. [23,24].

where the functions α_{\pm} satisfy

$$-\partial_{\pm}^2 \alpha_{\pm} = T_{\pm\pm} - \kappa t_{\pm} . \quad (2.37)$$

D. Semiclassical solutions

It was observed in Ref. [23] that the global causal nature of dynamical semiclassical geometries depends on the incoming energy flux. If the flux remains below a certain critical value,

$$T_{++}(\sigma^+) < \kappa/4 , \quad (2.38)$$

then no black hole is formed. We will describe such low-energy solutions later on. Let us first focus on the case when the incoming flux is above the critical value for some period of time, $0 < \sigma^+ < \tau$. The boundary curve then becomes spacelike and develops a curvature singularity. A global event horizon H_G separates the black hole region from the outside world. The geometry representing the black hole history in the semiclassical approximation is shown in Fig. 4. The event horizon intersects the spacelike singularity at the end point of the evaporation process, (y_E^+, y_E^-) . The line segment $y^- = y_E^-$, $y^+ < y_E^+$ is the global event horizon. The extension of this line to $y^+ > y_E^+$ was called the thunderpop in Ref. [23], and it divides the spacetime into two regions called I and II as shown in Fig. 4. Region II represents the spacetime after the last bit of Hawking radiation has gone past and therefore it is vacuumlike.⁶ Region I covers the rest of the spacetime.

The solution in region I is

$$\Omega = -y^+ [y^- + P_+(y^+) - P_{\infty}] + M(y^+) - \frac{\kappa}{4} \ln[-y^+(y^- - P_{\infty})] , \quad (2.39)$$

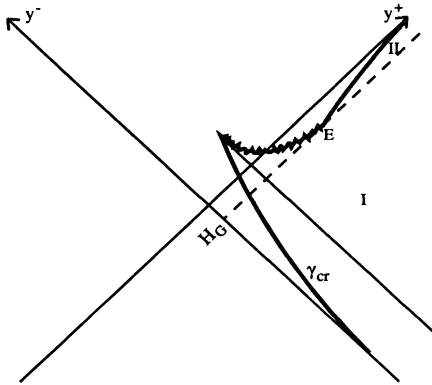


FIG. 4. Semiclassical black hole formation and evaporation in Kruskal coordinates.

⁶The sharply defined end point of the Hawking emission is presumably an artifact of the semiclassical approximation in this model. A more physical behavior would be for the outgoing flux to die out gradually. We will return to this point in Sec. III D.

and in region II it is the vacuum solution given by

$$\Omega = -y^+ y^- - \frac{\kappa}{4} \ln(-y^+ y^-) . \quad (2.40)$$

Note that the global event horizon is not at $y^- = 0$, as it was in the classical case, but rather at

$$y_E^- = -\frac{P_{\infty}}{e^{4M_{\infty}/\kappa} - 1} . \quad (2.41)$$

For a large black hole mass, y_E^- is exponentially close to zero. The line $y^- = 0$ still has special significance. First of all it is the asymptotic limit of the boundary curve γ_{cr} as $y^+ \rightarrow \infty$. Therefore, it defines the boundary of the region covered by the tortoise coordinates which are appropriate for asymptotic observers. Furthermore, if it were possible for signals to propagate through the singularity along lines of constant y^- and reappear in the final vacuumlike region, then $y^- = 0$ would indeed be the global horizon. We will call it the *ultimate horizon*. At any rate, for massive black holes the values of y^- at the ultimate and global horizons are extremely close.

If the incoming energy flux remains below its critical value at all times the boundary curve is everywhere timelike. Semiclassical solutions will have singularities there unless appropriate boundary conditions are imposed [23,24]. The curvature will be finite at the timelike boundary if and only if

$$\partial_+ \Omega|_{\Omega=\Omega_{cr}} = \partial_- \Omega|_{\Omega=\Omega_{cr}} = 0 . \quad (2.42)$$

These boundary conditions, along with the semiclassical equations of motion, are sufficient to uniquely determine both the shape of the boundary curve and the values of the semiclassical fields everywhere in spacetime, for a given incoming energy flux. We shall describe some of these solutions in Sec. III C. Despite having some attractive features these semiclassical solutions have some unphysical properties. This was part of our motivation to develop a more physical picture in terms of a “stretched horizon.”

III. THE STRETCHED HORIZON

Our postulates require us to build a theory in which a distant observer makes no reference to events inside a black hole. For this purpose it is very useful to introduce the idea of a stretched horizon H_S , which is a visible timelike curve, in front of the global event horizon of the black hole. Each point on H_S is identified with a point on H_G , so the stretched horizon can act as a “surrogate” for the global horizon in a phenomenological description of black hole evolution.

A. Definition and properties of the classical stretched horizon

We define the classical stretched horizon as follows. Consider the classical area function (2.2) along the global event horizon. For a black hole formed by gravitational collapse, depicted in Fig. 3, this area increases with y^+ until the black hole has settled to its final size. We define the stretched horizon by mapping each point m on the

event horizon back along a past-directed null line (away from the event horizon itself) to a point p at which

$$\mathcal{A}(p) = \mathcal{A}(m) + \delta, \quad (3.1)$$

where δ is an arbitrary small constant. This results in a timelike curve as indicated in Fig. 5.

Note that our definition differs from that given in Ref. [9]. There the shift in the area between the global event horizon and the stretched horizon scales like the horizon area itself. For a massive black hole our stretched horizon is thus much closer to the event horizon. Our definition is better suited for the semiclassical theory of two-dimensional gravity considered here, and may also be appropriate for the quantum description of black holes in four spacetime dimensions. For the classical black hole solutions (2.17) one finds the remarkably simple result that in Kruskal coordinates the stretched horizon curve is independent of the incoming energy flux, $T_{++}(y^+)$, and the curve is given by

$$-y^+ y^- = \delta. \quad (3.2)$$

The simplicity of the stretched horizon becomes even more apparent in the tortoise coordinates (2.20). While the event horizon lies at $t = \infty$ and $\sigma = -\infty$, the stretched horizon is at fixed spatial position

$$\sigma_S = \frac{1}{2} \ln \delta. \quad (3.3)$$

Thus the stretched horizon can receive and emit signals. Furthermore, to distant observers, clocks at the event horizon appear infinitely slowed, while they appear to run at a finite rate at the stretched horizon. For an eternal black hole of mass M_0 , proper time τ along H_S is related to coordinate time t by

$$d\tau = \left[\frac{\delta}{M_0 + \delta} \right]^{1/2} dt. \quad (3.4)$$

Finally, one can consider the Hawking temperature of a massive two-dimensional dilaton black hole, $T = 1/2\pi$,

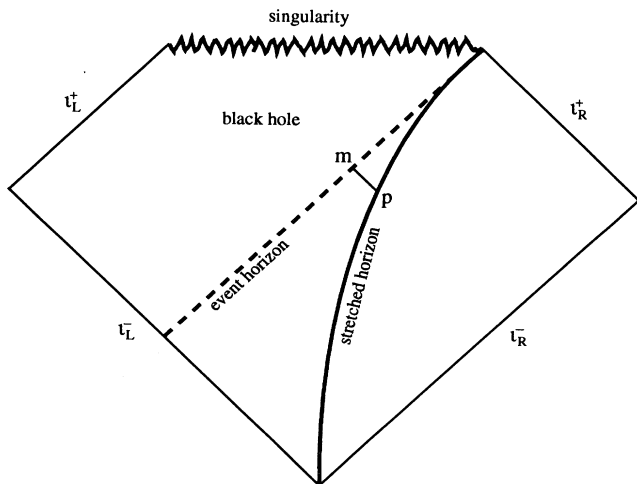


FIG. 5. Construction of the classical stretched horizon.

which is independent of the mass. Since temperature has units of energy, then in proper time units at the stretched horizon the temperature is

$$T_S = \frac{1}{2\pi} \frac{dt}{d\tau} = \frac{1}{2\pi} \left[\frac{M_0 + \delta}{\delta} \right]^{1/2}. \quad (3.5)$$

From (3.5) it appears that the local temperature at the stretched horizon increases with M_0 . This is a bit misleading, because the analogue of the Planck length in two-dimensional dilaton gravity depends on the local value of the dilaton field, according to $l_{pl} \sim e^\phi$. At the stretched horizon, the dilaton field satisfies $e^{-\phi} = \sqrt{M_0 + \delta}$, so (3.5) implies that, measured in Planck units, the temperature at the stretched horizon is independent of the mass. This result also holds for four-dimensional black holes as we will see in Sec. V A.

