

Renormalization group for nonrenormalizable theories: Einstein gravity with a scalar field

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We develop a renormalization-group formalism for nonrenormalizable theories and apply it to Einstein gravity theory coupled to a scalar field with the Lagrangian $L = \sqrt{g}[RU(\phi) - \frac{1}{2}G(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)]$, where $U(\phi)$, $G(\phi)$, and $V(\phi)$ are arbitrary functions of the scalar field. We calculate the one-loop counterterms of this theory and obtain a system of renormalization-group equations in partial derivatives for the functions U , G , and V playing the role of generalized charges which substitute for the usual charges in multicharge theories. In the limit of a large but slowly varying scalar field and small spacetime curvature this system gives the asymptotic behavior of the generalized charges compatible with the conventional choice of these functions in quantum cosmological applications. It also demonstrates in the over-Planckian domain the existence of the Weyl-invariant phase of gravity theory asymptotically free in gravitational and cosmological constants.

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I. INTRODUCTION

It is widely recognized that a consistent theory of quantum gravity is a matter of crucial importance with regard to the two main challenges of modern physics: the creation of a unified theory of fundamental interactions and construction of the theory of the quantum origin and evolution of the Universe. The principles of the latter theory, quantum cosmology, were founded many years ago by Dirac, Wheeler, and DeWitt [1] and have been further developed in recent years when the proposals for the quantum state of the Universe were put forward [2–4]. Among these proposals is the so-called no-boundary prescription of Hartle and Hawking [2,3], which is supposed to provide a smooth transition between the quantum birth of the Universe and the inflationary stage of its development. The latter property is very important, because the theory of the inflationary expansion of the very early Universe [5–12] has become an integral part of modern cosmology.

To study the physical effects of the proposed wave function of the inflationary Universe at a deeper level, one has to go beyond the tree-level approximation and, in the absence of full nonperturbative quantum field theory and quantum gravity, to calculate, as a first step, perturbative quantum corrections. This inspired the series of pa-

pers on the one-loop calculations in quantum cosmology [13–24], which, in particular, have shown that the normalizability property of the cosmological wave function and the partition function of the inflationary universes drastically change after the inclusion of loop corrections [17].

However, when resorting to perturbative calculations in quantum cosmology we should not forget about one of the stumbling blocks on the road to a consistent theory of quantum gravity—its nonrenormalizability [25,26]. It is well known that the origin of this fundamental problem consists in the fact that the gravitational coupling constant has a mass dimension -2 ($\hbar = c = 1$). Thus, the Feynman diagrams, which contain a growing number of graviton loops, lead formally to an infinite set of different counterterms to the gravitational Lagrangian, which cannot be eliminated by a renormalization procedure of the standard type, that is removed by the renormalization of a finite number of parameters [27]. As regards pure gravity theory, it was shown that at the one-loop level on mass shell no physically relevant divergencies remain; all of them can be absorbed into a field renormalizations [28]. However, the pure gravity is two-loop nonrenormalizable, even on mass shell [29]. The situation gets worse in the case of the interaction with matter, in particular, with the scalar field. This theory is nonrenormalizable [28] already in the one-loop approximation.

There are different approaches to the problem of nonrenormalizability in quantum gravity. One can consider Einstein gravity as a low-energy limit of a more general theory such as supergravity [30] or superstring theory [31,32]. Because of the presence of the additional sym-

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metry, one has a smaller number of types of divergencies in quantum supergravity and can hope to find it renormalizable. As concerns superstring theory, we can look forward to building a finite “theory of everything” from it. However, one should recognize that the questions of the renormalizability in supergravity theories and the finiteness of superstring ones still remain open.

Another approach to the question of renormalizability in gravity theory is connected with the idea of adding to the Lagrangian curvature-squared terms which allow one to carry out a resummation of the perturbation series and obtain an effective renormalizable theory [33–35]. However, pursuing this approach we stumble upon the residues of incorrect signs at propagator poles, which in turn imply the problem of the breakdown of unitarity [26].

In any case, it makes sense to try to work with the usual nonrenormalizable Einstein gravity by overcoming our fear of the infinite number of counterterms arising in the Lagrangian as a response to an infinite number of different types of ultraviolet divergencies. It is interesting that working with nonrenormalizable theories we can apply such a useful mathematical tool as renormalization-group equations. The idea of the possibility to apply the renormalization-group equations in the theory with a charge having a negative mass dimension was mentioned in Ref. [36]. Weinberg in Ref. [26] applied the concept of the asymptotic safety to the discussion of a renormalization group in quantum gravity. A theory is considered to be asymptotically safe if “essential” coupling parameters approach a fixed point as the momentum scale of their renormalization point goes to infinity. The condition of the asymptotic safety could be treated as a generalization of the notion of renormalizability, which fixes all but a finite number of essential coupling parameters of a theory.

The most general scheme of obtaining the renormalization-group equations in arbitrary nonrenormalizable theories was formulated recently by Kazakov [37]. In spite of the absence of the multiplicative renormalizability, the method proposed in [37] allows one to calculate all the higher singularities (the poles in the dimensional regularization scheme) from the generalized β functions describing the ultraviolet divergencies without subdividing them into those related to different parameters of the theory under consideration. However, although this formalism has an undoubted theoretical significance, it can be hardly used in the concrete applications.

Here, we develop the renormalization-group formalism adapted for the purposes of Einstein gravity interacting with a scalar field. This model seems especially interesting because it is the inflaton scalar field that provides the existence and subsequent termination of the de Sitter stage in the evolution of the Universe, which is widely recognized to be responsible for the formation of the large-scale cosmological structure consistent with the present-day observational data. The main idea of our approach consists in such rearrangement of an infinite set of counterterms in the Lagrangian that the groups of these terms having an analogous nature are combined

together into certain functions which we shall call the generalized charges (in contrast with the usual charges in the traditional theory of the renormalization group [38,39,27,40,41]). These generalized charges include implicitly all the divergencies which can appear in the theory.

We shall introduce the generalized charges as coefficients in the expansion for the action in powers of the curvature and in numbers of derivatives of the scalar field:¹

$$S[g, \phi] = \int d^4x g^{1/2} \left\{ -V(\phi) - \frac{1}{2} G(\phi) g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + R(g) U(\phi) + \dots \right\}, \quad (1.1)$$

where the ellipsis denotes all other possible terms containing all higher powers of curvatures and these derivatives. It is obvious that the action (1.1) contains an infinite number of such structures. Assuming the renormalization-point independence of the bare generalized charges (just like as in the usual approach to the renormalization-group theory), we can obtain an infinite system of generalized renormalization-group equations for an infinite set of such charges. It is certainly impossible to work with this system in practice. Therefore we shall have to restrict ourselves with a finite subsystem of generalized charges, assuming that other structures present in the action (1.1) are comparatively small in concrete physical applications. This can happen due to two different reasons. One situation allowing us to consider only the first few generalized charges $U(\phi)$, $G(\phi)$, and $V(\phi)$ is when the rest of the terms in (1.1) are negligible, because $\partial\phi$ and the space-time curvature are small enough to neglect the terms with their higher powers. Another situation corresponds to the setting of the physical problem with such an energy scale that the running coupling constants of the (\dots) structures in (1.1) are negligible at this scale—the property called asymptotic freedom in corresponding coupling constants and justifying the use of the perturbation theory.

The situation of the first type takes place in a wide class of modern cosmological applications in the theory of the early Universe driven by the large inflaton scalar field, having at the inflationary stage sufficiently small (compared to the Planckian scale) curvature and space-time gradients of the inflaton. As far as it concerns the situation of the asymptotic freedom, it can only follow from the properties of solutions of renormalization-group equations and cannot be *a priori* used without their analysis. Anyway, in this paper we shall assume either of these two possibilities as a justification for truncating the set of generalized charges to those of Eq. (1.1) and calculating the corresponding one-loop counterterms and β functions. It will turn out that there exists a particular solution of the renormalization-group equations demonstrating the asymptotic freedom, which can be used as

¹Our conventions are $\text{sgn } g_{\mu\nu} = +2$, $g = \det g_{\mu\nu}$, $R = g^{\mu\nu} R_{\mu\alpha\nu}^{\alpha} = g^{\mu\nu} (\partial_\alpha \Gamma_{\mu\nu}^{\alpha} - \dots)$, ∇_μ is a covariant derivative with respect to $g_{\mu\nu}$, $\partial/\partial x^\alpha = \partial_\alpha = ,_\alpha$.

an *a posteriori* argument in favor of this approach (irrespective of the approximation of small curvatures and field gradients).

Thus, using the first terms in action (1.1), we can find the one-loop counterterms in a rather general form for arbitrary functions $U(\phi)$, $G(\phi)$, and $V(\phi)$. They turn out to be of a very complicated nonpolynomial structure even on the mass shell [28,42]. Given these one-loop counterterms we can construct the generalized β functions and study a corresponding system of renormalization-group equations. In a certain sense this formalism occupies an intermediate position between the standard renormalization-group procedure [38,39,27,40,41] and that of Ref. [37]. The calculation of one-loop counterterms of theory (1.1) is a rather nontrivial problem which can be solved by the combination of the covariant Schwinger-DeWitt technique [43,44] and the background field method [45,46]. It is remarkable that gravitational theory coupled nonminimally to a scalar field can be simplified by means of the conformal transformation of the metric and scalar field [42,47–48], so that the nonminimal coupling between gravity and a scalar field disappears. Such a technique has been used in [42] for obtaining the divergent part of the one-loop effective action in Einstein gravity with the cosmological term, nonminimally coupled to the self-interacting scalar field

$$S[g, \phi] = \int d^4x g^{1/2} \left\{ \frac{1}{k^2} (R - 2\Lambda) - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{2} \xi R \phi^2 - \frac{\lambda}{4!} \phi^4 \right\}. \quad (1.2)$$

Theories (1.1) and (1.2) after the corresponding conformal transformations have the same form:

$$S[\mathcal{G}, \varphi] = \int d^4x \mathcal{G}^{1/2} \left\{ \frac{1}{k^2} R(\mathcal{G}) - \frac{1}{2} \mathcal{G}^{\mu\nu} \bar{\nabla}_\mu \varphi \bar{\nabla}_\nu \varphi - \bar{V}(\varphi) \right\}, \quad (1.3)$$

where \mathcal{G} and φ are the conformally transformed metric and the scalar field, $R(\mathcal{G}) \equiv \bar{R}$ the scalar curvature with respect to the metric $\mathcal{G}_{\mu\nu}$, $\bar{\square} \equiv \mathcal{G}^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu$, and $\bar{\nabla}_\mu$ the covariant derivative with respect to $\mathcal{G}_{\mu\nu}$. Thus, having one-loop divergencies for theory (1.3) expressed in terms of new field variables, we at the same time have the solution for theory (1.1). The inverse conformal transformation gives us the needed counterterms as functions of original variables.

After obtaining all the counterterms we calculate the corresponding β functions using our generalized formalism. The construction of the renormalization-group equations is carried out in analogy with the standard method, but, in contrast with the usual multiple-charge renormalization theory, this method results in the differential equations in partial derivatives with respect to the renormalization mass parameter t and the scalar field—the arguments of $U(\phi, t)$, $G(\phi, t)$, and $V(\phi, t)$. In view of

the complexity of β functions, even the truncated set of these equations for generalized charges turns out to be very complicated. Moreover, these equations require setting the Cauchy data, and at present we do not have exhaustive physical principles to fix it uniquely. Therefore, instead of a complete rigorous setting of the boundary-value problem, we shall study the admissible types of the asymptotic behavior for the generalized charges and compare them with the present-day phenomenological models widely used in the early-Universe implications.

