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Towards complete integrability of two-dimensional Poincaré gauge gravity

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It is shown that gravity on the line can be described by the two-dimensional (2D) Hilbert-Einstein Lagrangian supplemented by a kinetic term for the coframe and a translational *boundary* term. The resulting model is equivalent to a Yang-Mills theory of local *translations* and frozen Lorentz gauge degrees. We will show that this restricted Poincaré gauge model in two dimensions is completely integrable. *Exact* wave, charged black hole, and "dilaton" solutions are then readily found. In vacuum, the integrability of the *general* 2D Poincaré gauge theory is formally proved along the same line of reasoning.

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I. INTRODUCTION

Recently, two-dimensional (2D) models of gravity have attracted some attention as a conceptual "laboratory" for future studies of gravity in higher dimensions and as a basis of string theory. As is well known, the Hilbert-Einstein Lagrangian (s=signature)

$$V_{\rm HE}=(-1)^s\tfrac{1}{2}R^{\{\}\alpha\beta}\wedge\eta_{\alpha\beta} \tag{1.1}$$

of general relativity (GR) does not yield any Einstein-type equations in two spacetime dimensions. [For $n=2$, no inverse fundamental length l^{-1} occurs in (1.1) as a coupling constant; in n dimensions this factor would be l^{2-n}.] Therefore, in the approach of Teitelboim [1] and Jackiw (TJ model) [2,3], one had to resort to a dynamical model with constraints in which the field equation of constant or even vanishing [4] (scalar) curvature is enforced by means of a Lagrange multiplier. This *teleparallelism constraint* of the TJ model will be put in this paper in its proper perspective: Effectively, it yields a gauge theory of spacetime *translations.*

In fact, in $n=4$ dimensions, a theory of gravity with the constraint of vanishing Riemann-Cartan curvature $R^{\alpha\beta}$ is known as teleparallelism theory [5,6]. It is a gauge theory of local translations [7,8] and empirically indistinguishable from Einstein's general relativity theory. Moreover, teleparallelism theory remains nontrivial in $n=2$ dimensions and, as it turns out, has many salient features of the TJ model. In the context of string theory, 2D teleparallel models were actually studied previously [9–11].

In this paper we demonstrate the *complete integrability* of 2D teleparallelism in vacuum. In accordance with old mechanical knowledge on *generalized coordinates* [12], the Lagrange multiplier λ of the constraint $R^{\alpha\beta}=0$ converts into one of the two coordinates of our exact black-hole solution.

The coupling to gauge, scalar, and spinor matter is also studied. It is a peculiar but common feature of two dimensions that all these fields have a vanishing 2D spin current $\tau_{\alpha\beta}$. Thus the material energy-momentum current is symmetric and covariantly conserved with respect to the Riemannian connection. This already indicates that in two dimensions a decoupling from the Lorentz connection $\Gamma^{\alpha\beta}$ occurs. It considerably facilitates the integration of gravitationally coupled matter.

Constrained dynamical systems tend to become liberated classically or, ultimately, by quantum fluctuations. Nevertheless we will show for the first time that the *general* Poincaré gauge (PG) field equations [13] can be formally solved in two dimensions. For a *complete proof* of integrability, the gauge field momenta have to be invertible with respect to torsion T^α and curvature $R^{\alpha\beta}$. This puts only very mild restrictions on the form of the gravitational gauge Lagrangian. As an application we demonstrate that the general $R+T^2+R^2$ Lagrangian is completely integrable and has black-hole-type solutions [14,15]. In contrast with a previous proof of Katanaev and Volovich [16] (see also Ref. [17]), we do not have to rely on specific gauges, such as the conformal gauge for the coframe.