Let us continue examining the classical behavior of the stretched horizon. Consider the evolution of the area \mathcal{A} on H_S . Parametrizing H_S by y^+ and substituting the definition (3.2) of H_S into the gravitational collapse solution (2.17) we find

$$\mathcal{A}_S(y^+) = M(y^+) + \delta - y^+ [P_+(y^+) - P_\infty], \quad (3.6)$$

which, when differentiated twice, gives

$$\frac{d^2 \mathcal{A}_S}{(dy^+)^2} = -T_{++}(y^+). \quad (3.7)$$

Transforming to tortoise coordinates, one can parametrize H_S by the tortoise time $t = \ln y^+ - \frac{1}{2} \ln \delta$. Equation (3.7) then becomes

$$\frac{d^2 \mathcal{A}_S}{dt^2} - \frac{d\mathcal{A}_S}{dt} = - \left[\frac{dy^+}{dt} \right]^2 T_{++}(y^+) \equiv -T_{++}(\sigma^+) \quad (3.8)$$

where the quantity $T_{++}(\sigma^+)$ is the incoming physical flux of energy as seen by a distant observer. There are two interesting features of (3.8). The first has to do with the nature of the boundary conditions on the solutions of the equation. In general, the stretched horizon will begin to grow even before any energy crosses it. From (3.8) we see that before T_{++} becomes nonzero, \mathcal{A}_S has the solution

$$\mathcal{A}_S(t) = Ce^t. \quad (3.9)$$

The choice of the constant C is dictated by *final* conditions. As $t \rightarrow \infty$, a black hole is present with mass M_∞ . The area of the stretched horizon of such a black hole is

$$\lim_{t \rightarrow \infty} \mathcal{A}_S(t) = M_\infty + \delta, \quad (3.10)$$

and thus (3.10) is the boundary condition one must impose on the solution of (3.8). This means that the initial state of the stretched horizon must be tuned in conjunction with the incoming matter distribution so that (3.10) is satisfied. This strange feature has been referred to in the membrane paradigm literature as the “teleological boundary condition” [9]. We will show in Sec. III D how the equations can, in fact, be given a more conventional

and causal interpretation.

The second interesting feature of (3.8) is the dissipative term $d\mathcal{A}_S/dt$. It breaks time-reversal symmetry much like a friction term in ordinary mechanics. The presence of dissipative terms in mechanics is generally associated with the production of heat and the increase of thermal entropy. In the classical case, the temperature of the black hole is zero, but in the semiclassical case the temperature increases when the black hole is formed, and the stretched horizon appears to radiate like a thermally excited blackbody.

In the limit of large black holes, we can also consider the theory of a massless matter field f propagating in the black hole background. We find that the equations governing the matter fields interacting with the stretched horizon also exhibit dissipation. In the case of a four-dimensional black hole interacting with electromagnetic fields, the phenomenon of dissipation is described by attributing an ohmic resistance to the membrane. A similar description can be given for two-dimensional dilaton black holes. Indeed, the theory of a massless field f bears a useful resemblance to ordinary classical electrodynamics, with f playing the role of the vector potential. We work in the tortoise coordinates (2.20) and define the “electric” and “magnetic” fields E and B by

$$E = -\nabla_t f, \quad B = \nabla_\sigma f. \quad (3.11)$$

The equation of motion for the f field is

$$\nabla^2 f = -4\pi J, \quad (3.12)$$

where we have introduced a source j . Writing this equation in terms of the fields E and B , we obtain an inhomogeneous “Maxwell” equation

$$\nabla_t E + \nabla_\sigma B = -4\pi J. \quad (3.13)$$

We can also obtain the homogeneous “Maxwell” equation $\nabla_\sigma E + \nabla_t B = 0$.

Now we consider the interaction of the f field with the stretched horizon, which, from the point of view of an external observer, is a boundary absorbing all incoming waves. This behavior can be modeled by attributing a resistance to the stretched horizon:

$$\nabla_t E + \nabla_\sigma B = -4\pi\rho_S^{-1}\delta(\sigma - \sigma_S)E. \quad (3.14)$$

An incoming f wave will be completely absorbed if and only if $\rho = 4\pi$. This is the analogue of the surface electrical resistivity of a four-dimensional black hole. The power absorbed by the stretched horizon is $\rho_S^{-1}(\partial_t f)^2$, which can be thought of as ohmic heating. When quantum corrections are included, the heat is radiated back as Hawking radiation.

B. The semiclassical stretched horizon

In defining the stretched horizon of a semiclassical black hole, we find it more convenient to refer to the ultimate horizon at $y^- = 0$ than the event horizon (2.41). For a large black hole, the difference is negligible. We also replace the classical area function (2.2) by its semiclassical counterpart (2.35). This leads to the following

condition for points on the stretched horizon:

$$-y^+ y^- - \frac{\kappa}{4} \ln \left[1 - \frac{y^-}{P_\infty} \right] = \delta, \quad (3.15)$$

where we have used the black-hole solution (2.39). If the incoming energy is large, then y^-/P_∞ will be very small on H_S , except in the extremely early stages of its evolution. Thus, we will drop the log term in the definition. In the classical case, δ is an arbitrary small number. In the semiclassical theory, there is a natural choice, $\delta = \kappa/4$, for which the area of the stretched horizon vanishes in the asymptotic past and future when there is no black hole. This implies that the stretched horizon will coincide with the boundary curve γ_{cr} in these limits. Thus, we define the stretched horizon to be the set of points satisfying the condition

$$-y^+ y^- = \kappa/4. \quad (3.16)$$

In tortoise coordinates, the stretched horizon is given by the curve

$$\hat{\sigma}_S(t) = \frac{1}{2} \ln \left[\frac{\kappa}{4} \right] = \sigma_S. \quad (3.17)$$

Let us now consider the black hole evolution in tortoise coordinates as shown in Fig. 6. The incoming flux is assumed to be vanishing outside the interval $0 < \sigma^+ < \tau$. For $y^+ < 1$, i.e., $\sigma^+ < 0$, we have the initial linear dilaton vacuum and the boundary curve is given by

$$\hat{\sigma}_{\text{cr}}(t) = \ln \left\{ \frac{P_\infty e^t}{2} \left[\left(1 + \frac{\kappa e^{-2t}}{P_\infty^2} \right)^{1/2} - 1 \right] \right\}. \quad (3.18)$$

In the remote past, this curve tends to

$$\hat{\sigma}_{\text{cr}}(t) \rightarrow \sigma_S - \frac{P_\infty}{\sqrt{\kappa}} e^t. \quad (3.19)$$

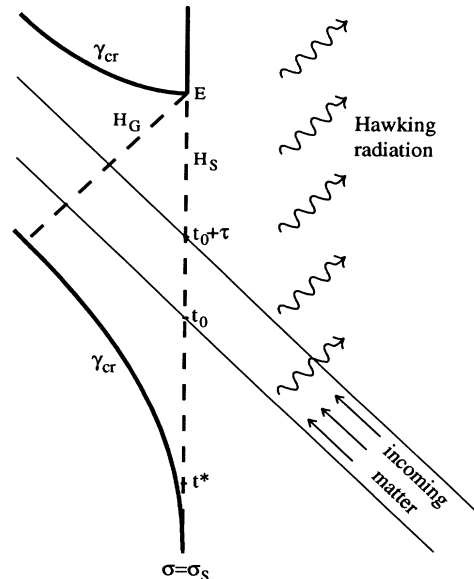


FIG. 6. Black hole evolution in tortoise coordinates.

We see that the boundary begins to separate from the stretched horizon at a time

$$t^* = -\ln P_\infty + \frac{1}{2} \ln \kappa . \quad (3.20)$$

The incoming matter arrives at the stretched horizon at time $t_0 = -\sigma_S$. Consequently, we see a period of time $\sim \ln P_\infty$ during which the boundary moves in anticipation of the infalling matter. It continues to move toward $\sigma = -\infty$ with a velocity which approaches that of light.

A second boundary curve passes through the naked singularity at the end point of the black-hole evaporation. Behind the stretched horizon the second boundary curve is spacelike and coincides with the curvature singularity. This is shown in Fig. 6. The semiclassical viewpoint is that the infalling matter becomes trapped between these boundary lines and disappears into a spatially disconnected region. However, our postulates do not require us to pay any attention to this region, as it lies behind the stretched horizon.