In this way we shall study two asymptotic forms of $U(\phi)$, $G(\phi)$, and $V(\phi)$. The first one has a power-logarithmic dependence on the scalar field in the high-energy limit of large values of ϕ . In this limit we find a two-parameter family of solutions for generalized charges $U(\phi)$, $G(\phi)$, and $V(\phi)$. It is worth noticing that the functions $V(\phi) = \lambda\phi^4$, $G(\phi) = 1$, and $U(\phi) = 1 - \xi\phi^2/2$, usually used in phenomenological models, satisfy the obtained restrictions and, hence, sustain our renormalization-group analysis. At the same time, the models without self-interaction of the scalar field or with a minimal coupling to gravity are ruled out by this analysis which, thus, can serve for selecting intrinsically consistent models. It is also interesting to note that the motivation for considering the nonminimally coupled scalar field with a self-interaction follows also from the requirement of the normalizability of the cosmological wave function and a reasonable probability distribution of inflationary cosmologies [17]. Thus the requirements of the high-energy (ultraviolet) quantum consistency of the theory match with the requirements of a reasonable dynamical scenario in the early Universe and lead to certain selection rules for admissible phenomenological Lagrangians. The second asymptotic form of generalized charges which we consider here involves an exponential dependence on the scalar field. These models are of special interest in the theory of the early Universe, because they imply a power-law inflation intensively discussed in current literature [49–54]. It turns out that these models also satisfy the consistency conditions within the renormalization-group approach, which impose certain relations between the asymptotic expressions for generalized charges and lead to their one-parameter family.

A remarkable property of the obtained pure power in ϕ solution for the truncated set of $U(\phi)$, $G(\phi)$, and $V(\phi)$ is that this solution turns out to be *exact* (valid for all values of ϕ) and describing in the ultraviolet limit the Weyl-invariant theory of coupled metric and the dilaton field, the latter being a purely gauge mode of the local conformal group. This theory turns out to be asymptotically free in the effective renormalized gravitational and cosmological constants, and, thus, seems to justify in the high-energy domain the truncation of the above type even irrespective of the inflationary context with small curvatures and space-time field gradients. The structure of the renormalized Lagrangian of the theory shows that at intermediate energies the dynamically excited dilaton mode, presumably, breaks in view of its ghost nature the over-Planckian Weyl invariance and leads to the low-energy theory. The latter must be described by the as

yet unknown nontrivial solution of the full system of generalized renormalization-group equations, containing the new dimensionful parameters reflecting the broken Weyl and scale invariances of the theory. We discuss the possible structure of these solutions in connection with setting the Cauchy problem for the generalized renormalization-group equations and with the low-energy stability of the theory.

It is worth pointing out here the relation of the above technique to recent work on applications of the renormalization group to quantum field theory on curved space-time background (we cite here only several references [55–63] in a very extensive bibliography on this subject). In these papers the gravitational field was basically considered at the classical level and the problem of its nonrenormalizability did not arise: the curvature-squared terms, generated by the renormalization procedure for the matter fields, were usually interpreted as a polarization of their vacuum, contributing to Einstein's equations, but not as the first terms in an infinite series of local interactions. With regard to these papers, our work can be considered as a means to justify the truncation of this series for physical problems with slowly varying fields and simultaneously to find a correct (generally nonpolynomial) structure of the first few interactions, encoded in the generalized charges of the above type.

This paper is organized as follows. Section II contains the calculation of one-loop divergencies for the above models. In Sec. III we review the standard renormalization-group method and Kazakov's formalism and apply it to the calculation of the generalized β functions and the corresponding renormalization-group equations. In Secs. IV and V we analyze the asymptotic properties of their solutions in the context of some cosmological models, find the over-Planckian Weyl invariant phase of gravity theory, and discuss the possible prospects for the further development of this approach and its physical implications.

II. ONE-LOOP DIVERGENCIES IN THE GENERALIZED MODEL OF COUPLED GRAVITATIONAL AND SCALAR FIELDS

A. Nonlinear minimally coupled scalar field

The one-loop effective action for gauge theories has the following form in the condensed notation of DeWitt [46]:

$$i W_{\text{one loop}} = -\frac{1}{2} \text{Tr} \ln \frac{\delta^2 S^{\text{tot}}[\phi]}{\delta\phi^A \delta\phi^B} + \text{Tr} \ln Q_\alpha^\beta, \quad (2.1)$$

where ϕ^A is the full set of fields, $S^{\text{tot}}[\phi] = S[\phi] + S_\chi[\phi]$ is the total action of the theory including the gauge-breaking term $S_\chi[\phi]$, $Q_\beta^\alpha = \nabla_\beta^A (\delta\chi^\alpha / \delta\phi^A)$ is the ghost operator defined in terms of the generators of gauge transformations of field variables ∇_α^A and gauge conditions χ^β entering $S_\chi[\phi]$, and Tr is the functional trace.

As was mentioned in the Introduction, the first three terms of action (1.1) and action (1.2) can be transformed into the form

$$S[\mathcal{G}, \varphi] = \int d^4x \mathcal{G}^{1/2} \left\{ \frac{1}{k^2} R(\mathcal{G}) - \frac{1}{2} \mathcal{G}^{\mu\nu} \bar{\nabla}_\mu \varphi \bar{\nabla}_\nu \varphi - \bar{V}(\varphi) \right\} \quad (2.2)$$

by means of a conformal transformation of the metric and the scalar field. So we shall carry out all the calculations for action (2.2) and then use this transformation to the initial field variables in order to get the divergent part of the one-loop effective action for (1.1) and (1.2).

For the calculation of the divergent part $W_{\text{one loop}}^{\text{div}}$ we shall use the background field method [45,46] and the Schwinger-DeWitt technique [43,44] applicable to a wide class of differential operators, to which belong the inverse gauge propagator $\delta^2 S^{\text{tot}}[\phi] / \delta\phi^A \delta\phi^B$ and ghost operator Q_β^α of Eq. (2.1). Obtaining these operators in the background-field method looks as follows. We first split $\mathcal{G}_{\mu\nu}$ and φ into background fields $(\mathcal{G}_{\mu\nu}^{(0)}, \varphi^{(0)})$ and quantum disturbances $(h_{\mu\nu}, f)$:

$$\mathcal{G}_{\mu\nu} = \mathcal{G}_{\mu\nu}^{(0)} + h_{\mu\nu}, \quad \varphi = \varphi^{(0)} + f. \quad (2.3)$$

Then, under such a splitting, we introduce a background-covariant gauge-breaking term in the total action and expand this action in powers of $(h_{\mu\nu}, f)$, so that the kernel of the quadratic term in this expansion will give rise to the inverse propagator $\delta^2 S^{\text{tot}}[\phi] / \delta\phi^A \delta\phi^B$. In what follows we shall omit the superscript zero in the notation of the background fields $\mathcal{G}_{\mu\nu}^{(0)}$ and $\varphi^{(0)}$ —the functional arguments of the effective action. In such notation the gauge-breaking term can be written as

$$S_\chi = -\frac{1}{2k^2} \int d^4x \mathcal{G}^{1/2} \mathcal{G}^{\alpha\beta} \chi_\alpha \chi_\beta, \quad (2.4)$$

where the background-covariant gauge conditions, which we choose here, are the following functions linear in quantum disturbances:

$$\chi_\alpha \equiv \bar{\nabla}^\mu h_{\mu\alpha} - \frac{1}{2} \bar{\nabla}_\alpha h, \quad (2.5)$$

with

$$h = \mathcal{G}^{\mu\nu} h_{\mu\nu} \quad (2.6)$$

and covariant derivatives $\bar{\nabla}^\mu = \mathcal{G}^{\mu\nu} \bar{\nabla}_\nu$ defined with respect to metric $\mathcal{G}_{\mu\nu}$. As a result, the part of the total action $S^{\text{tot}}[\phi]$ quadratic in quantum perturbations can be represented in the form

$$S_2^{\text{tot}} = \frac{1}{2} \int d^4x \mathcal{G}^{1/2} \psi^A F_{AB} \psi^B. \quad (2.7)$$

Here $\psi^A \equiv (h_{\mu\nu}, f)$ and the matrix differential operator $F_{AB}(\bar{\nabla})$ is given by

$$F_{AB}(\bar{\nabla}) = C_{AB} \bar{\square} + 2\Gamma_{AB}^\sigma \bar{\nabla}_\sigma + W_{AB}, \quad \bar{\square} \equiv \mathcal{G}^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu, \quad (2.8)$$

where the coefficient of the covariant \mathcal{G} -metric D'Alambertian is given by the matrix

$$C_{AB} = \begin{pmatrix} \frac{1}{4k^2} C^{\mu\nu, \alpha\beta} & 0 \\ 0 & 1 \end{pmatrix}, \quad (2.9)$$

$$C^{\mu\nu, \sigma\rho} = \frac{1}{4} (\mathcal{G}^{\mu\sigma} \mathcal{G}^{\nu\rho} + \mathcal{G}^{\mu\rho} \mathcal{G}^{\nu\sigma} - \mathcal{G}^{\mu\nu} \mathcal{G}^{\rho\sigma}). \quad (2.10)$$

To resort to the universal algorithms of the Schwinger-DeWitt technique, we should transform the operator (2.8) to the minimal form. For this purpose we, first, go over to a unit matrix coefficient of the higher-derivative term in (2.8) by multiplying it with the matrix C^{AD} inverse to C_{DB} :

$$\hat{F}(\bar{\nabla})_B^A = C^{AD} F_{DB}(\bar{\nabla}) = \bar{\square} \hat{I} + 2\hat{\Gamma}^\sigma \bar{\nabla}_\sigma + \hat{W}, \quad (2.11)$$

$$C^{AD} C_{DB} = \delta_B^A. \quad (2.12)$$

Here and in what follows we shall use a caret to denote the matrix acting in the space of fields ψ^A and having one contravariant and one covariant index: $\hat{I} = \delta_B^A$, $\hat{\Gamma}^\sigma = \hat{\Gamma}_A^\sigma$, $\hat{W} = \hat{W}_B^A$. So the matrix unity and the matrix C^{AD} above look like

$$\hat{I} = \delta_B^A = \begin{pmatrix} \delta_{\mu\nu}^{\alpha\beta} & 0 \\ 0 & 1 \end{pmatrix}, \quad (2.13)$$

$$C^{AD} = \begin{pmatrix} k^2 C_{\alpha\beta, \mu\nu} & 0 \\ 0 & 1 \end{pmatrix}, \quad (2.14)$$

$$C_{\alpha\beta, \mu\nu} = \mathcal{G}_{\alpha\mu} \mathcal{G}_{\beta\nu} + \mathcal{G}_{\alpha\nu} \mathcal{G}_{\beta\mu} - \mathcal{G}_{\alpha\beta} \mathcal{G}_{\mu\nu}, \quad (2.15)$$

$$\delta_{\mu\nu}^{\alpha\beta} = \delta_{(\mu}^{\alpha} \delta_{\nu)}^{\beta}, \quad (2.16)$$

where the indices in brackets imply their symmetrization with the factor 1/2. Note that this transformation does not change the divergent part of the one-loop effective action, because the matrix coefficient C_{AB} gives the contribution to the effective action proportional to $\delta^4(x, x)$ and canceled by the local measure [43].

Then we introduce a new covariant derivative

$$\mathcal{D}_\mu = \bar{\nabla}_\mu + \hat{\Gamma}_\mu, \quad (2.17)$$

which absorbs the part of (2.11) linear in derivatives. As a result, the operator $\hat{F}(\mathcal{D})$ takes the minimal form

$$\hat{F}(\mathcal{D}) = \mathcal{G}^{\mu\nu} \mathcal{D}_\mu \mathcal{D}_\nu \hat{I} + \hat{P} - \frac{1}{6} \bar{R} \hat{I}, \quad (2.18)$$

where the scalar-curvature term $-\frac{1}{6} \bar{R} \hat{I} \equiv -\frac{1}{6} R(\mathcal{G}) \hat{I}$ has been extracted from the potential term of the operator for reasons of convenience.