Now let us turn to the outgoing Hawking radiation. Using the $(- -)$ constraint, Eq. (2.34), one finds the outgoing Hawking flux

$$T_{--}(t, \sigma) = \frac{\kappa}{4} \left[1 - \frac{1}{(1 - P_\infty e^{(t-\sigma)})^2} \right] \Theta(t_E - t + \sigma - \sigma_S) . \quad (3.21)$$

where

$$\begin{aligned} t_E &= \ln(e^{4M_\infty/\kappa} - 1) - \ln(P_\infty) + \sigma_S \\ &\approx 4M_\infty/\kappa - \ln(P_\infty) + \sigma_S . \end{aligned}$$

The outgoing flux has its leading, albeit somewhat fuzzy, edge along the null curve

$$\sigma - t \approx \ln(P_\infty) . \quad (3.22)$$

Tracing this line back to the stretched horizon we find that it intersects the stretched horizon at the time

$$t^* = t_0 - \ln P_\infty . \quad (3.23)$$

If we interpret the outgoing thermal radiation as originating on the stretched horizon, it begins at a time well before the incoming matter arrives. For early times, the semiclassical area of the stretched horizon is approximately

$$\mathcal{A}_S \approx \frac{1}{2} P_\infty^2 e^{2t} . \quad (3.24)$$

The radiation begins *just as the area of the stretched horizon begins to increase*. The radiation has turned on by the time the area (and entropy) of the stretched horizon have increased to their values at t^* , given by $\mathcal{A}^* \approx \kappa/8$.

The correspondence between the onset of Hawking radiation and the excitation of the stretched horizon is unexpected. From a strictly local point of view, nothing special is happening at this point.

It is straightforward to generalize the equation (3.8) governing the evolution of the area of the stretched horizon. Using (2.39) and (3.16) and parametrizing H_S by y^+ , we find

$$\begin{aligned} \mathcal{A}_S(y^+) &= M(y^+) - y^+ [P_+(y^+) - P_\infty] \\ &\quad + \frac{\kappa}{4} \left[1 - \ln \left[\frac{\kappa}{4} + y^+ P_\infty \right] \right] . \end{aligned} \quad (3.25)$$

Differentiating twice with respect to y^+ and transforming to tortoise coordinates gives

$$\begin{aligned} \frac{d^2 \mathcal{A}_S}{dt^2} - \frac{d \mathcal{A}_S}{dt} &= -T_{++}(t, \sigma_S) \\ &\quad + \frac{\kappa}{4} \left[\frac{1}{\{1 + (\kappa/4P_\infty) \exp[-(\sigma_S - t)]\}^2} \right] . \end{aligned} \quad (3.26)$$

Once the stretched horizon area is significantly greater than $\kappa/4$, the second term on the right-hand side can be simplified to $\kappa/4$, giving

$$\frac{d^2 \mathcal{A}_S}{dt^2} - \frac{d \mathcal{A}_S}{dt} = -T_{++}(t, \sigma_S) + \frac{\kappa}{4} . \quad (3.27)$$

The second term on the right-hand side of (3.27) represents the effects of the outgoing Hawking radiation on the evolution of \mathcal{A}_S . For example, we see that a stationary solution is possible if $T_{++}(t, \sigma) = \kappa/4$. In this case, the incident energy flux is just sufficient to balance the outgoing thermal radiation. In Sec. IV, we will see that things are somewhat more complicated, and that \mathcal{A}_S has a Brownian motion superimposed on its average motion.

Let us now review the process of formation and evaporation as seen by a distant observer using tortoise coordinates. The infalling matter is scheduled to begin passing the stretched horizon at time t_0 . However, well before this, at time $t_0 - \ln P_\infty$, the stretched horizon begins to separate from the boundary, and its area increases by an amount of order $\kappa/8$. Assuming the standard connection between entropy and area, this is the point at which the stretched horizon becomes thermally excited. The distant observer sees the onset of Hawking radiation originating from this point. At the time t_0 , the infalling matter is swallowed behind the stretched horizon, which continues to radiate. If we assume that there are microphysical degrees of freedom which underlie the thermodynamic description, t_0 is the first opportunity for them to feel the infalling matter. Therefore, at least for the initial time of order $\ln P_\infty$, no information can be stored in the Hawking radiation [4].

After the infalling matter is absorbed, the area begins to decrease. The acceleration term in (3.27) goes to zero, and \mathcal{A}_S satisfies

$$d \mathcal{A}_S / dt = -\kappa/4 . \quad (3.28)$$

The area, entropy, and mass of the black hole tend linearly to zero. The entire process from t_0 to the end point at which \mathcal{A}_S returns to its initial value takes a time $t \approx 4M/\kappa$, during which a constant flux of Hawking radi-

ation is emitted by the stretched horizon. As we shall see in Sec. IV, the entire semiclassical evolution is accompanied by random Brownian fluctuations, which introduce an uncertainty of order \sqrt{M} to the lifetime of the process.

C. Incident flux below the black hole threshold

We now consider the case in which the incident energy flux remains below the critical value $\kappa/4$ for all time. The resulting geometry and outgoing flux of radiation was obtained in Ref. [24]. Here we will transcribe some of those results into tortoise coordinates. Assume that the incoming energy flux $T_{++}(\sigma^+)$ vanishes outside the interval $0 < \sigma^+ < \tau$. The timelike boundary curve γ_{cr} is obtained by solving the equations of motion (2.33) subject to the RST boundary conditions (2.42). In tortoise coordinates, γ_{cr} satisfies the equation

$$e^{\hat{\sigma}_{\text{cr}}(t)} - \frac{\kappa}{4} e^{-\hat{\sigma}_{\text{cr}}(t)} = - \int_{\hat{\sigma}_{\text{cr}}(t)}^{\infty} ds e^{-s} T_{++}(t+s). \quad (3.29)$$

As in the black-hole case, in the remote past the boundary curve tends to a fixed spatial position,

$$\sigma_S = \frac{1}{2} \ln \left[\frac{\kappa}{4} \right], \quad (3.30)$$

which we will continue to call the location of the stretched horizon, even though no black hole is formed. The boundary begins to move exponentially in anticipation of the incoming matter, as it did in the black-hole case, but this time γ_{cr} remains timelike throughout the evolution, and eventually returns to the stretched horizon at the time $t_f = \tau - \sigma_S$. This is shown in Fig. 7.

The outgoing flux is obtained by applying the reflection conditions [24]

$$T_{--}(\sigma^-) - \frac{\kappa}{4} = \left[\frac{d\hat{\sigma}_{\text{cr}}^+}{d\sigma^-} \right]^2 \left[T_{++}(\hat{\sigma}_{\text{cr}}^+) - \frac{\kappa}{4} \right], \quad (3.31)$$

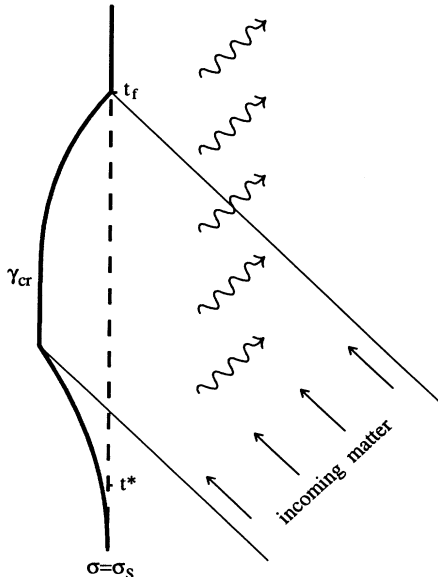


FIG. 7. Subcritical flux of incident matter.

where $\hat{\sigma}_{\text{cr}}^+(\sigma^-)$ denotes the boundary curve γ_{cr} parametrized by σ^- . Note that this prescription for reflecting energy flux does not involve boundary conditions imposed directly on the matter fields.

It is instructive to consider a constant incoming energy flux, $T_{++}(\sigma^+) = \bar{T}$, of duration τ . The early time behavior of the boundary curve is given by

$$\hat{\sigma}_{\text{cr}}(t) = \sigma_S - \frac{P_\infty}{\sqrt{\kappa}} e^t. \quad (3.32)$$

The boundary curve continues to recede from the stretched horizon until the incoming flux intersects γ_{cr} , as shown in Fig. 7. If \bar{T} is much smaller than $\kappa/4$, the boundary curve never moves appreciably away from the stretched horizon, but if \bar{T} is close to $\kappa/4$, γ_{cr} moves deep into the region of negative σ . The maximum coordinate distance between γ_{cr} and H_S is

$$\sigma_{\text{max}} = \frac{1}{2} \ln \left[\frac{\kappa}{4} - \bar{T}(1 - e^{-\tau}) \right], \quad (3.33)$$

which occurs at time $t_{\text{max}} = -\sigma_{\text{max}}$. After that the boundary curve begins to return to the stretched horizon. How fast it returns depends on the parameters \bar{T} and τ . In the limit of very long duration of the incoming energy flux, $\tau \gg 1$, the boundary remains practically stationary for a long time at its maximum distance, but eventually it returns and arrives back at the stretched horizon at time $t_f = \tau - \sigma_S$. If, on the other hand, the duration of the incoming energy flux is relatively short the boundary curve rapidly returns and approaches H_S with a velocity

$$v \approx \frac{\bar{T}}{\kappa/2 - \bar{T}}. \quad (3.34)$$

When the incoming energy flux goes to the critical value $\kappa/4$ this velocity approaches the speed of light.

In Ref. [24], it was speculated that the critical boundary might behave like a moving mirror reflecting the f_i fields. We can now see that this can only be consistent in the limit of small incoming energy flux, $\bar{T} \ll \kappa/4$. For if $\bar{T} \approx \kappa/4$, the incoming radiation would be met by a very relativistic mirror, which would greatly blueshift the reflected radiation. In addition, accelerated mirrors create incoherent quantum radiation of net positive energy. The result would be far more energy output than the total incoming energy. It is therefore clear that only a tiny fraction of the incident energy can be coherently reflected by the boundary when $\bar{T} \approx \kappa/4$.

The outgoing flux of energy can be calculated from (3.31) and one finds that almost all of the energy is radiated back before the incoming signal could have been reflected from the boundary curve γ_{cr} . The following odd rule gives a better account of the energy output as determined by the RST boundary conditions in the case $\bar{T} \approx \kappa/4$: assume that at time $t^* = -\ln(P_\infty/\sqrt{\kappa})$, when the boundary curve γ_{cr} separates from H_S , the stretched horizon becomes thermally excited to a temperature $T \approx 1/2\pi$. Assume that the hot horizon emits thermal radiation at a fixed rate until time t_f . The total radiated energy will be

$$E_{\text{out}} \approx \frac{\kappa}{4}(t_f - t^*), \quad (3.35)$$

which accounts for about the right amount of energy output. We will provide further motivation for this alternative viewpoint in the following subsection.

Another interesting point concerns the fate of the conserved charges associated with the global $O(N)$ symmetry of the matter fields. Only when the energy flux \bar{T} is much less than $\kappa/4$ can the boundary curve consistently behave like a mirror, reflecting both the energy and the conserved charges. When $\bar{T} \approx \kappa/4$, almost all the energy is radiated before the charges have an opportunity to reflect. If a large total charge of order $(t_f - t^*)$ came in and was reflected, it would have to be carried by a small energy, of order κ . In other words, it would have to be carried out in the form of quanta with energy of order $\kappa/(t_f - t^*)$ which would take a very long time. It is easy to see, however, that as \bar{T} approaches $\kappa/4$, the reflected region does not spread out as it would have to if it were composed of quanta of ever lower energy. Therefore, only a small amount of conserved charge can be reflected. In addition to thermalizing the incident energy, the process must also destroy the conservation of quantum numbers.

Another failure of the boundary to behave like a mirror can be illustrated by considering an interruption in an otherwise uniform incoming flux—a glitch. If the boundary behaved like a mirror, a brief sharp interruption would be expected in T_{--} where the glitch reflects off the boundary, but an explicit calculation shows that this is not the case.

D. A causal description of the stretched horizon

Our aim is a self-contained description of black hole evolution as seen by a distant observer in which no reference need be made to events behind the stretched horizon. It will also become clear that such a formulation has significant advantages in the low-energy sector, compared to a semiclassical description which focuses on boundary conditions imposed at the boundary curve γ_{cr} . In particular, the stretched horizon offers a unified view, in which it is no longer necessary to treat the cases of large and small incident energy flux separately.

We saw earlier that the stretched horizon begins to expand in what appears to be a teleological manner before the incoming matter arrives. One might be concerned that this would preclude a conventional causal Hamiltonian description of the quantum stretched horizon. We do not believe this to be the case. From a formal point of view, the cause of the horizon expansion is a gravitational dressing which is attached to the incoming energy flux.

Consider the initial state description in tortoise coordinates. Suppose an incoming flux of energy is described by $T_{++}(\sigma^+)$. The functions P_+ and M are given in tortoise coordinates by

$$\begin{aligned} P_+(\sigma^+) &= \int_{-\infty}^{\sigma^+} du e^{-u} T_{++}(u), \\ M(\sigma^+) &= \int_{-\infty}^{\sigma^+} du T_{++}(u), \end{aligned} \quad (3.6)$$

and the field Ω is given by

$$\begin{aligned} \Omega &= e^{2\sigma} + [P_\infty - P_+(\sigma^+)]e^{\sigma^+} + M(\sigma^+) \\ &\quad - \frac{\kappa}{4}[\sigma^+ + \ln(P_\infty + e^{-\sigma^-})]. \end{aligned} \quad (3.37)$$

Let us subtract from Ω the functional form $\bar{\Omega}$ which it would have, in the absence of any incoming matter,

$$\bar{\Omega} = e^{2\sigma} - \frac{\kappa}{2}\sigma. \quad (3.38)$$

We obtain

$$\omega \equiv \Omega - \bar{\Omega} = \omega_{\text{in}}(\sigma^+) + \omega_{\text{out}}(\sigma^-), \quad (3.39)$$

where

$$\begin{aligned} \omega_{\text{in}}(\sigma^+) &= [P_\infty - P_+(\sigma^+)]e^{\sigma^+} + M(\sigma^+), \\ \omega_{\text{out}}(\sigma^-) &= -\frac{\kappa}{4} \ln(1 + P_\infty e^{\sigma^-}). \end{aligned} \quad (3.40)$$

We see that the free field ω consists of an incoming part and an outgoing part. We can use the outgoing ω field to determine the outgoing energy-momentum flux.

Since the incoming part is completely determined by the incoming energy flux, we will consider it to be a “dressing” of the incoming matter. It can be written

$$\omega_{\text{in}}(\sigma^+) = \int du T_{++}(u) \mathcal{W}(\sigma^+ - u), \quad (3.41)$$

where

$$\mathcal{W}(\sigma^+ - u) = \Theta(u - \sigma^+) e^{(\sigma^+ - u)} + \Theta(\sigma^+ - u). \quad (3.42)$$

In other words, a bit of energy δM arriving along the curve $\sigma^+ = u$ must be accompanied by an ω dressing which has the value $\mathcal{W}(\sigma^+ - u)\delta M$. The ω dressing precedes the incoming f_i flux and is the first thing that strikes the stretched horizon. By time-reversal symmetry, a bit of outgoing energy δM departing along the curve $\sigma^- = v$ also has an ω dressing given by $\mathcal{W}(v - \sigma^-)\delta M$.

In a complete quantum theory, the outgoing state would be described by a vector in the physical state space of f_i particles, from which it would be possible to compute the expectation value of T_{--} . This would not be feasible, however, even if we knew the exact nature of the microstructure of the stretched horizon. As we shall now see, thermodynamic arguments can give information about ω_{out} , which is sufficient to compute T_{--} .

The ω dressing of the incoming matter satisfies

$$(\partial_+ - \partial_+^2)\omega_{\text{in}}(\sigma^+) = T_{++}. \quad (3.43)$$

By time reversal we obtain a relation between ω_{out} and T_{--} :

$$(\partial_- + \partial_-^2)\omega_{\text{out}}(\sigma^-) = -T_{--}. \quad (3.44)$$

This can be written as a condition at the stretched horizon:

$$\frac{d\omega_{\text{out}}}{dt} = -T_{--}(t, \hat{\partial}_S(t)) - \frac{d^2\omega_{\text{out}}}{dt^2}. \quad (3.45)$$

The outgoing thermal flux T_{--} is assumed to originate

at the thermally excited stretched horizon. The entropy of the stretched horizon is given by $\omega = \omega_{\text{in}} + \omega_{\text{out}}$, since at the stretched horizon, $\omega = \mathcal{A}$.