The ghost operator Q_α^β corresponding to the gauge-breaking term (2.4) and (2.5) also has the form (2.18). It is defined by the gauge transformation of the gauge (2.5) under the transformations $\Delta^f h_{\mu\nu}$ of quantum disturbances,

$$\begin{aligned} h_{\mu\nu} &\rightarrow h_{\mu\nu} + \Delta^f h_{\mu\nu}, \quad \Delta^f h_{\mu\nu} = 2\bar{\nabla}_{(\mu} f_{\nu)}, \\ \chi^\alpha &\rightarrow \chi^\alpha + Q_\mu^\alpha(\bar{\nabla}) f^\mu \end{aligned} \quad (2.19)$$

(where f^μ is an arbitrary vector function) and reads

$$Q_\beta^\alpha(\bar{\nabla}) = \bar{\square} \delta_\mu^\alpha - \bar{R}_\mu^\alpha. \quad (2.20)$$

The calculation of one-loop divergences for the func-

tional determinants of the minimal operators $\hat{F}(\mathcal{D})$ (2.18) and $Q_\mu^\alpha(\bar{\nabla})$ (2.20) can be performed by the following universal algorithm. Let $\hat{\tilde{F}} \equiv \hat{\tilde{F}}_B^A$ be the second-order minimal operator of the form (2.18),

$$\hat{\tilde{F}} = \bar{\square} \hat{I} + \hat{P} - \frac{1}{6} R(\hat{g}) \hat{I}, \quad (2.21)$$

acting on some set of fields $\varphi = \varphi^A$, $\hat{\tilde{F}}\varphi = \hat{\tilde{F}}_B^A \varphi^B$, where $\hat{\tilde{P}} = \hat{\tilde{P}}_B^A$ is an arbitrary matrix, $\hat{I} = \delta_B^A$, $\bar{\square} = \bar{g}^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu$ is the 2ω -dimensional D'Alambertian and $\bar{\nabla}_\mu$ is the covariant derivative with any torsion-free connection covariantly conserving the metric $\bar{g}_{\mu\nu}$. Let the commutator of these covariant derivatives be given by the action of the matrix $\hat{\mathcal{R}}_{\mu\nu}$:

$$(\bar{\nabla}_\mu \bar{\nabla}_\nu - \bar{\nabla}_\nu \bar{\nabla}_\mu)\varphi = \hat{\mathcal{R}}_{\mu\nu} \varphi, \quad \hat{\mathcal{R}}_{\mu\nu} = \hat{\mathcal{R}}_{\mu\nu}^A. \quad (2.22)$$

Then the logarithmically divergent part of the one-loop effective action for the operator (2.21) reads [44]

$$\frac{i}{2} \text{Tr} \ln \hat{\tilde{F}} |^{\text{div}} = \frac{1}{32\pi^2(2-\omega)} \int d^4x \bar{g}^{1/2} \text{tr} \hat{a}_2, \quad (2.23)$$

where $\omega \rightarrow 2$ is half the dimensionality of space-time playing the role of the parameter in the dimensional regularization and the DeWitt coefficient \hat{a}_2 is defined by the expression

$$\begin{aligned} \hat{a}_2 &= \frac{1}{180} \left\{ R_{\alpha\beta\mu\nu}^2(\hat{g}) - R_{\mu\nu}^2(\hat{g}) + \bar{\square} R(\hat{g}) \right\} \\ &+ \frac{1}{2} \hat{\tilde{P}}^2 + \frac{1}{12} \hat{\mathcal{R}}_{\mu\nu}^2 + \frac{1}{6} \bar{\square} \hat{\tilde{P}}. \end{aligned} \quad (2.24)$$

Therefore, the divergent part $W_{\text{one loop}}^{\text{div}}$ of the one-loop effective action (2.1) has the form

$$\begin{aligned} W_{\text{one loop}}^{\text{div}} &= \frac{1}{32\pi^2(2-\omega)} \int d^4x \mathcal{G}^{1/2} \text{tr} \hat{a}_2 \\ &- \frac{1}{16\pi^2(2-\omega)} \int d^4x \mathcal{G}^{1/2} a_2^\mu. \end{aligned} \quad (2.25)$$

Here \hat{a}_2 and a_2^μ are the DeWitt coefficients of the operators $\hat{F}(\mathcal{D})$ and $Q_\mu^\alpha(\bar{\nabla})$ correspondingly. Let us first calculate the first term of this equation.

The metric $\bar{g}_{\mu\nu}$ and the curvatures of the algorithm (2.24) corresponding to the operator (2.18) are given by

$$\bar{g}^{\mu\nu} = \mathcal{G}^{\mu\nu}, \quad \bar{g}_{\mu\nu} = \mathcal{G}_{\mu\nu} = (\mathcal{G}^{\mu\nu})^{-1}, \quad (2.26)$$

$$R_{\alpha\beta\mu\nu}(\bar{g}) = \bar{R}_{\alpha\beta\mu\nu}, \quad R_{\mu\nu}(\bar{g}) = \bar{R}_{\mu\nu}, \quad R(\bar{g}) = \bar{R}, \quad (2.27)$$

while its covariant derivatives \mathcal{D}_μ and the potential term \hat{P} can be obtained from the matrix components of the nonminimal operator $F_{AB}(\bar{\nabla})$ (2.8) which have the form

$$\Gamma_{AB}^\sigma = \begin{pmatrix} 0 & C^{\mu\nu, \lambda\sigma} \varphi_{, \lambda} \\ -C^{\alpha\beta, \lambda\sigma} \varphi_{, \lambda} & 0 \end{pmatrix}, \quad (2.28)$$

$$W_{AB} = \begin{pmatrix} C^{\mu\nu, \lambda\sigma} P_{\lambda\sigma}^{\alpha\beta} & -\frac{1}{2} \mathcal{G}^{\mu\nu} \frac{\partial \bar{V}}{\partial \varphi} \\ -C^{\alpha\beta, \lambda\sigma} \bar{\nabla}_\lambda \bar{\nabla}_\sigma \varphi - \frac{1}{2} \mathcal{G}^{\alpha\beta} \frac{\partial \bar{V}}{\partial \varphi} & -\frac{\partial^2 \bar{V}}{\partial \varphi^2} \end{pmatrix}, \quad (2.29)$$

$$\begin{aligned}
P_{\lambda\sigma}^{\alpha\beta} = \frac{1}{k^2} & \left\{ 2\bar{R}_{(\lambda}{}^{\alpha}{}_{\sigma)}{}^{\beta} + 2\delta_{(\lambda}^{(\alpha} \bar{R}_{\sigma)}^{\beta)} - \delta_{\lambda\sigma}^{\alpha\beta} \bar{R} \right. \\
& - \mathcal{G}_{\lambda\sigma} \bar{R}^{\alpha\beta} - \mathcal{G}^{\alpha\beta} \bar{R}_{\lambda\sigma} + \frac{1}{2} \mathcal{G}_{\lambda\sigma} \mathcal{G}^{\alpha\beta} \bar{R} \left. \right\} \\
& + \frac{1}{2} \varphi_{,\mu} \varphi_{,\nu} \mathcal{G}^{\mu\nu} \delta_{\lambda\sigma}^{\alpha\beta} - 2\delta_{(\lambda}^{(\alpha} \varphi_{,\sigma)} \varphi^{,\beta)} + \bar{V} \delta_{\lambda\sigma}^{\alpha\beta} \\
& + \frac{1}{2} \mathcal{G}_{\lambda\sigma} \varphi^{,\alpha} \varphi^{,\beta} + \frac{1}{2} \mathcal{G}^{\alpha\beta} \varphi_{,\lambda} \varphi_{,\sigma} \\
& - \frac{1}{4} \mathcal{G}^{\alpha\beta} \mathcal{G}_{\lambda\sigma} \varphi^{,\mu} \varphi_{,\mu}. \tag{2.30}
\end{aligned}$$

They give rise to matrices of the operator (2.11)

$$\begin{aligned}
\hat{\Gamma}^{\sigma} &= \begin{pmatrix} 0 & k^2 \delta_{\rho\tau}^{\lambda\sigma} \varphi_{,\lambda} \\ -C^{\mu\nu,\lambda\sigma} \varphi_{,\lambda} & 0 \end{pmatrix}, \tag{2.31} \\
\hat{W} &= \begin{pmatrix} P_{\rho\tau}^{\mu\nu} & -2k^2 \bar{\nabla}_{(\rho} \bar{\nabla}_{\tau)} \varphi + k^2 \mathcal{G}_{\rho\tau} \frac{\partial \bar{V}}{\partial \varphi} \\ -\frac{1}{2} \mathcal{G}^{\mu\nu} \frac{\partial \bar{V}}{\partial \varphi} & -\frac{\partial^2 \bar{V}}{\partial \varphi^2} \end{pmatrix}, \tag{2.32}
\end{aligned}$$

leading to the following expression for the potential term \hat{P} in the minimal form of the operator (2.18):

$$\begin{aligned}
\hat{P} &= \hat{W} - (\bar{\nabla}_{\sigma} \hat{\Gamma}^{\sigma}) - \mathcal{G}_{\mu\nu} \hat{\Gamma}^{\mu} \hat{\Gamma}^{\nu} + \frac{1}{6} \bar{R} \hat{I} \\
&= \begin{pmatrix} A_{\mu\nu}^{\alpha\beta} + \frac{1}{6} \bar{R} \delta_{\mu\nu}^{\alpha\beta} & B_{\mu\nu} \\ E^{\alpha\beta} & D + \frac{1}{6} \bar{R} \end{pmatrix}, \tag{2.33} \\
A_{\mu\nu}^{\alpha\beta} &= k^2 P_{\mu\nu}^{\alpha\beta} + \frac{1}{2} k^2 \delta_{(\mu}^{(\alpha} \varphi_{,\nu)} \varphi^{,\beta)} - \frac{1}{4} k^2 \mathcal{G}^{\alpha\beta} \varphi_{,\mu} \varphi_{,\nu}, \tag{2.34}
\end{aligned}$$

$$B_{\mu\nu} = k^2 \left(\mathcal{G}_{\mu\nu} \frac{\partial \bar{V}}{\partial \varphi} - \bar{\nabla}_{\mu} \bar{\nabla}_{\nu} \varphi \right), \tag{2.35}$$

$$E^{\alpha\beta} = -\frac{1}{2} \mathcal{G}^{\alpha\beta} \frac{\partial \bar{V}}{\partial \varphi} - \frac{1}{2} \bar{\nabla}^{(\alpha} \bar{\nabla}^{\beta)} \varphi + \frac{1}{4} \mathcal{G}^{\alpha\beta} \bar{\square} \varphi, \tag{2.36}$$

$$D = -\frac{1}{2} \frac{\partial^2 \bar{V}}{\partial \varphi^2} + k^2 \mathcal{G}^{\rho\tau} \varphi_{,\rho} \varphi_{,\tau}, \tag{2.37}$$

and the corresponding commutator of covariant derivatives $(\mathcal{D}_{\alpha} \mathcal{D}_{\beta} - \mathcal{D}_{\beta} \mathcal{D}_{\alpha}) \psi = \hat{\mathcal{R}}_{\alpha\beta} \psi$:

$$\begin{aligned}
\hat{\mathcal{R}}_{\alpha\beta} &= \hat{\mathcal{R}}_{\alpha\beta}^0 + 2\bar{\nabla}_{[\alpha} \hat{\Gamma}_{\beta]} + 2\hat{\Gamma}_{[\alpha} \hat{\Gamma}_{\beta]} \\
&= \begin{pmatrix} X_{\rho\tau,\alpha\beta}^{\mu\nu} & Y_{\rho\tau,\alpha\beta} \\ Z_{\alpha\beta}^{\mu\nu} & 0 \end{pmatrix}, \tag{2.38}
\end{aligned}$$

where $\hat{\mathcal{R}}_{\alpha\beta}^0$ satisfies the commutation relation (2.22) with the derivatives replaced by $\bar{\nabla}_{\alpha}$, the square brackets imply the antisymmetrization in indices with the factor 1/2 and the blocks of the resulting matrix read

$$X_{\rho\tau,\alpha\beta}^{\mu\nu} = 2\delta_{(\rho}^{(\mu} \bar{R}_{\tau)\alpha\beta}^{\nu)} - 2k^2 \mathcal{G}_{\sigma[\alpha} \mathcal{G}_{\beta]\kappa} C^{\mu\nu,\lambda\sigma} \delta_{\rho\tau}^{\epsilon\kappa} \bar{\nabla}_{\lambda} \varphi \bar{\nabla}_{\epsilon} \varphi, \tag{2.39}$$

$$Y_{\rho\tau,\alpha\beta} = 2k^2 \delta_{\rho\tau}^{\lambda\sigma} \mathcal{G}_{\sigma[\alpha} \bar{\nabla}_{\beta]} \bar{\nabla}_{\lambda} \varphi, \tag{2.40}$$

$$Z_{\alpha\beta}^{\mu\nu} = 2C^{\mu\nu,\lambda\sigma} \mathcal{G}_{\sigma[\beta} \bar{\nabla}_{\alpha]} \bar{\nabla}_{\lambda} \varphi. \tag{2.41}$$