In thermal equilibrium, T_{--} should be a well-defined function of the thermodynamic state of the stretched horizon, and thus of the entropy ω_S . More generally, for nonequilibrium processes such as the onset and end of Hawking evaporation, the flux may depend on the detailed time history. Nevertheless, we will consider a simplified model in which T_{--} depends only on the instantaneous value of ω_S and the sign of its time derivative. In other words, we shall allow for the possibility that $T_{--}(\omega)$ has different functional forms at the beginning and end of the evaporation. Thus, we assume the radiated flux is a function of ω ,

$$T_{--} = T_{--}(\omega) = T_{--}(\omega_{\text{in}} + \omega_{\text{out}}). \quad (3.46)$$

Substituting this into (3.45) gives

$$\dot{\omega}_{\text{out}} + \ddot{\omega}_{\text{out}} = -T_{--}(\omega_{\text{out}} + \omega_{\text{in}}). \quad (3.47)$$

Since $\omega_{\text{in}}(t)$ is known in terms of T_{++} , we have obtained a differential equation for ω_{out} .

To obtain agreement with the semiclassical theory for large black holes, we assume that $T_{--}(\omega)$ approaches the value $\kappa/4$ for $\omega \gg \kappa$. Of course, $T_{--}(\omega)$ should be zero for $\omega = 0$, which is the ground state of the stretched horizon. Furthermore, if one requires that the onset of the Hawking radiation, for a massive black hole, agrees with the semiclassical result (3.21), then one is led to a specific form for $T_{--}(\omega)$ during the heating phase at early times. This can be constructed as follows. Define a function $z(\omega)$ by

$$\omega = z(\omega) - \frac{\kappa}{4} \ln \left[1 + \frac{4}{\kappa} z(\omega) \right], \quad (3.48)$$

then choose

$$T_{--}(\omega) = \frac{\kappa}{4} \left[1 - \frac{1}{[1 + (4/\kappa)z(\omega)]^2} \right]. \quad (3.49)$$

For massive black holes, it can be shown that $\omega(t)$ as given by (3.39) solves (3.47) until near the end of the evaporation.

We can now see an important advantage of the stretched horizon formulation, concerning the endpoint of Hawking evaporation. In the semiclassical RST model [23], overall energy conservation could only be achieved by having a negative energy ‘‘thunderpop’’ at the end of the evaporation process. The negative energy was bounded and small, but nevertheless an embarrassment [25]. In contrast, consider the sum of (3.43) and (3.44), evaluated at the stretched horizon,

$$\dot{\omega}_{\text{in}} - \ddot{\omega}_{\text{in}} + \dot{\omega}_{\text{out}} + \ddot{\omega}_{\text{out}} = T_{++} - T_{--}. \quad (3.50)$$

Integrating both sides of (3.50) over time reveals that energy is conserved, i.e.,

$$\int dt T_{++} = \int dt T_{--} \quad (3.51)$$

as long as ω begins *and* ends at zero. This is assured in the remote past, when the stretched horizon coincides with the boundary curve. For the late time evolution of ω , we return to (3.47). From (3.40) we see that the late value of ω_{in} is the total infalling mass, so $\omega_{\text{in}} \rightarrow M_\infty$. Inserting this into (3.47) leads to the following differential equation for ω_{out} at late time:

$$\ddot{\omega}_{\text{out}} + \dot{\omega}_{\text{out}} + T_{--}(\omega_{\text{out}} + M_\infty) = 0. \quad (3.52)$$

This is the equation for the damped motion of a particle subject to a restoring force, with equilibrium position at $\omega_{\text{out}} + M_\infty = 0$. Provided the motion is overdamped, we find that ω , and therefore T_{--} , tend smoothly to zero at late times. This places a condition on $T_{--}(\omega)$, namely that it goes to zero no slower than $\omega/4$. Comparing with (3.48) and (3.49) we see that T_{--} must depend differently on ω during the cooling and heating phases.

IV. BROWNIAN MOTION OF THE HORIZON⁷

Lagrangian mechanics and thermodynamics are quite different descriptions of a system. According to the usual principles of Lagrangian mechanics, the motion of any system is reversible and the concepts of heat and entropy have no place. Thermodynamics, on the other hand, is the theory of the irreversible dissipation of organized energy into heat. The thermodynamic description arises from the coarse graining of the mechanical description, in which configurations which are macroscopically similar are considered identical.

The equations of semiclassical gravity are peculiarly thermodynamic near the stretched horizon. In this section, we will see that they include another effect that generally occurs in thermodynamic systems, namely, random fluctuation and diffusion. That such an effect should occur was pointed out to us by Seiberg and Shenker [29]. Specifically, we shall see that the area of a two-dimensional dilaton black hole undergoes Brownian motion and diffuses away from its semiclassical value. This phenomenon can be independently understood from thermodynamics and quantum field theory.

We begin by recalling the Einstein relation between specific heat and energy fluctuations. The average energy and squared energy of a system in thermal equilibrium with a heat bath are

$$\langle E \rangle = -\frac{1}{Z} \frac{\partial Z}{\partial \beta}, \quad (4.1)$$

$$\langle E^2 \rangle = \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2}, \quad (4.2)$$

where Z is the partition function and β is the inverse temperature. From (4.1) and (4.2) it follows that

$$\frac{\partial \langle E \rangle}{\partial \beta} = [\langle E \rangle^2 - \langle E^2 \rangle] = -\text{Var}(E), \quad (4.3)$$

⁷The material in this section is based on work done in collaboration with Seiberg, Shenker, and Tuttle [28].

where

$$\text{Var}(X) = \langle (X - \langle X \rangle)^2 \rangle$$

denotes the variance of the quantity X . This can be expressed in terms of the specific heat C , defined by

$$C = \frac{\partial \langle E \rangle}{\partial T} = - \frac{1}{T^2} \frac{\partial \langle E \rangle}{\partial \beta}, \quad (4.4)$$

so that

$$T^2 C = \text{Var}(E). \quad (4.5)$$

In particular, since the variance of any quantity is positive definite, the specific heat is also positive definite. When applied to a four-dimensional Schwarzschild black hole, (4.4) gives nonsense because the specific heat is negative. This is a sign of instability.

In Sec. III B, we obtained a dynamical equation (3.27) for the time dependence of the horizon area \mathcal{A} . In thermal equilibrium, the incoming and outgoing average energy fluxes are both equal to $\kappa/4$. In this case (3.27) has a static solution for each value of the average area. Since the Hawking temperature of two-dimensional black holes is independent of the mass in the semiclassical approximation, the specific heat of a black hole is infinite. By (4.5), the root-mean-square fluctuations of the mass, and therefore the area, are also infinite. This means that the thermal fluctuations will so smear the horizon that the different mass static black hole solutions would be replaced by a single ensemble for all masses and areas.

More generally, a time-dependent semiclassical black hole will have a Brownian motion superimposed on the semiclassical solution. Among other effects, this will cause a statistical fluctuation in the elapsed time before the black hole ceases to radiate. The fluctuation will be of order \sqrt{M} , where M is the initial black hole mass.

Physically, we can understand this as follows: fluctuations in the thermal flux of energy at the horizon cause the black hole mass to randomly increase and decrease with time. For an ordinary system, with positive specific heat, such fluctuations are self-regulating. A momentary increase (decrease) in the energy of the system causes an increase (decrease) in its temperature, which in turn causes heat to flow back to (from) the reservoir, thus restoring equilibrium. In the present case, the temperature does not respond to the energy fluctuation. Therefore, there is no tendency to return to the original energy balance. The mass and area just random walk away from their original values.

If a black hole of area \mathcal{A}_0 is created at time $t=0$ and is subsequently illuminated with thermal radiation at the Hawking temperature $T=1/2\pi$, then at time $t>0$ the mass of the black hole will have random walked:

$$\langle [\mathcal{A}(t) - \mathcal{A}_0]^2 \rangle \propto t. \quad (4.6)$$

The exact coefficient in (4.6) can be computed from a knowledge of fluctuations in the thermal energy of the matter fields in the surrounding bath. For the case of N massless fields, the result is

$$\langle [\mathcal{A}(t) - \mathcal{A}_0]^2 \rangle = \frac{N}{24\pi^2} t. \quad (4.7)$$

A rough translation of this result into Kruskal coordinates can be made by observing that y^\pm are exponentials of tortoise coordinates. Equation (4.7) suggests that in terms of an infrared cutoff in Kruskal coordinates $\ln R \propto t$, the fluctuations in the horizon area satisfy

$$\text{Var}(\mathcal{A}) \approx \frac{N}{24\pi^2} \ln R. \quad (4.8)$$

Now let us consider the semiclassical field equations (2.33). In particular, the scalar field Ω satisfies an inhomogeneous free field equation in Kruskal coordinates given by

$$\partial_+ \partial_- \Omega = -1. \quad (4.9)$$

The static black hole solutions to (4.9) have the form

$$\Omega_M = M - y^+ y^- \quad (4.10)$$

and the semiclassical area of the event horizon of a massive black hole is given by

$$\mathcal{A} \approx \Omega_M(0) = M. \quad (4.11)$$

The area of the stretched horizon is a bit larger but this difference will not be important in this section.

Now let us consider the quantum fluctuations about (4.10). Define $\Delta = \Omega - \Omega_M$. The fluctuation Δ satisfies a free wave equation,

$$\partial_+ \partial_- \Delta = 0. \quad (4.12)$$

This suggests that Δ is a canonical massless free field. As such, it has fluctuations which are logarithmically infrared divergent:

$$\langle \Delta^2(0) \rangle \approx \frac{\kappa}{2\pi^2} \ln R, \quad (4.13)$$

where R is the Kruskal coordinate infrared cutoff. This estimate of the fluctuations in the horizon area precisely agrees with the thermodynamic result (4.7). It should be pointed out that there are technical subtleties involved in the quantization of this model, and the above result has not been rigorously established. However, the agreement with thermodynamics strongly suggests that Δ behaves like a canonical field [28].

V. CONSEQUENCES OF THE POSTULATES

A. Microstructure of the stretched horizon

Consider a quantum field theory in a two-dimensional spacetime with a strictly timelike boundary. Suppose the boundary is stationary at $\sigma=0$, except for a brief time interval $[t_a, t_b]$, during which it moves toward negative σ (left) and then returns. The fields are defined to the right of the boundary. The boundary may have additional degrees of freedom.

Without loss of generality we can pretend that the boundary is permanently at $\sigma=0$ by assigning it extra degrees of freedom during the interval $[t_a, t_b]$. During this period the system has field degrees of freedom on the negative σ axis. Nothing prevents us from formally considering these degrees of freedom to belong to the boundary

at $\sigma=0$.

In the case of subcritical flux, where the boundary is always timelike and in causal contact with distant observers, we can perform a similar formal trick, regardless of the nature of the boundary degrees of freedom. They, as well as the fields behind the stretched horizon, can be formally assigned to the stretched horizon. We gain nothing from this except the assurance that a set of stretched horizon degrees of freedom can be defined. Note that this procedure in no way influences the experiences of an observer crossing the stretched horizon.

Up to now we have assumed nothing radical. The fact that outside observers see an apparently real stretched horizon is surprising but derivable from conventional semiclassical assumptions. At this point we will make a radical departure from traditional thought about black holes, which is required by our three postulates. We propose that, for the purposes of a distant observer, a *consistent set of quantum mechanical degrees of freedom continue to describe the stretched horizon even when the critical flux is exceeded*.

We postulate no details about these degrees of freedom, but some general properties are required by Postulate 3. According to standard thermodynamic reasoning, the entropy of a large system is the logarithm of the density of states. For both two- and four-dimensional black holes, the entropy is proportional to the area. We therefore require that the dimension of the Hilbert space of a stretched horizon with area \mathcal{A} is of order $\exp(\mathcal{A})$. For a four-dimensional black hole, this suggests that the number of degrees of freedom per unit area is a universal intensive property independent of the total mass of the black hole.

This universality of stretched horizon properties is general. Define the stretched horizon of a four-dimensional Schwarzschild black hole to have an area *one Planck unit greater* than the global event horizon. The local rate of clocks at the stretched horizon is easy to compute. The analogue of (3.4) has the form

$$\frac{d\tau}{dt} \sim \frac{M_P}{M}, \quad (5.1)$$

where M_P is the Planck mass and M is the mass of the black hole. The local proper temperature at the stretched horizon, T_S , is related to the asymptotically measured Hawking temperature T_H by

$$T_S = \frac{M}{M_P} T_H. \quad (5.2)$$

Using the standard Hawking temperature $T_H \sim M_P^2/M$ gives the universal value

$$T_S \sim M_P. \quad (5.3)$$

The total energy of the black hole, measured in proper units at the stretched horizon, is

$$M_S = M \frac{dt}{d\tau} \sim \frac{M^2}{M_P}. \quad (5.4)$$

Dividing this by the area of the stretched horizon we find the surface energy density to be

$$M_S/\mathcal{A} \sim M_P^3. \quad (5.5)$$

In the semiclassical theory defined in Sec. III, the temperature of a black hole is completely independent of its mass. Thus, as a black hole evaporates, its energy flux is exactly constant, until the instant it disappears. An immediate consequence of Postulate 3 is that the temperature of a two-dimensional dilaton black hole cannot be strictly constant when the mass tends to zero. However, there can be a maximum temperature, which is quickly saturated as energy increases. Suppose there are discrete energy levels with a density $\rho(E)$ which behaves asymptotically as

$$\rho(E) \sim \exp(2\pi E) \text{ as } E \rightarrow \infty. \quad (5.6)$$

The partition function

$$Z(\beta) = \sum_{\text{states}} e^{-\beta E} \quad (5.7)$$

converges for all $\beta > 2\pi$. For large β , Z can be approximated by the first few terms:

$$Z \approx 1 + e^{-\beta E_1} + \dots, \quad (5.8)$$

and the average energy is given by

$$\langle E \rangle = -\frac{\partial \ln(Z)}{\partial \beta} \approx E_1 e^{-\beta E_1} + \dots. \quad (5.9)$$

As the $\langle E \rangle$ tends to infinity, the temperature tends to the value $T=1/2\pi$ in agreement with the semiclassical limit. As the energy tends to zero, the temperature $T=1/\beta$ also tends to zero.

In the semiclassical approximation, the black hole radiates a bit more energy than the system originally had [23]. This was compensated by a final ‘‘thunderpop’’ of negative energy. From the present point of view, a more plausible behavior is that as the black hole nears the end point of the evaporation process, its temperature and luminosity tend to zero and do not overshoot. Note that this is precisely the behavior exhibited by solutions of (3.52).

Another unphysical consequence of the semiclassical theory concerns static solutions, corresponding to a uniform subcritical energy flux, as the limit of critical flux is approached. According to the semi-classical theory a static solution exists for every value of the energy flux, $T_{++} = T_{--} = \bar{T} < \kappa/4$. The semiclassical area (2.35), evaluated at the stretched horizon (3.16), is given by

$$\mathcal{A}_S = \bar{T} + \left[\frac{\kappa}{4} - \bar{T} \right] \left[\ln \left[\frac{\kappa}{4} - \bar{T} \right] - \ln \frac{\kappa}{4} \right]. \quad (5.10)$$

As \bar{T} goes to $\kappa/4$ the area approaches $\mathcal{A} = \kappa/4$. On the other hand, the statistical theory of the previous section requires the mean-square value of the area to diverge in this limit. The model in Sec. III D, based on the thermodynamics of the stretched horizon, does exhibit that behavior. In general, thermodynamic boundary conditions at the stretched horizon, as in Sec. III D, yield a more consistent physical description than semiclassical boundary conditions imposed at the critical curve, γ_{cr} .

The large horizon fluctuations as the temperature ap-

proaches $T=1/2\pi$ are reminiscent of critical behavior.⁸ As the critical temperature is approached the area of the stretched horizon fluctuates more and more, until the horizon swallows up all of space. In the case of a second-order phase transition, correlated domains fluctuate and grow until one domain swallows up the whole sample. The failure of semiclassical theory to correctly account for the horizon fluctuations is analogous to the failure of mean-field theory in critical phenomena.