Using the above expressions in algorithm (2.24), we obtain the contribution of the first term in Eq. (2.25):

$$\begin{aligned}
\frac{i}{2} \text{Tr} \ln \hat{F}(\mathcal{D}) \Big|_{\text{div}} &= \frac{1}{32\pi^2(2-\omega)} \int d^4x \mathcal{G}^{1/2} \left\{ \frac{191}{180} \bar{R}_{\alpha\beta\mu\nu}^2 - \frac{551}{180} \bar{R}_{\alpha\beta}^2 + \frac{119}{72} \bar{R}^2 \right. \\
&+ \frac{5}{4} k^4 (\mathcal{G}^{\alpha\beta} \varphi_{,\alpha} \varphi_{,\beta})^2 + k^2 \mathcal{G}^{\alpha\beta} \varphi_{,\alpha} \varphi_{,\beta} \left(-\frac{1}{3} \bar{R} + k^2 \bar{V} - 2 \frac{\partial^2 \bar{V}}{\partial \varphi^2} \right) \\
&\left. - \frac{13}{3} k^2 \bar{R} \bar{V} - \frac{1}{6} \bar{R} \frac{\partial^2 \bar{V}}{\partial \varphi^2} + \frac{5}{4} k^4 \bar{V}^2 - 2k^2 \left(\frac{\partial \bar{V}}{\partial \varphi} \right)^2 + \frac{1}{2} \left(\frac{\partial^2 \bar{V}}{\partial \varphi^2} \right)^2 \right\}. \tag{2.42}
\end{aligned}$$

For the ghost operator (2.20) we have the following quantities participating in the algorithm (2.24) for the trace of the vector-field DeWitt coefficient a_2^{μ} :

$$\tilde{g}_{\mu\nu} = \mathcal{G}_{\mu\nu}, \quad \tilde{\nabla}_{\mu} = \bar{\nabla}_{\mu}, \tag{2.43}$$

$$\hat{P} = \bar{R}_{\alpha}^{\mu} + \frac{1}{6} \delta_{\alpha}^{\mu} \bar{R}, \tag{2.44}$$

$$\hat{\mathcal{R}}_{\alpha\beta} = (\mathcal{R}_{\alpha\beta})_{\mu\nu}^{\lambda\gamma} = -2\delta_{(\nu}^{(\gamma} \bar{R}_{\mu)\alpha\beta}^{\lambda)}. \tag{2.45}$$

Therefore, the ghost contribution to (2.25) equals

$$\text{Tr} \ln [\bar{\square} \delta_{\alpha}^{\mu} + \bar{R}_{\alpha}^{\mu}]^{\text{div}} = \frac{1}{16\pi^2(2-\omega)} \int d^4x \mathcal{G}^{1/2} \left\{ -\frac{11}{180} \bar{R}_{\alpha\beta\mu\nu}^2 + \frac{43}{90} \bar{R}_{\alpha\beta}^2 + \frac{2}{9} \bar{R}^2 \right\}, \tag{2.46}$$

and the total divergent part of the one-loop effective action for the theory with minimally coupled nonlinear scalar field (2.2) reads²

$$\begin{aligned}
W_{\text{one loop}}^{\text{div}} &= \frac{1}{32\pi^2(2-w)} \int d^4x \mathcal{G}^{1/2} \left\{ \frac{43}{60} \bar{R}_{\alpha\beta}^2 + \frac{1}{40} \bar{R}^2 + \frac{5}{4} k^4 (\mathcal{G}^{\alpha\beta} \varphi_{,\alpha} \varphi_{,\beta})^2 \right. \\
&+ k^2 \mathcal{G}^{\alpha\beta} \varphi_{,\alpha} \varphi_{,\beta} \left(-\frac{1}{3} \bar{R} + k^2 \bar{V} - 2 \frac{\partial^2 \bar{V}}{\partial \varphi^2} \right) + \bar{R} \left(-\frac{13}{3} k^2 \bar{V} - \frac{1}{6} \frac{\partial^2 \bar{V}}{\partial \varphi^2} \right) \\
&\left. + \frac{5}{2} k^4 \bar{V}^2 - 2k^2 \left(\frac{\partial \bar{V}}{\partial \varphi} \right)^2 + \frac{1}{2} \left(\frac{\partial^2 \bar{V}}{\partial \varphi^2} \right)^2 \right\}. \tag{2.47}
\end{aligned}$$

²We used the fact that $\int d^4x \mathcal{G}^{1/2} \{ \bar{R}_{\alpha\beta\mu\nu}^2 - 4\bar{R}_{\alpha\beta}^2 + \bar{R}^2 \}$ is the topological invariant which can be reduced to the surface integral.

B. Reduction method of conformal transformations

The method reducing the calculations in theories (1.1) and (1.2) to those of (2.2) consists in the following simple observation which we shall first demonstrate on the example of theory (1.2). It is a well-known fact [47,48] that under the conformal transformation

$$g_{\mu\nu} = \Omega^{-2} \mathcal{G}_{\mu\nu}, \quad \Omega^2 = 1 + b\phi^2, \quad b = -\frac{1}{2} k^2 \xi, \quad (2.48)$$

$$R = \bar{R} + 6\Omega\Box\Omega - 12\mathcal{G}^{\mu\nu}\Omega_{,\mu}\Omega_{,\nu}, \quad \bar{R} \equiv R(\mathcal{G}), \quad (2.49)$$

the action (1.2), $S[g, \phi]$, takes in terms of the new metric $\mathcal{G}_{\mu\nu}$ the following form free from the nonminimal interaction between the scalar and gravitational fields:

$$S[g, \phi] = \int d^4x \mathcal{G}^{1/2} \left\{ \frac{1}{k^2} \bar{R} - \frac{1}{2} \Omega^{-4} (1 - a\phi^2) \mathcal{G}^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - W(\phi) \right\}, \quad (2.50)$$

$$W(\phi) = \Omega^{-4} \left(\frac{2\Lambda}{k^2} + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right), \quad (2.51)$$

where $a \equiv \frac{1}{2} k^2 \xi (1 - 6\xi)$. However, the kinetic term of the Lagrangian in (2.50) contains an essential nonlinearity in ϕ . To eliminate it we introduce the new scalar field φ related to the old one by means of the differential equation

$$\Omega^{-4} (1 - a\phi^2) \mathcal{G}^{\mu\nu} \phi_{,\mu} \phi_{,\nu} = \alpha \mathcal{G}^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu}, \quad (2.52)$$

where $\alpha = \text{sgn}[\Omega^{-4} (1 - a\phi^2)]$. Later on we shall confine ourselves with $\alpha = +1$, because the negative sign corresponds to the ghost nature of a scalar field.

Relation (2.52) obviously holds if $\varphi = \varphi(\phi)$ satisfies the equation

$$\left(\frac{d\varphi}{d\phi} \right)^2 = \Omega^{-4} (1 - a\phi^2), \quad (2.53)$$

which has a solution

$$\varphi = \begin{cases} -\frac{\sqrt{a}}{b} \arcsin(\sqrt{a}\phi) - \frac{\sqrt{-(a+b)}}{2b} \ln \frac{1+z}{1-z}, \\ z = \phi \sqrt{-(a+b)/(1-a\phi^2)}, \quad 0 \leq \xi \leq \frac{1}{6}, \\ \frac{\sqrt{|a|}}{b} \operatorname{arcsinh}(\sqrt{|a|}|\phi|) - \frac{\sqrt{|a|-b}}{2b} \ln \frac{1+z}{1-z}, \\ z = \phi \sqrt{(|a|-b)/(1-|a|\phi^2)}, \quad \xi < 0 \text{ and } \xi > \frac{1}{6}, \end{cases} \quad (2.54)$$

where the constant of integration is defined by the condition $\varphi(\phi)|_{\phi=0} = 0$. From this solution it is obvious that the inverse expression $\phi = \phi(\varphi)$ cannot be obtained analytically. This fact does not, however, present any difficulty because for the calculation of one-loop divergencies of theory (1.2) [as well as for (1.1)] we shall not need an explicit expression for the potential $\bar{V}(\varphi)$, which can be written formally as

$$\bar{V}(\varphi) = W(\phi)|_{\phi=\phi(\varphi)}. \quad (2.55)$$

Now the calculation of one-loop divergences in theory (1.2) reduces to using the result (2.47) for a simplified model of minimal nonlinear scalar field (2.2) with the metric $\mathcal{G}_{\mu\nu}$ and the field φ reparametrized back to the original field variables. This reparametrization can be done by using the relations

$$\bar{R} = \Omega^{-2} R - 6\Omega^{-6} b \phi_{,\alpha} \phi^{,\alpha} - 6\Omega^{-4} b \phi \Box \phi, \quad (2.56)$$

$$\int d^4x \mathcal{G}^{1/2} \left\{ \frac{43}{60} \bar{R}_{\alpha\beta}^2 + \frac{1}{40} \bar{R}^2 \right\} = \int d^4x g^{1/2} \left\{ \frac{43}{60} R_{\alpha\beta}^2 + \frac{1}{40} R^2 - \frac{19}{6} b \Omega^{-4} R \phi_{,\alpha} \phi_{,\beta} - \frac{19}{6} b \Omega^{-2} R \phi \Box \phi + \frac{19}{2} \Omega^{-8} b^2 (\phi_{,\alpha} \phi^{,\alpha})^2 + 19 \Omega^{-6} b^2 \phi_{,\alpha} \phi^{,\alpha} \phi \Box \phi + \frac{19}{2} \Omega^{-4} b^2 \phi^2 (\Box \phi)^2 \right\}, \quad (2.57)$$

$$\int d^4x \mathcal{G}^{1/2} (\mathcal{G}^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta})^2 = \int d^4x g^{1/2} \Omega^{-8} (1 - a\phi^2) (\phi_{,\alpha} \phi^{,\alpha})^2, \quad (2.58)$$

$$\int d^4x \mathcal{G}^{1/2} \bar{R} \mathcal{G}^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} = \int d^4x g^{1/2} (1 - a\phi^2) \{ \Omega^{-4} R \phi_{,\alpha} \phi^{,\alpha} - 6 \Omega^{-8} b (\phi_{,\alpha} \phi^{,\alpha})^2 - 6 \Omega^{-6} b \phi_{,\alpha} \phi^{,\alpha} \phi \Box \phi \}, \quad (2.59)$$

and also expression (2.55) for the scalar potential \bar{V} in the old variables, which allows one to calculate in the same variables the derivatives

$$\begin{aligned} \frac{\partial \bar{V}}{\partial \varphi} &\equiv \left. \frac{\partial \bar{V}}{\partial \varphi} \right|_{\varphi=\varphi(\phi)} = \left(\frac{\partial W(\phi)}{\partial \phi} \right) \left(\frac{\partial \varphi}{\partial \phi} \right)^{-1} \\ &= \Omega^{-4} \phi \frac{(4b\lambda/k^2) + m^2 + (\lambda/6)\phi^2 - m^2 b \phi^2}{\sqrt{1 - a\phi^2}}, \end{aligned} \quad (2.60)$$

$$\begin{aligned} \frac{\partial^2 \bar{V}}{\partial \varphi^2} &\equiv \left. \frac{\partial^2 \bar{V}}{\partial \varphi^2} \right|_{\varphi=\varphi(\phi)} = \left(\frac{\partial^2 W(\phi)}{\partial \phi^2} \frac{\partial \varphi}{\partial \phi} - \frac{\partial W(\phi)}{\partial \phi} \frac{\partial^2 \varphi}{\partial \phi^2} \right) \left(\frac{\partial \varphi}{\partial \phi} \right)^{-3} \\ &= \frac{\Omega^{-4}}{(1 - a\phi^2)^2} \left\{ \frac{8b\Lambda}{k^2} + m^2 + \phi^2 \left(\frac{24b^2\Lambda}{k^2} + \frac{1}{2}\lambda - 6m^2b \right) \right. \\ &\quad \left. + \phi^4 \left(-\frac{32ab^2\Lambda}{k^2} - \frac{1}{6}\lambda b - \frac{1}{3}\lambda a + m^2b^2 + 6abm^2 \right) + \phi^6 \left(\frac{2}{3}ab\lambda - 2ab^2m^2 \right) \right\}. \end{aligned} \quad (2.61)$$