B. Thermal entropy vs entropy of entanglement

Given our assumptions about the microstructure of the stretched horizon, it is evident that no real loss of information takes place during black hole evaporation. Nevertheless, it is far from clear how the large amount of initial data is stored in outgoing thermalized radiation. Our discussion of this subject will follow the very illuminating study by Page [27].

Let us begin by distinguishing two kinds of entropy. The first, which is of purely quantum origin, we call *entropy of entanglement*. Consider a quantum system composed of two parts, A and B . In what follows, B will refer to the stretched horizon and A to the radiation field outside the stretched horizon. Assume that the Hilbert space of state vectors \mathcal{H} is a tensor product space: $\mathcal{H}=\mathcal{H}_A\otimes\mathcal{H}_B$. If $\{|a\rangle\}$ is an orthonormal basis for \mathcal{H}_A and $\{|b\rangle\}$ is an orthonormal basis for \mathcal{H}_B , then a general ket $|\psi\rangle$ in \mathcal{H} may be written

$$|\psi\rangle=\sum_{a,b}\psi(a,b)|a\rangle\otimes|b\rangle. \quad (5.11)$$

The density matrix of the subsystem A , in the basis $\{|a\rangle\}$, is

$$\rho_A(a,a')=\sum_b\psi(a,b)\psi^*(a',b), \quad (5.12)$$

and that of B is

$$\rho_B(b,b')=\sum_a\psi(a,b)\psi^*(a,b'). \quad (5.13)$$

Note that the composite system $A\cup B$ is in a pure state.

The entropies of entanglement of subsystems A and B are defined by

$$\begin{aligned} S_E(A) &= -\text{Tr}[\rho_A\ln(\rho_A)], \\ S_E(B) &= -\text{Tr}[\rho_B\ln(\rho_B)]. \end{aligned} \quad (5.14)$$

It is easy to prove that $S_E(A)=S_E(B)$ if the composite system is in a pure state. The entropy of entanglement of a subsystem is only zero if $|\psi\rangle$ is an uncorrelated product state. The entropy of entanglement is *not* the entropy with which the second law of thermodynamics is concerned; S_E can increase or decrease with time. A final point is that if the dimension of \mathcal{H}_B is D_B , then the maximum value of $S_E(B)$ [and, therefore, of $S_E(A)$] is

$$S_E(B)_{\max}=-\ln(D_B). \quad (5.15)$$

We have assumed in (5.15) that $D_B\leq D_A$.

The second kind of entropy is *entropy of ignorance*. Sometimes we assign a density matrix to a system, not because it is quantum entangled with a second system, but because we are ignorant about its state, and we assign a probability to each state. For example, if we know nothing about a system, we assign it a density matrix proportional to the unit matrix. If we know only its energy, we assign a density matrix which is vanishing everywhere except the allowed energy eigenspace. Thermal entropy is of this type: it arises because of practical inability to follow the fine-grained details of a system. For a system in thermal equilibrium with a reservoir, we assign a Maxwell-Boltzmann density matrix

$$\rho_{\text{MB}}=Z^{-1}\exp(-\beta H), \quad (5.16)$$

and define the thermal entropy by

$$S_T=-\text{Tr}[\rho_{\text{MB}}\ln(\rho_{\text{MB}})]. \quad (5.17)$$

Now let us consider the evolution of both kinds of entropy during the formation and evaporation of a two-dimensional black hole. Let us begin with the thermal entropy of the stretched horizon, which we assume is equal to its area. As we have seen, the area begins to increase exponentially with t before the infalling matter arrives, reaching its maximum at the arrival time. The area then decreases linearly with t until the black hole disappears. This is illustrated in Fig. 8. The thermal entropy of the outgoing radiation begins to increase due to the emission from the excited stretched horizon. Shortly after the radiation begins, the temperature approaches $T=1/2\pi$, so that the rate of change of the thermal entropy of the radiation is constant throughout most of the process. This is also shown in Fig. 8.

Now consider the entropy of entanglement. Initially, the stretched horizon is in its ground state, with minimal area, and the radiation field is described by a pure state. The entropy of entanglement starts at zero. As soon as the stretched horizon area begins to increase, f quanta are emitted. Typically, the state of the f quanta will be correlated to the state of the stretched horizon, so that

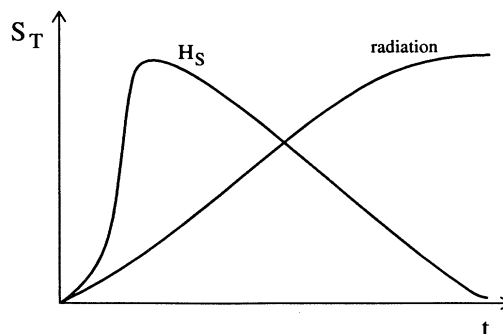


FIG. 8. Thermal entropy of stretched horizon and radiation field as a function of time.

⁸Laughlin has also suggested similarities between black-hole behavior and phase transitions [30].

S_E will start to increase. However, S_E will generally be bounded by the logarithm of the dimension of the Hilbert space describing the stretched horizon, which we have assumed is proportional to the area. In other words, at any time,

$$S_E(H_S) \leq S_T(H_S) = \mathcal{A}(t). \quad (5.18)$$

Thus, the entropy of entanglement is bounded and must return to zero as the area of the stretched horizon returns to its vacuum value. Page has argued that in the beginning the entropy of entanglement is likely to approximately follow the thermal entropy of the radiation field, so that the history of S_E should look like Fig. 9.

Evidently, as time elapses, subtle differences develop between the coarse-grained thermal density matrix of the radiation and the exact description. Postulates 1 and 3 dictate that the entropy of entanglement return to zero in a more or less definite way as the black hole evaporates. In particular, there is no room for a stable or very long-lived remnant storing the incident information.

To understand the difference between the thermal and exact density matrices of the final outgoing radiation, consider a time about half-way through the evaporation process, when the thermal entropy and the entropy of entanglement are still not too different. The total fine-grained entropy of the combined system of stretched horizon and radiation is zero, but the radiation is correlated to the degrees of freedom of the stretched horizon. More time elapses, and the stretched horizon emits more quanta. The previous correlations between the stretched horizon and the radiation field are now replaced by correlations between the early part of the radiation and the newly emitted quanta. In other words, the features of the exact radiation state that allow S_E to return to zero are *long-time correlations spread over the entire time occupied by the outgoing flux of energy*. The local properties of the radiation are expected to be thermal. For example, the average energy density, short-time radiation field correlations, and similar quantities that play an important role

in the semiclassical dynamics should be thermal. The long-time correlations which restore the entropy to zero are not important to average coarse-grained behavior, and are just the features which are not found unless a suitable microphysical description is provided for the stretched horizon.

C. Discussion

We will conclude with some speculation about the nature of the stretched horizon microstructure for four-dimensional black holes. If we consider nearly spherical black holes, a stretched horizon can be defined as follows. Consider a radial incoming null geodesic which crosses the global horizon where its area is \mathcal{A} . Proceed backward along such geodesics until the surface with area one Planck unit larger is encountered. By using such ingoing geodesics, we can map every point of the global horizon to a point on the stretched horizon.

The global horizon is composed of a bundle of light rays that can be thought of as a two-dimensional fluid on the global horizon [9]. The points of this fluid can be mapped to the stretched horizon, thereby defining a fluid flow on that surface. Classically, the fluid behaves as a continuous, viscous fluid with conventional shear viscosity and negative bulk viscosity. A natural candidate for the microphysics of the stretched horizon is to replace the continuous classical fluid with a fluid of discrete "atoms."

As we have seen, the intensive thermodynamic variables of the stretched horizon are universal and do not depend on the size or mass of the black hole. This demands that the surface density of atoms also is independent of the area. When incoming energy flux or outgoing Hawking radiation causes the area of a patch of the stretched horizon to change, points of the fluid will pop into and out of existence in order to keep the density constant.

Finally we would like to point to a feature of (3+1)-dimensional black holes which is not shared by the (1+1)-dimensional theory. This feature adds plausibility to the claim that the stretched horizon is in thermal equilibrium during most of the evaporation. Consider an observer at the stretched horizon who counts the number of particles emitted per unit proper time. Since the stretched horizon is always at the Planck temperature the number of particles emitted per unit area per unit proper time is of order one in Planck units. If all these particles made it out to infinity, then a distant observer would estimate a number of particles emitted per unit time, which is obtained by multiplying by the black hole area and the time dilation factor:

$$\frac{dN}{dt} \approx M^2 \frac{d\tau}{dt} \approx M. \quad (5.19)$$

On the other hand, the number per unit time of particles that actually emerge to infinity is obtained by multiplying the black-hole luminosity $L \approx 1/M^2$ by the inverse ener-

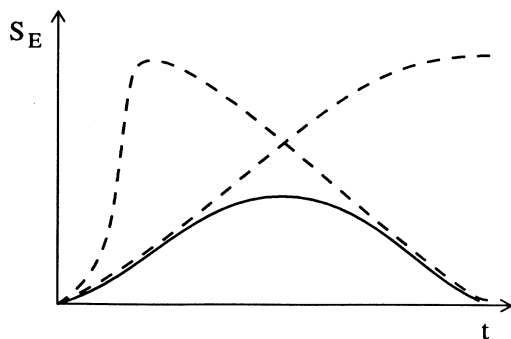


FIG. 9. Entanglement entropy of radiation and stretched horizon. The dashed curves indicate the thermal entropies of Fig. 8.

gy of a typical thermal particle at the Hawking temperature. The result is

$$\frac{dN}{dt} \sim \frac{1}{M}. \quad (5.20)$$

Therefore, it seems that most of the particles emitted from the stretched horizon do not get to infinity. In fact, only those particles which are emitted with essentially zero angular momentum can overcome the gravitational attraction of the black hole, and the rest fall back [9]. This gives rise to a thermal atmosphere above the stretched horizon which only slowly evaporates and whose repeated interaction with the stretched horizon insures thermal equilibrium. Such a thermal atmosphere can be obtained in two dimensions by including massive degrees of freedom, and this may indeed be necessary for a fully consistent description of two-dimensional black hole evaporation.

If the considerations of this paper are correct then black holes catalyze a very different phenomenon than that envisioned by Hawking [16]. To begin with, an incoming pure state of matter composed of low-energy particles falls into its own gravitational well. The matter is blueshifted relative to stationary observers so that when it arrives at the stretched horizon it has Planckian wavelengths. Thereupon it interacts with the “atoms” of the stretched horizon leading to an approximately thermal state. The subsequent evaporation yields approximately thermal radiation but with nonthermal long-time correlations. These nonthermal effects depend not only on the incoming pure state but also on the precise nature of the Planck-scale “atoms” and their interaction with the blueshifted matter. The evaporation products then climb out of the gravitational well and are redshifted to low energy. The result is remarkable. The very low-energy Hawking radiation from a massive black hole has nonthermal correlations, which contain detailed information about Planck-scale physics [3,4]. The phenomenon is reminiscent of the imprinting of Planckian fluctuations onto the microwave background radiation by inflation.

The view of black holes that we have presented is, of course, incomplete. As we have emphasized, the reality

of the membrane cannot be an invariant which all observers agree upon. Furthermore, although conventional quantum field theory in an evaporating black hole background seems to lead to a description in which a single-state vector describes the interior and exterior of the black hole, this description must be wrong if our postulates are correct. Precisely what is wrong is not clear to us, but we wish to emphasize that the event space for an experiment should only contain physically measurable results.

In many respects, the situation seems comparable to that of the early part of the century. The contradictions between the wave and particle theories of light seemed irreconcilable, but careful thought could not reveal any logical contradiction. Experiments of one kind or the other revealed either particle or wave behavior, but not both. We suspect that the present situation is similar. An experiment of one kind will detect a quantum membrane, while an experiment of another kind will not. However, no possibility exists for any observer to know the results of both. Information involving the results of these two kinds of experiments should be viewed as *complementary* in the sense of Bohr.

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- [1] S. W. Hawking, *Commun. Math. Phys.* **43**, 199 (1975); *Phys. Rev. D* **14**, 2460 (1976).
 - [2] For an early critique of Hawking's proposal, see D. N. Page, *Phys. Rev. Lett.* **44**, 301 (1980).
 - [3] G. 't Hooft, *Nucl. Phys.* **B335**, 138 (1990), and references therein.
 - [4] E. Verlinde and H. Verlinde, “A Unitary S-matrix for 2D Black Hole Formation and Evaporation,” Princeton Report No. PUPT-1380, IASSNS-HEP-93/8, 1993 (unpublished); K. Schoutens, E. Verlinde, and H. Verlinde, *Phys. Rev. D* **48**, 2670 (1993).
 - [5] J. D. Bekenstein, *Phys. Rev. D* **9**, 3292 (1974).
 - [6] R. S. Hanni and R. Ruffini, *Phys. Rev. D* **8**, 3259 (1973); P. Hajicek, *Commun. Math. Phys.* **36**, 305 (1974); R. L. Znajek, *Mon. Not. R. Astron. Soc.* **185**, 833 (1978).
 - [7] T. Damour, *Phys. Rev. D* **18**, 3598 (1978); in *Proceedings of the Second Marcel Grossmann Meeting on General Relativity*, edited by R. Ruffini (North-Holland, Amsterdam, 1982), p. 587; R. L. Znajek, “Black Holes, Apparent Horizons, Entropy, and Temperature,” University of Cambridge report, 1981 (unpublished); “A Hartle-Hawking Formula for Apparent Horizons,” University of Cambridge report, 1981 (unpublished).
 - [8] K. S. Thorne and D. A. MacDonald, *Mon. Not. R. Astron. Soc.* **198**, 339 (1982); W. H. Zurek and K. S. Thorne, *Phys. Rev. Lett.* **54**, 2171 (1985); R. H. Price and K. S. Thorne, *Phys. Rev. D* **33**, 915 (1986).
 - [9] K. S. Thorne, R. H. Price, and D. A. MacDonald, *Black*

Holes: The Membrane Paradigm (Yale University Press, New Haven, CT, 1986), and references therein.

- [10] G. 't Hooft (private communication).
- [11] C. G. Callan, S. B. Giddings, J. A. Harvey, and A. Strominger, *Phys. Rev. D* **45**, R1005 (1992).
- [12] S. B. Giddings and A. Strominger, *Phys. Rev. D* **46**, 627 (1992).
- [13] T. Banks, A. Dabholkar, M. R. Douglas, and M. O'Loughlin, *Phys. Rev. D* **45**, 3607 (1992).
- [14] J. G. Russo, L. Susskind, and L. Thorlacius, *Phys. Lett. B* **292**, 13 (1992).
- [15] B. Birnir, S. B. Giddings, J. A. Harvey, and A. Strominger, *Phys. Rev. D* **46**, 638 (1992).
- [16] S. W. Hawking, *Phys. Rev. Lett.* **69**, 406 (1992).
- [17] L. Susskind and L. Thorlacius, *Nucl. Phys.* **B382**, 123 (1992).
- [18] A. Strominger, *Phys. Rev. D* **46**, 4396 (1992).
- [19] J. G. Russo and A. A. Tseytlin, *Nucl. Phys.* **B382**, 259 (1992); T. Burwick and A. Chamseddine, *ibid.* **B384**, 411 (1992).
- [20] A. Bilal and C. G. Callan, *Nucl. Phys.* **B394**, 73 (1993).
- [21] S. P. de Alwis, *Phys. Lett. B* **289**, 278 (1992); **300**, 330 (1993); *Phys. Rev. D* **46**, 5429 (1992).
- [22] S. B. Giddings and A. Strominger, *Phys. Rev. D* **47**, 2454 (1993).
- [23] J. G. Russo, L. Susskind, and L. Thorlacius, *Phys. Rev. D* **46**, 3444 (1992).
- [24] J. G. Russo, L. Susskind, and L. Thorlacius, *Phys. Rev. D* **47**, 533 (1993).
- [25] Y. Park and A. Strominger, *Phys. Rev. D* **47**, 1569 (1993).
- [26] D. Lowe, *Phys. Rev. D* **47**, 2446 (1993); S. W. Hawking and J. M. Stewart, *Nucl. Phys.* **B400**, 393 (1993); T. Piran and A. Strominger, *Phys. Rev. D* (to be published).
- [27] D. N. Page, "Black Hole Information," University of Alberta Report No. Alberta-Thy-23-93, 1993 (unpublished), and references therein.
- [28] N. Seiberg, S. Shenker, L. Susskind, L. Thorlacius, and J. Tuttle (unpublished).
- [29] N. Seiberg and S. Shenker (private communication).
- [30] R. B. Laughlin (private communication).