Finally, the one-loop divergencies of theory (1.2) take the form

$$\begin{aligned} W_{\text{one loop}}^{\text{div}}[g, \phi] &= \frac{1}{32\pi^2(2-\omega)} \int d^4x g^{1/2} \left\{ \frac{43}{60} R_{\alpha\beta}^2 + \frac{1}{40} R^2 \right. \\ &\quad \left. + \left[\frac{19}{2} b^2 + \frac{5}{4} k^4 (1 - a\phi^2)^2 + 2k^2 b(1 - a\phi^2) \right] \Omega^{-8} (\phi, \alpha \phi, \alpha)^2 \right. \\ &\quad \left. + [19b^2 + 2k^2 b(1 - a\phi^2)] \Omega^{-6} \phi, \alpha \phi, \alpha \phi \square \phi + \frac{19}{2} \Omega^{-4} b^2 \phi^2 (\square \phi)^2 - \frac{19}{6} b \Omega^{-2} R \phi \square \phi \right. \\ &\quad \left. - \left[\frac{19}{6} b + \frac{1}{3} k^2 (1 - a\phi^2) \right] \Omega^{-4} R \phi, \alpha \phi, \alpha \right. \\ &\quad \left. + b \left(26k^2 W + \frac{\partial^2 \bar{V}}{\partial \varphi^2} \right) \phi \square \phi + k^2 \Omega^{-2} W [k^2 (1 - a\phi^2) + 26b] \phi, \alpha \phi, \alpha \right. \\ &\quad \left. + \Omega^{-2} \frac{\partial^2 \bar{V}}{\partial \varphi^2} [-2k^2 (1 - a\phi^2) + b] \phi, \alpha \phi, \alpha \right. \\ &\quad \left. + \Omega^2 \left(-\frac{13}{3} k^2 W(\phi) - \frac{1}{6} \frac{\partial^2 \bar{V}}{\partial \varphi^2} \right) R + \Omega^4 \left[5k^4 W^2(\phi) - 2k^2 \left(\frac{\partial \bar{V}}{\partial \varphi} \right)^2 + \frac{1}{2} \left(\frac{\partial^2 \bar{V}}{\partial \varphi^2} \right)^2 \right] \right\}. \end{aligned} \quad (2.62)$$

From these formulas it is clear that the one-loop divergencies have a very complicated structure (the counterterms are nonpolynomial in the scalar field ϕ). Consequently, the theory with the Lagrangian (1.2), as well as other theories which include gravity interacting with matter fields, is nonrenormalizable not only due to the counterterms quadratic in curvature, but also because of these nonpolynomial structures.

Let us consider one limiting case for the theory (1.2) when $\Lambda = \lambda = m = \xi = 0$. We have then $a = b = 0$, $\Omega^2 = 1$. Hence, the new field variables coincide with old ones:

$$\mathcal{G}_{\mu\nu} \equiv g_{\mu\nu}, \quad \varphi = \phi. \quad (2.63)$$

Action (2.2) takes the form

$$S[g, \phi] = \int d^4x g^{1/2} \left\{ \frac{1}{k^2} R - \frac{1}{2} g^{\mu\nu} \phi, \mu \phi, \nu \right\}, \quad (2.64)$$

and the divergent part of the one-loop effective action is defined by the expression

$$W_{\text{one loop}}^{\text{div}} = \frac{1}{32\pi^2(2-\omega)} \int d^4x g^{1/2} \left\{ \frac{43}{60} R_{\alpha\beta}^2 + \frac{1}{40} R^2 + \frac{5}{4} k^4 (g^{\alpha\beta} \phi, \alpha \phi, \beta)^2 - \frac{1}{3} k^2 g^{\alpha\beta} \phi, \alpha \phi, \beta R \right\}, \quad (2.65)$$

which, on mass shell, i.e., taking into account the equations of motion

$$\square \phi = 0, \quad R = \frac{1}{2} k^2 g^{\alpha\beta} \phi, \alpha \phi, \beta, \quad (2.66)$$

coincides with the well-known result of 't Hooft and Veltman [28]. Just as has been expected, expression (2.62) obtained is conformally invariant for the case of $k^2 \rightarrow \infty$, $m = 0$, and $\xi = 1/6$:

$$W_{\text{one loop}}^{\text{div}}[g, \phi] = \frac{1}{32\pi^2(2-\omega)} \int d^4x g^{1/2} \left\{ \frac{43}{60} \left(R_{\alpha\beta}^2 - \frac{1}{3} R^2 \right) - \frac{79}{6} \lambda \left(\phi \square \phi - \frac{1}{6} R \phi^2 \right) + \frac{91}{72} \lambda^2 \phi^4 \right\}. \quad (2.67)$$

For the generalized theory of nonminimal nonlinear scalar field (1.1) the analogue of the reparametrization (2.48) and (2.53) looks like

$$g_{\mu\nu} = \Omega^{-2} \mathcal{G}_{\mu\nu}, \quad \phi = \phi(\varphi), \quad (2.68)$$

$$\Omega^2 = U(\phi), \quad (2.69)$$

$$\left(\frac{d\varphi}{d\phi} \right)^2 = U^{-2}(\phi) \left[U(\phi) G(\phi) + 3 \left(\frac{dU}{d\phi} \right)^2 \right], \quad (2.70)$$

$$\bar{V}(\varphi) = U^{-2}(\phi) V(\phi)|_{\phi=\phi(\varphi)}. \quad (2.71)$$

It also reduces the theory to a simplified model (2.2) and allows us to write one of the main results of the present paper—the following answer for one-loop divergences in theory (1.1):

$$\begin{aligned} W_{\text{one loop}}^{\text{div}} &= \frac{1}{32\pi^2(2-\omega)} \int d^4x g^{1/2} \left\{ \frac{5}{2} U^{-2} V^2 - 2 U^2 \left(\frac{\partial \bar{V}}{\partial \varphi} \right)^2 + \frac{1}{2} U^2 \left(\frac{\partial^2 \bar{V}}{\partial \varphi^2} \right)^2 \right. \\ &+ \left[\left(\frac{45}{2} U^{-3} (U')^2 + U^{-2} G \right) V - 13 U^{-2} U' V' - \left(\frac{25}{4} U^{-1} (U')^2 + 2G + \frac{1}{2} U' \frac{d}{d\phi} \right) \frac{\partial^2 \bar{V}}{\partial \varphi^2} \right] \phi_{,\mu} \phi^{,\mu} \\ &- \left[\frac{13}{3} U^{-1} V + \frac{1}{6} U \frac{\partial^2 \bar{V}}{\partial \varphi^2} \right] R + \frac{43}{60} R_{\alpha\beta}^2 + \frac{1}{40} R^2 + \frac{43}{60} U^{-2} (U')^2 R^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - \frac{19}{12} U^{-1} U' R \square \phi \\ &- \left(\frac{5}{12} U^{-2} (U')^2 + \frac{19}{12} U^{-1} U'' + \frac{1}{3} U^{-1} G \right) R \phi_{,\mu} \phi^{,\mu} \\ &+ \left[\frac{711}{20} U^{-4} (U')^4 + \frac{13}{12} U^{-3} (U')^2 U'' - \frac{113}{120} U^{-2} (U'')^2 - \frac{199}{60} U^{-2} U' U''' + 13 U^{-3} (U')^2 G \right. \\ &\quad \left. - 2 U^{-2} U'' G - 3 U^{-2} U' G' + \frac{5}{4} U^{-2} G^2 \right] (\phi_{,\mu} \phi^{,\mu})^2 \\ &- \left[\frac{26}{5} U^{-2} U' U'' + \frac{111}{20} U^{-3} (U')^3 + 6 U^{-2} U' G \right] \phi_{,\alpha} \phi_{,\beta} \phi^{;\alpha\beta} \\ &\left. + \frac{43}{60} U^{-2} (U')^2 (\phi_{;\alpha\beta})^2 + \frac{199}{120} U^{-2} (U')^2 (\square \phi)^2 \right\}. \quad (2.72) \end{aligned}$$

Here primes are used to denote the derivatives of the generalized charges with respect to ϕ ,

$$U' = dU/d\phi, \quad U'' = d^2U/d\phi^2, \quad U''' = d^3U/d\phi^3, \quad G' = dG/d\phi, \quad V' = dV/d\phi,$$

etc., and the derivatives of the potential \bar{V} with respect to a new scalar field φ are given by

$$\frac{\partial \bar{V}}{\partial \varphi} = \frac{-2U^{-2} U' V + U^{-1} V'}{[U G + 3(U')^2]^{1/2}}, \quad (2.73)$$

$$\begin{aligned} \frac{\partial^2 \bar{V}}{\partial \varphi^2} &= \frac{1}{[U G + 3(U')^2]^2} \left[12 U^{-2} (U')^4 V - 9 U^{-1} (U')^3 V' \right. \\ &+ 3 (U')^2 V'' - 3 U' U'' V' + 5 U^{-1} (U')^2 G V - 2 U'' G V \\ &\left. + U G V'' - \frac{7}{2} U' G V' + U' G' V - \frac{1}{2} U G' V' \right]. \quad (2.74) \end{aligned}$$

It is the expression (2.72) which will be applied within the generalized renormalization-group approach to quantum gravity interacting with the scalar field.

III. RENORMALIZATION-GROUP EQUATIONS IN NONRENORMALIZABLE THEORIES

A. Renormalization group in multicharge theories

The idea of the renormalization-group theory can be expressed in terms of bare quantities (coupling constants, masses, fields) and counterterms to Lagrangians. A basic property of these bare quantities consists in the fact that being infinite they, after being substituted into corresponding Feynman diagrams, provide the cancellation of all ultraviolet divergences and give us finite values for all observable physical quantities, such as cross sections for scattering processes, physical masses, and so on. These finite charges and masses are called the renormalized ones, and the procedure of eliminating the ultraviolet divergences is called the renormalization.

The renormalization procedure requires at the intermediate stages some regularization which allows one to avoid ill-defined quantities during the elimination of divergences [27]. At the final stage of calculations one removes the regularization and obtains finite results. However, as a remnant of all these operations with infinities, one gets a certain ambiguity in the final results, which can be parametrized by a mass-dimensional parameter μ . The origin of this parameter is different in various regularization schemes. In the minimal subtraction scheme of dimensional regularization [64,65], which we shall use in this paper, μ appears as a dimension-correcting parameter.

Now, we can go to the definition of the renormalizability of quantum field-theoretical models. A quantum field model is called renormalizable if ultraviolet divergences in all orders of the perturbation theory can be canceled by inserting into the Lagrangian of the theory a finite number of bare charges. In other words, there are only a finite number of field structures for which we obtain divergent coefficients. All these divergences can be canceled by adding to the initial "naive" classical Lagrangian a finite number of counterterms. In the opposite case, when there is an infinite number of divergent structures, the theory is called nonrenormalizable.

We have already mentioned that the renormalization-procedure ambiguity should not affect the values of physically observable quantities. This requirement, at least in the case of renormalizable theories, can be rewritten as a requirement of the independence of bare quantities on the renormalization mass parameter μ^2 (see [41]). This condition implies certain equations regulating the dependence of renormalized charges (or other quantities) on the renormalization mass parameter. These equations are usually called renormalization-group equations, because different reparametrizations of the procedure of eliminating the ultraviolet divergences constitutes a group. Solving these renormalization-group equations in some perturbative approximation gives an opportunity to make a

partial summation of the perturbation series.

To begin with, we shall write down the renormalization-group equation for a usual renormalizable theory with one charge (coupling constant) in the minimal subtraction scheme of the dimensional regularization. In this theory a bare charge g_b can be expressed through a renormalized one g as

$$g_b = (\mu^2)^\varepsilon \left[g + \sum_{n=1}^{\infty} \frac{a_n(g)}{\varepsilon^n} \right], \quad (3.1)$$

where $\varepsilon \equiv 4 - 2\omega$ is a parameter of dimensional regularization. Introducing the notion of a β function as the following derivative of the renormalized charge at fixed value of the bare charge,

$$\mu^2 \frac{dg}{d\mu^2} \Big|_{g_b} = -\varepsilon g + \beta(g), \quad (3.2)$$

and differentiating (3.1) with respect to μ^2 , we make the bare charge g_b independent of μ^2 by imposing the equation

$$0 = \varepsilon \left[g + \sum_{n=1}^{\infty} \frac{a_n(g)}{\varepsilon^n} \right] + [-\varepsilon g + \beta(g)] \left[1 + \sum_{n=1}^{\infty} \frac{a'_n(g)}{\varepsilon^n} \right]. \quad (3.3)$$

Then, equating the coefficients of equal powers of ε , we find

$$\beta(g) = \left(g \frac{\partial}{\partial g} - 1 \right) a_1(g) \quad (3.4)$$

and

$$\left(g \frac{\partial}{\partial g} - 1 \right) a_n(g) = \beta(g) \frac{\partial}{\partial g} a_{n-1}(g). \quad (3.5)$$

Thus, we see that knowing the coefficient $a_1(g)$ at the pole $\frac{1}{\varepsilon}$, we can determine the coefficients at higher poles by Eq. (3.5). In addition, we see that the β function defined by (3.2) is determined by the coefficient of the first-order pole in ε . The knowledge of the β function, on the other hand, gives, because of the equation

$$\mu^2 \frac{dg}{d\mu^2} = \beta(g), \quad (3.6)$$

the dependence of g on μ^2 , which can be transformed into the dependence of g and relevant Green's functions on the energy scale factor.

The generalization of this method to multicharge theories is straightforward [66–68]. Let us suppose that we have the set of charges g_i where $i = 1, \dots, N$. Then Eqs. (3.1) and (3.2) turn into the sets of equations

$$g_{bi} = (\mu^2)^\varepsilon \left[g_i + \sum_{n=1}^{\infty} \frac{a_{i n}(g_1, \dots, g_N)}{\varepsilon^n} \right], \quad (3.7)$$

$$\mu^2 \frac{dg_i}{d\mu^2} \Big|_{g_{bi}} = -\varepsilon g_i + \beta_i(g_1, \dots, g_N). \quad (3.8)$$

After differentiating Eq. (3.7) with respect to μ^2 and substituting into the corresponding expression definition

(3.8), we compare the coefficients of equal powers of ε and get the following expressions for β functions:

$$\beta_i(g_1, \dots, g_N) = -a_{i1}(g_1, \dots, g_N) + \sum_{j=1}^N g_j \frac{\partial a_{i1}(g_1, \dots, g_N)}{\partial g_j}. \quad (3.9)$$

Thus, instead of one renormalization-group equation (3.6), we get the whole system of coupled equations:

$$\mu^2 \frac{dg_i(\mu^2)}{d\mu^2} = \beta_i(g_1, \dots, g_N). \quad (3.10)$$

B. Functional charges and renormalization-group equations in partial derivatives

Although all this concerns renormalizable models, it was shown [37] that the renormalization-group equations could be generalized to theories with Lagrangians of arbitrary form, including nonrenormalizable ones. The main idea consists in the assumption that the bare Lagrangian, which can include an infinite number of counterterms, does not depend on the renormalization mass parameter μ^2 . This Lagrangian can be represented in the form

$$L^b = (\mu^2)^\varepsilon \left[L + \sum_{n=1}^{\infty} \frac{A_n L}{\varepsilon^n} \right], \quad (3.11)$$

where the counterterms $A_n L$ are functionals of the renormalized Lagrangian. Introducing the following definition of the β function,

$$\mu^2 \frac{dL}{d\mu^2} = -\varepsilon L + \beta_L,$$

and differentiating (3.11) with respect to μ^2 , we have the relations

$$\beta_L = \left(L \frac{\delta}{\delta L} - 1 \right) A_{1L},$$

$$\left(L \frac{\delta}{\delta L} - 1 \right) A_n L = \beta(L) \frac{\delta}{\delta L} A_{n-1} L.$$

Thus, the generalized β function of the Lagrangian is determined by the coefficient of the first-order pole in (3.11). The recurrent relations give the higher-order poles, so that the only independent function is the coefficient of the first-order pole.

In spite of the theoretical significance of this formalism, it can hardly be used in the concrete theories, such as quantum gravity. Therefore our purpose is to develop a formalism, which is less abstract than the formalism proposed in [37], but at the same time is convenient for treating concrete models, in particular, Einstein gravity theory interacting with a scalar field. In the usual renormalization-group formalism we deal with charges, masses, and renormalization-field constants which are all simple functions of μ^2 —the coefficients of some special field structures [for example, in the φ^4 model $\lambda(\mu^2)$ is a coefficient of φ^4 , m^2 is a coefficient of φ^2 , etc.]. In the

formalism of [37] one does not subdivide the Lagrangian into some simpler structures. Our formalism occupies the intermediate position between the two approaches mentioned above. Instead of the usual numerical charges related to fixed field structures, we introduce the generalized functional charges—generally arbitrary functions of the scalar field, which appear in the Lagrangian as coefficients of certain powers of space-time derivatives of the scalar field, powers of space-time curvature, and covariant derivatives of the curvature. Thus, in the generalized model (1.1) we consider as such charges the functions $U(\phi)$, $G(\phi)$, and $V(\phi)$ which do not contain the dependence on $\partial_\mu \phi$. They enter the Lagrangian in the combinations $RU(\phi)$, $G(\phi)\partial_\mu \phi \partial_\nu \phi g^{\mu\nu}$, and $V(\phi)$ which contain no more than second derivatives of field variables. It is obvious that we must include into the Lagrangian also terms which are quadratic in curvatures and have a fourth power in derivatives of a scalar field, and, generally, also an infinite number of different generalized charges which correspond to terms with a growing number of derivatives in the bare Lagrangian. As a result we would have an infinite system of renormalization-group equations with an infinite number of unknown functional variables. However we shall restrict ourselves only to these three terms and justify it by considering only those physical problems which can be characterized by intensive but slowly varying fields and small curvatures. As is discussed in the next section, this approximation makes sense at certain stages of the early inflationary Universe.

Thus, giving up all higher-derivative terms, we truncate our system of renormalization-group equations and reduce it to three equations with three unknown functions. Let us deduce them. In analogy with the usual formalism, we introduce the bare quantities

$$U_b = (\mu^2)^\varepsilon \left[U + \sum_{n=1}^{\infty} \frac{A_n U}{\varepsilon^n} \right],$$

$$G_b = (\mu^2)^\varepsilon \left[G + \sum_{n=1}^{\infty} \frac{A_n G}{\varepsilon^n} \right], \quad (3.12)$$

$$V_b = (\mu^2)^\varepsilon \left[V + \sum_{n=1}^{\infty} \frac{A_n V}{\varepsilon^n} \right],$$

where $A_n U$, $A_n G$, and $A_n V$ are the counterterms which correspond to structures U , G , and V , respectively. We should also define the generalized β functions

$$\beta_U = \mu^2 \frac{\partial U}{\partial \mu^2} + \varepsilon U, \quad (3.13)$$

$$\beta_G = \mu^2 \frac{\partial G}{\partial \mu^2} + \varepsilon G, \quad (3.14)$$

$$\beta_V = \mu^2 \frac{\partial V}{\partial \mu^2} + \varepsilon V. \quad (3.15)$$

Now differentiating Eqs. (3.12) with respect to μ^2 and assuming the independence of the bare quantities on μ^2 , we have the equation

$$\begin{aligned}
0 &= \mu^2 \frac{\partial U_b}{\partial \mu^2} \\
&= \left[\varepsilon U + \mu^2 \frac{\partial U}{\partial \mu^2} + \sum_{n=1}^{\infty} \mu^2 \frac{1}{\varepsilon^n} \frac{\delta A_n U}{\delta U} \frac{\partial U}{\partial \mu^2} + \sum_{n=1}^{\infty} \varepsilon \frac{A_n U}{\varepsilon^n} \right] (\mu^2)^\varepsilon \\
&\quad + \left[\sum_{n=1}^{\infty} \mu^2 \frac{1}{\varepsilon^n} \frac{\delta A_n U}{\delta G} \frac{\partial G}{\partial \mu^2} + \sum_{n=1}^{\infty} \mu^2 \frac{1}{\varepsilon^n} \frac{\delta A_n U}{\delta V} \frac{\partial V}{\partial \mu^2} \right] (\mu^2)^\varepsilon, \tag{3.16}
\end{aligned}$$

and corresponding equations for G_b and V_b . Substituting into these equations the proposed definitions (3.13)–(3.15) for β_U , β_G , and β_V and also equating the coefficients of equal powers of ε , one can obtain (from terms of zeroth power in ε)

$$\beta_U = -A_{1U} + \frac{\delta A_{1U}}{\delta U} U + \frac{\delta A_{1U}}{\delta G} G + \frac{\delta A_{1U}}{\delta V} V, \tag{3.17}$$

$$\beta_G = -A_{1G} + \frac{\delta A_{1G}}{\delta U} U + \frac{\delta A_{1G}}{\delta G} G + \frac{\delta A_{1G}}{\delta V} V, \tag{3.18}$$

$$\beta_V = -A_{1V} + \frac{\delta A_{1V}}{\delta U} U + \frac{\delta A_{1V}}{\delta G} G + \frac{\delta A_{1V}}{\delta V} V. \tag{3.19}$$

It should be emphasized that in contrast to usual multicharge theories the counterterm coefficients $A_{nU,G,V}$ are not simply functions of the generalized charges, but their local one-point functionals (in the sense of parametric dependence on one point ϕ of the configuration space of a scalar field) $A_{nU,G,V} = A_{nU,G,V}(\phi)[U(\phi'), G(\phi'), V(\phi')]$, and therefore the multiplication of functional derivatives with the corresponding charges in the above equations has a functional nature which should read as

$$\frac{\delta A_n}{\delta U} U = \int d\phi' \frac{\delta A_{1U}(\phi)}{\delta U(\phi')} U(\phi')$$

(similar equations hold for the derivatives with respect to the other charges). However, as is easily seen from Eq. (2.72), $A_{nU,G,V}$ are local functionals on the configuration space of a scalar field, and, therefore, the functional derivatives of the above type represent the differential operators with respect to ϕ :

$$\frac{\delta A_{1U}(\phi)}{\delta U(\phi')} = F(\partial/\partial\phi) \delta(\phi - \phi'),$$

etc. That is why we shall keep the condensed notation of Eqs. (3.16)–(3.19) bearing in mind this differential-operator structure of functional derivative coefficients. In this way Eqs. (3.17)–(3.19) generalize relations (3.9) for the usual β functions in multicharge models.

Now, to calculate the generalized β functions, we read from $W_{\text{one loop}}^{\text{div}}$ (2.72), which was found in the preceding section, the counterterms renormalizing the initial structures V , G , and U in the Lagrangian [the negatives of the coefficients of the corresponding structures in (2.72)]:

$$A_{1V} = \frac{1}{32\pi^2} \left[\frac{5}{2} U^{-2} V^2 - 2 U^2 \left(\frac{\partial \bar{V}}{\partial \varphi} \right)^2 + \frac{1}{2} U^2 \left(\frac{\partial^2 \bar{V}}{\partial \varphi^2} \right)^2 \right], \tag{3.20}$$

$$A_{1G} = \frac{1}{32\pi^2} \left[\left(45 U^{-3} (U')^2 + 2 U^{-2} G \right) V - 26 U^{-2} U' V' - \left(\frac{25}{2} U^{-1} (U')^2 + 4G + U' \frac{d}{d\phi} \right) \left(\frac{\partial^2 \bar{V}}{\partial \varphi^2} \right) \right], \tag{3.21}$$

$$A_{1U} = \frac{1}{32\pi^2} \left[\frac{13}{3} U^{-1} V + \frac{1}{6} U \left(\frac{\partial^2 \bar{V}}{\partial \varphi^2} \right) \right], \tag{3.22}$$

and substitute these expressions into (3.17)–(3.19). Then, taking into account Eqs. (2.73) and (2.74) for $\partial \bar{V}/\partial \varphi$ and $\partial^2 \bar{V}/\partial \varphi^2$, one can obtain the needed β functions in an explicit form. The calculation of the functional-derivative terms in (3.17)–(3.19) can be essentially simplified due to the observation that these terms actually represent the homogeneous transformation of all three functional arguments of A_{1V} , A_{1G} , and A_{1U} . From the above expressions it follows, however, that A_{1V} , A_{1G} , and A_{1U} are homogeneous functionals of zeroth order in these arguments, and, therefore, these terms do not contribute to β functions. Thus, in our model (and in our approximation) the β functions reduce to the counterterms themselves,

$$\beta_V = -A_{1V}, \quad \beta_G = -A_{1G}, \quad \beta_U = -A_{1U}, \tag{3.23}$$

which we present, in view of their complexity, in the Appendix as explicit functions of the generalized charges and their derivatives with respect to ϕ .

Thus the truncated system of renormalization-group equations has the form where we explicitly show the dependence of β functions on generalized charges and their derivatives:

$$\frac{\partial U}{\partial t} = \beta_U(U, U', U'', V, V', V'', G, G'), \tag{3.24}$$

$$\frac{\partial G}{\partial t} = \beta_G(U, U', U'', U''', V, V', V'', V''', G, G', G''), \tag{3.25}$$

$$\frac{\partial V}{\partial t} = \beta_V(U, U', U'', V, V', V'', G, G'). \quad (3.26)$$

In contrast with the usual renormalization-group equations, we work with functions which depend not only on parameter t , but also have a nontrivial and unknown algebraic dependence on the field variables. It makes this system of equations much more complicated than the usual one and much more rich: instead of ordinary differential equations they represent the differential equations *in partial derivatives*. One can say that due to the introduction of these generalized functional charges we rearranged our bare Lagrangian and could take into account an infinite number of elementary terms which arise in the process of renormalization. However, if in the usual renormalization-group equations, we investigate the dependence of effective charges on $t = \ln \mu^2$, here we have an additional problem, the study of the functional structure of our generalized charges. These two tasks, the investigation of a behavior of generalized charges at different t (that is a behavior at different scales) and the investigation of the functional structure of these charges, are combined together in the solution of differential equations in partial derivatives with respect to t and ϕ .

In view of the extremal complexity of β functions which we present in the Appendix the general solution of these equations is hardly available for exhaustive analysis without any simplifying assumptions about the structure of $U(\phi)$, $G(\phi)$, and $V(\phi)$. So the kind of information about these functions we shall be able to extract here will consist of the admissible large-field asymptotic behavior of the generalized charges $U(\phi)$, $G(\phi)$, and $V(\phi)$, which will turn to be compatible with the conventional choice of these functions in numerous quantum gravitational models. This asymptotic behavior unexpectedly hints us the existence of a simple *exact* solution describing the high-energy Weyl invariant and asymptotically free phase of the gravity theory considered in the next section.

IV. ASYMPTOTIC FREEDOM, WEYL INVARIANCE, AND OTHER IMPLICATIONS OF THE GENERALIZED RENORMALIZATION-GROUP THEORY

The implications of phenomenological particle-theory Lagrangians in the theory of the early Universe can be basically characterized by the conditions in which a large but slowly varying scalar field generates the effective cosmological constant which drives the inflationary stage of the Universe. This means that one mainly needs local terms in the Lagrangian of the theory in the limit of large ϕ , discarding the terms of high powers in its space-time derivatives. The magnitude of the corresponding space-time curvatures in such inflationary models is also supposed to be much below the Planck scale. Altogether these two properties exactly match with the assumptions of our approximation in the generalized renormalization-group theory, which allowed us to truncate the system of equations for generalized charges. Now, to make the formalism of generalized renormalization-group equations in

partial derivatives handlable, we can go even further and consider only the large-field behavior of these charges. For this purpose we shall look for the solution of our system of equations in the asymptotic form

$$U = u(t) \phi^{x_1} (\ln \phi)^{x_2}, \quad (4.1)$$

$$G = g(t) \phi^{y_1} (\ln \phi)^{y_2}, \quad (4.2)$$

$$V = v(t) \phi^{z_1} (\ln \phi)^{z_2}, \quad (4.3)$$

for $\phi \rightarrow \infty$. Note that such a combined power-logarithmic behavior is natural in field theory, because the emergence of logarithms in the effective potentials is a well-known phenomenon underlying the effects of symmetry breaking and phase transitions [69].

Substituting the chosen ansatz (4.1)–(4.3) into the system (3.24)–(3.26), we can compare in the limit $\phi \rightarrow \infty$ the largest powers of ϕ on the left- and right-hand sides of equations. This gives us the following two-parameter family of asymptotics (4.1)–(4.3) with arbitrary parameters (x_1, x_2) ,

$$y_1 = x_1 - 2, \quad y_2 = x_2, \quad z_1 = 2x_1, \quad z_2 = 2x_2, \quad (4.4)$$

and the following ordinary differential renormalization-group equations for the coefficients $u(t)$, $g(t)$ and $v(t)$:

$$\frac{du}{dt} = -\frac{1}{32\pi^2} \frac{13v}{3u}, \quad (4.5)$$

$$\frac{dg}{dt} = -\frac{1}{32\pi^2} \frac{v}{u^2} (2g - 7x_1^2 u), \quad (4.6)$$

$$\frac{dv}{dt} = -\frac{1}{32\pi^2} \frac{5v^2}{2u^2}. \quad (4.7)$$

In the traditional particle-physics models and their applications in the theory of inflationary cosmology, the parameters of the functions (4.1)–(4.3) in the Lagrangian (1.2) have the values³ $x_1 = 2$, $y_1 = 0$, $z_1 = 4$, which, obviously, satisfy the obtained restrictions (4.4) and, therefore, do not contradict the generalized renormalization-group theory.

The other interesting case, which can be considered, is related to the choice of U , G , and V in the exponential form. Such a choice originates from certain multidimensional theories which undergo dimensional reduction to an effective four-dimensional theory and result in a linear combination of exponential potentials [49]. They are interesting from the viewpoint of cosmological applications because they provide the power-law inflationary scenario [49–54]. Thus we assume that, for $\phi \rightarrow \infty$,

$$U(\phi) = u(t) \exp(\lambda_1 \phi), \quad (4.8)$$

$$G(\phi) = g(t) \exp(\lambda_2 \phi), \quad (4.9)$$

$$V(\phi) = v(t) \exp(\lambda_3 \phi), \quad (4.10)$$

and, by using the same procedure as above, obtain the

³Usually the potentials describing self-interaction of inflaton scalar field have a more complicated polynomial structure providing the possibility of symmetry breaking; however, in the limit $\phi \rightarrow \infty$ the term $\lambda\phi^4$ dominates.

following relations for λ_i :

$$\lambda_2 = \lambda_1, \quad \lambda_3 = 2\lambda_1, \quad (4.11)$$

similar to restrictions (4.4) and forming another one-parameter family of high-energy asymptotics.

It is interesting that the above-mentioned homogeneity of the one-loop counterterms in the generalized charges, which allowed us easily to calculate the β functions (3.23), and the homogeneity properties of β functions themselves give a one-parameter family of the *exact* solutions (4.1)–(4.3) with $x_2 = y_2 = z_2 = 0$ and the coefficients $u(t)$, $g(t)$, and $v(t)$ satisfying the set of *exact* equations (4.5)–(4.7). A trivial integration of the latter then gives the particular family of exact generalized charges:

$$V(\phi, t) = -\frac{192}{37} \pi^2 C^2 t^{15/37} \phi^{2x_1}, \quad (4.12)$$

$$G(\phi, t) = -\left(3x_1^2 C t^{26/37} - C_1 t^{12/37}\right) \phi^{x_1-2}, \quad (4.13)$$

$$U(\phi, t) = C t^{26/37} \phi^{x_1}, \quad (4.14)$$

where C and C_1 are two integration constants. Apart from negative overall coefficients in (4.12) and (4.13) this solution corresponds to the asymptotic freedom in the high-energy limit, because the effective gravitational constant $1/U$ vanishes in this limit, $t \rightarrow \infty$, and the growth of the nonlinear scalar potential (4.12) is compensated by the even faster growth of the coefficient of the scalar kinetic term G (4.13) (which means that the contribution of the higher-order Feynman graphs with scalar loops will be highly suppressed by the powers of a scalar field propagator proportional to $1/G$). The negative sign in (4.12), however, means that the scalar potential is negative, which apparently corresponds to the well-known property of the pure $\lambda\phi^4$ theory being asymptotically free only for the wrong sign of λ . This makes the only fixed point $2g = 7x_1^2 u$ of Eq. (4.6) unstable and, moreover, implies in view of Eqs. (4.13) and (4.14) that the effective gravitational constant and the kinetic term of the scalar field are of opposite signs, whence either the graviton or the scalar boson are supposed to be a ghost particle.

A possible qualitative interpretation of this seemingly unreasonable solution might consist in the following observation. Note that in the high-energy limit $t \rightarrow \infty$ there holds a relation $G(\phi, t) = -3x_1^2 U(\phi, t)/\phi^2$ between the asymptotic behaviors of the generalized charges (4.13) and (4.14). By redefining the old scalar field from ϕ to a new one $\varphi = \phi^{x_1/2}$ one can use this relation to show that the renormalized action (1.1) in this limit takes the form

$$\begin{aligned} S[g, \phi] = \int d^4x g^{1/2} \left\{ -\frac{192}{37} \pi^2 C^2 t^{15/37} \varphi^4 \right. \\ \left. + C t^{26/37} [R(g) \varphi^2 + 6(\nabla\varphi)^2] \right. \\ \left. + O(t^{12/37}) \right\}, \\ \varphi = \phi^{x_1/2}, \quad t \rightarrow \infty, \quad (4.15) \end{aligned}$$

where $O(t^{12/37})$ includes both the second term of Eq. (4.13) and the $O(t^0)$ terms of Eq. (2.72) of higher powers in space-time derivatives and curvatures. But this action is conformally invariant under the local Weyl transformations of the metric and scalar field

$$g'_{\mu\nu} = g_{\mu\nu} \Omega^2, \quad \varphi' = \varphi \Omega^{-1}, \quad (4.16)$$

whence it follows that a particular (monomial in ϕ) solution (4.12)–(4.14) of our renormalization-group equations describes in the ultraviolet limit a conformally invariant phase of the gravity theory.⁴ The wrong sign of the kinetic term of the field φ and its quartic interaction in (4.15) does not mean the physical instability of the theory, because this field is unphysical and represents a purely gauge mode of local transformations (4.16), which can (and must be) be gauged away by either imposing the conformal gauge condition $\varphi = 1$ or absorbing this field into the redefinition of the metric field

$$\mathcal{G}_{\mu\nu} = g_{\mu\nu} \varphi^2 = g_{\mu\nu} \phi^{x_1}. \quad (4.17)$$

In terms of this new metric the action (4.15) takes the form

$$\begin{aligned} S[g, \phi] = \int d^4x \mathcal{G}^{1/2} \left\{ C t^{26/37} \left[R(\mathcal{G}) - \frac{192}{37} \pi^2 C t^{-11/37} \right] \right. \\ \left. + O(t^{12/37}) \right\} \quad (4.18) \end{aligned}$$

of the asymptotically free Einstein theory with the positive (for $C > 0$) gravitational and cosmological constants

$$k^2 = C^{-1} t^{-26/37}, \quad \Lambda = \frac{192}{37} \pi^2 C t^{-11/37}, \quad (4.19)$$

both vanishing in the high-energy limit and providing the smallness of higher-order quantum perturbation corrections.

At lower energy scales the theory (4.15) loses its Weyl invariance, the scalar field φ (or ϕ) becomes dynamical, and due to its ghost nature induces the instability of the Weyl- (and scale) invariant phase. Therefore, for intermediate energies this rules out the above simple exact solutions of monomial type in ϕ . In full accordance with the loss of conformal invariance, the possible alternative solutions of our renormalization-group equations will have a polynomial structure in ϕ^{x_1} which necessarily induces in the theory the extra dimensional scale—the dimensional coefficients of different powers of ϕ (note that in the above formalism of monomial functional charges the scalar field and the constant C were subject to only one-dimensional restriction: the gravitational constant $1/U \sim 1/C\phi^{x_1}$ had to be of the squared length dimensionality). But these dimensionful quantities can enter the theory with

⁴A similar high-energy Weyl invariance of the nonminimal coupling between the scalar field and space-time curvature was found in Refs. [57,60,62,63] in the context of the renormalization-group theory for quantized matter in the external gravitational field.

generalized functional charges only through certain dimensional parameters of the quantum state of the theory, such as the value of the scalar field in a stable vacuum of the theory with broken symmetry. These parameters can constitute at least a part of the full initial data in the Cauchy problem for our renormalization-group equations, which is supposed to select a unique solution for the generalized functional charges. Presumably, this Cauchy problem has to be posed at some intermediate- or low-energy scale which corresponds to the low-energy physics of the observable Universe described by excitations over some stable vacuum state with broken conformal and scale invariance. Such a stable vacuum state and its dimensionful parameters are, in their turn, usually determined from the condition of stationarity of the corresponding effective potential (or more generally of the full effective action) with respect to the mean scalar and other fields. Thus the Cauchy problem for the generalized renormalization-group equations is related to the effective equations selecting a stable quantum state. This approach might represent a plausible development of the proposed formalism in application to the above model of quantum gravity theory, but it goes beyond the scope of this paper.

V. DISCUSSION AND CONCLUSIONS

Unfortunately, at the moment the picture, obtained thus far, does not imply much predictive power at intermediate-energy scales and does not give the quantitative mechanism of transition between the possible high-energy Weyl-invariant phase of the theory and our low-energy realm. One should bear in mind that the above interpretation has basically a qualitative nature, because at present we do not have a rigorous formalism incorporating the dynamical transition of the theory “defreezing” purely gauge modes into the physical dynamical ones. This problem is analogous to the issue of a rigorous quantization of classical gauge modes acquiring the dynamical content at the quantum level due to anomalies, which now has a well established status only in simple low-dimensional field theories.⁵ Nevertheless, our approach would seem to give certain selection rules for admissible Lagrangians of the inflaton scalar field nonminimally coupled to gravity and predict the existence of its nontrivial high-energy phase with very attractive features of asymptotic freedom and Weyl invariance. Just to summarize the difficulties of the above

model and of the whole formalism, let us briefly consider the questions of principal arising in the proposed generalized renormalization-group technique, which can serve as a guiding principle for possible further development of this approach.

The fundamental problem, which remains beyond the reach of our considerations, is the setting of the boundary-value problem for the renormalization-group equations (3.24)–(3.26). Since the β functions on their right-hand sides involve the generalized charges, functions of t and ϕ , as well as their derivatives with respect to ϕ , these boundary conditions consist in the Cauchy data, that is the functions of ϕ at some “moment” of t . These functions replace the initial values of usual charges in multicharge theories at some fixed energy scale. To see it, notice that our generalized charges are actually the result of partial summation in the theory with an infinite number of usual (numerical) charges: the expansion of $U(\phi)$, $G(\phi)$, and $V(\phi)$ in powers of ϕ recovers the infinite set of these usual charges as coefficients of this expansion. Therefore, the infinity of their initial values can be encoded in the functions of ϕ , which comprise the initial data for our renormalization-group equations. Unfortunately, we do not have at present exhaustive physical principles to fix this data, except the considerations, briefly mentioned above and relating this Cauchy problem to the search for stable quantum states of the theory.

Another approach to these equations, actually dominating the renormalization-group theory, consists in the analyses of the fixed points of (3.24)–(3.26) and does not essentially require the knowledge of this initial data. Again, a nontrivial generalization of the usual equations for fixed points,

$$\beta_U = 0, \quad \beta_G = 0, \quad \beta_V = 0, \quad (5.1)$$

is that in our case these equations are not algebraic, but rather represent ordinary differential equations of high order in derivatives with respect to ϕ . The analyses of these equations, which goes beyond the scope of this paper, would give the answers to the problem of the high-energy behavior of this conventionally nonrenormalizable theory, the ultraviolet or infrared stability of the fixed points, structure of the renormalization-group flows, etc. These equations will also require the constants of integration (the Cauchy problem of lower functional dimensionality) which again might be read off the stable quantum states in the theory.

This analysis would raise the basic conceptual issue behind the approximate nature of our generalized renormalization-group approach—the justification for the truncation of the system of charges to a finite set of the first few ones $U(\phi)$, $G(\phi)$, $V(\phi)$, etc. Our truncation was based on the physical assumption that in concrete problems under consideration the contribution of higher-order charges is negligible because of the slowly varying nature of the scalar field and small curvatures of space-time. This assumption can at best be justified only at the heuristic level, for virtual quantum disturbances of fields always probe in the renormalized Lagrangian arbitrarily high powers of their derivatives. The fundamental

⁵In connection with this one should mention an interesting approach extending the methods of the two-dimensional string models to the quantization of the conformal factor in four-dimensional gravity theory, undertaken in [70,71]. These references also contain the renormalization-group construction of a stable conformally invariant phase in the infrared limit of gravity theory—the domain which might be also attained within our approach via the as yet unknown solutions of the generalized renormalization-group equations.

solution of this problem would consist in the formulation of the generalized condition of asymptotic freedom or safety, which basically would reduce to a statement that at fixed points of the generalized renormalization-group flows all higher-order functional charges go to zero and thus justify our approximation. Anyway, the approach of this paper and a particular model demonstrating its complexity raise more questions than physically sensible predictions, but, probably, pave a path to more constructive attempts to renormalize conventionally non-renormalizable theories.

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APPENDIX

We list here the expressions for the β functions (3.23), with due regard for equations (3.20)–(3.22) and (2.73) and (2.74) for the counterterms:

$$\begin{aligned}
\beta_V = & -\frac{1}{32\pi^2 [UG + 3(U')^2]^4} \left\{ \frac{117}{2} U^{-2} (U')^8 V^2 + 108 U^{-1} (U')^7 V V' \right. \\
& -\frac{27}{2} (U')^6 (V')^2 + \frac{9}{2} U^2 (U')^2 (U'')^2 (V')^2 + \frac{9}{2} U^2 (U')^4 (V'')^2 \\
& -36 (U')^5 U'' V V' + 36 (U')^6 V V'' - 9 U^2 (U')^3 U'' V' V'' \\
& +27 U (U')^4 U'' (V')^2 - 27 U (U')^5 V' V'' \\
& +G \left(114 U^{-1} (U')^6 V^2 - \frac{45}{2} U (U')^4 (V')^2 + 129 (U')^5 V V' \right. \\
& -24 (U')^4 U'' V^2 + 27 U (U')^4 V V'' + 3 U (U')^3 U'' V V' \\
& -\frac{39}{2} U^2 (U')^3 V' V'' + 6 U^2 U' (U'')^2 V V' + \frac{21}{2} U^2 (U')^2 U'' (V')^2 \\
& \left. -3 U^3 U' U'' V' V'' - 6 U^2 (U')^2 U'' V V'' + 3 U^3 (U')^2 (V'')^2 \right) \\
& +G' \left(-3 U^2 (U')^2 U'' V V' + \frac{3}{2} U^3 U' U'' (V')^2 + 3 U^2 (U')^3 V V'' \right. \\
& -\frac{3}{2} U^3 (U')^2 V' V'' + 12 (U')^5 V^2 - 15 U (U')^4 V V' + \frac{9}{2} U^2 (U')^3 (V')^2 \\
& +G^2 \left(\frac{151}{2} (U')^4 V^2 - \frac{95}{8} U^2 (U')^2 (V')^2 + \frac{109}{2} U (U')^3 V V' \right. \\
& +2 U^2 (U'')^2 V^2 + \frac{1}{2} U^4 (V'')^2 - 10 U (U')^2 U'' V^2 \\
& +5 U^2 (U')^2 V V'' - 2 U^3 U'' V V'' - \frac{7}{2} U^3 U' V' V'' + 7 U^2 U' U'' V V' \\
& \left. + (G')^2 \left(\frac{1}{2} U^2 (U')^2 V^2 + \frac{1}{8} U^4 (V')^2 - \frac{1}{2} U^3 U' V V' \right) \right. \\
& +G G' \left(5 U (U')^3 V^2 - 6 U^2 (U')^2 V V' - 2 U^2 U' U'' V^2 \right. \\
& \left. +U^3 U'' V V' + \frac{7}{4} U^3 U' (V')^2 + U^3 U' V V'' - \frac{1}{2} U^4 V' V'' \right) \\
& \left. +G^3 (22 U (U')^2 V^2 + 8 U^2 U' V V' - 2 U^3 (V')^2) + \frac{5}{2} U^2 G^4 V^2 \right\},
\end{aligned}$$

$$\begin{aligned}
\beta_G = & -\frac{1}{32\pi^2 [UG + 3(U')^2]^3} \left\{ 837 U^{-3} (U')^8 V - \frac{855}{2} U^{-2} (U')^7 V' \right. \\
& - \frac{171}{2} U^{-1} (U')^6 V'' + \frac{171}{2} U^{-1} (U')^5 U'' V + 27 (U')^4 U'' V'' \\
& - 27 (U')^3 (U'')^2 V' + 9 (U')^4 U''' V' - 9 (U')^5 V'' \\
& + G \left(-\frac{1617}{4} U^{-1} (U')^5 V' + \frac{1701}{2} U^{-2} (U')^6 V + 57 U^{-1} (U')^4 U'' V \right. \\
& - \frac{177}{2} (U')^4 V'' - 24 (U')^2 (U'')^2 V + 69 (U')^3 U'' V' \\
& + 9 U (U')^2 U'' V'' + 3 U U' (U'')^2 V' + 3 U (U')^2 U''' V' \\
& \left. - 6 U (U')^3 G V''' + 6 (U')^3 U''' G V \right) \\
& + G' \left(-\frac{57}{2} U^{-1} (U')^5 V + \frac{39}{4} (U')^4 V' + 15 (U')^3 U'' V - 12 U (U')^2 U'' V' \right. \\
& \left. + \frac{9}{2} U (U')^3 V'' \right) + G'' \left(-3 (U')^4 V + \frac{3}{2} U (U')^3 V' \right) \\
& + G^2 \left(\frac{607}{2} U^{-1} (U')^4 V + 35 (U')^2 U'' V - \frac{497}{4} (U')^3 V' - 32 U (U')^2 V'' \right. \\
& \left. + \frac{35}{2} U U' U'' V' + 2 U U' U''' V - U^2 U' V''' \right) \\
& + G G' \left(-\frac{35}{2} (U')^3 V - 3 U U' U'' V + \frac{29}{4} U (U')^2 V' + \frac{3}{2} U^2 U' V'' \right) \\
& + (G')^2 (2 U (U')^2 V - U^2 U' V') + G G'' \left(-U (U')^2 V + \frac{1}{2} U^2 U' V' \right) \\
& + G^3 (43 (U')^2 V - 12 U U' V' + 8 U U'' V - 4 U^2 V'') \\
& \left. + G^2 G' (-4 U U' V + 2 U^2 V') + 2 U G^4 V \right\}, \\
\beta_U = & -\frac{1}{32\pi^2 [UG + 3(U')^2]^2} \left\{ 41 U^{-1} (U')^4 V - \frac{3}{2} (U')^3 V' - \frac{1}{2} U U' U'' V' \right. \\
& + \frac{1}{2} U (U')^2 V'' + G \left(\frac{161}{6} (U')^2 V - \frac{1}{3} U U'' V - \frac{7}{12} U U' V' + \frac{1}{6} U^2 V'' \right) \\
& \left. + G' \left(\frac{1}{6} U U' V - \frac{1}{12} U^2 V' \right) + \frac{13}{3} U G^2 V \right\}.
\end{aligned}$$

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