Our paper is organized as follows. In Sec. II the geometrical structure of Riemann-Cartan spacetime and some of its peculiarities in two dimensions are exhibited for both signatures of the metric. The transition from the Hilbert-Einstein Lagrangian to teleparallelism is motivated in Sec. III. The resulting field equations are reduced in Sec. IV in order to facilitate the proof of complete integrability in Sec. V. In general, we obtain a black-hole solution, whereas a constant torsion leads to the 2D "gravitational waves" of Sec. VI. The generalization to charged black holes is straightforward. As shown in Sec. VII, the gravitationally coupled Yang-Mills system is still completely integrable. The coupling to scalar fields is notoriously difficult; nevertheless, an exact dilaton-type solution has been obtained in Sec. VIII in the static massless case. For the Dirac field of Sec. IX, a complete decoupling from the gravitational field equations occurs at least for massless fermions. In Sec. X, the conserved Noether currents are presented such that the identification of the integration constant as the mass of

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the 2D black hole becomes finally established in Sec. XI. For an arbitrary PG gauge Lagrangian V, the general field equations are formally completely solved in Sec. XII. The former role of the Lagrange multiplier as a coordinate is now taken over by the gauge-field momentum conjugate to the curvature. Lagrangians with invertible gauge-field momenta turn out to be completely integrable. This new result is exemplified for the $R+T^2+R^2$ Lagrangian in Sec. XIII.

II. RIEMANN-CARTAN SPACETIME IN N AND IN TWO DIMENSIONS

The geometrical arena consists of an n-dimensional differentiable manifold M together with a metric

$$g=g_{ij}dx^i \otimes dx^j \tag{2.1}$$

and an *orthonormal* frame and coframe field, respectively:

$$e_\alpha = e^i{}_\alpha \partial_i, \quad \vartheta^\beta = e_j{}^\beta dx^j . \tag{2.2}$$

They are reciprocal to each other with respect to the *interior product* $\rfloor$, i.e.,

$$e_\alpha \rfloor \vartheta^\beta = e^i{}_\alpha e_i{}^\beta = \delta^\beta_\alpha . \tag{2.3}$$

In the following we adhere to the conventions (cf. Ref. [18]) that $\alpha,\beta,\gamma,\ldots=0,1,\ldots,n-1$ are holonomic or world indices, ∂_i are the tangent vectors, and $\wedge$ denotes the exterior product.

Anholonomic indices are lowered by means of the metric. The metric components with respect to an orthonormal frame read

$$o_{\alpha\beta}=e^i{}_\alpha e^j{}_\beta g_{ij} , \qquad (o_{\alpha\beta})=\mathrm{diag}(\underbrace{-1}_{s},\underbrace{1,\ldots,1}_{n-s}) . \tag{2.4}$$

For $s=1$ we have a Minkowskian and for $s=0$ we have a Euclidean signature. In order to be able to relate the pointwise attached tangent spaces to each other in a differentiable manner, we introduce a linear *connection* $\Gamma=\Gamma_\alpha{}^\beta L^\alpha{}_\beta$ with values in the Lie algebra of the n-dimensional rotation group SO(n) or "Lorentz" group SO($s,n-s$), respectively. With respect to a holonomic basis, the connection one-forms can be expanded as

$$\Gamma^{\alpha\beta}=\Gamma_i{}^{\alpha\beta}dx^i=-\Gamma^{\beta\alpha} . \tag{2.5}$$

Similarly as in the four-dimensional Poincaré gauge theory [13], the coframe ϑ^α and the connection $\Gamma^{\alpha\beta}$ are regarded as *gauge potentials* of local translations and local Lorentz transformations, respectively. The corresponding field strengths are given by the *torsion* two-form

$$T^\alpha:=D\vartheta^\alpha=d\vartheta^\alpha+\Gamma_\beta{}^\alpha\wedge\vartheta^\beta=\tfrac{1}{2}T_{ij}{}^\alpha dx^i\wedge dx^j \tag{2.6}$$

and the *curvature* two-form

$$R^{\alpha\beta}:=d\Gamma^{\alpha\beta}-\Gamma^{\alpha\gamma}\wedge\Gamma_\gamma{}^\beta=\tfrac{1}{2}R_{ij}{}^{\alpha\beta}dx^i\wedge dx^j = -R^{\beta\alpha} . \tag{2.7}$$

In a Riemann-Cartan (RC) spacetime in an orthonormal frame, the curvature, like the connection, is antisymmetric in α and β. For the irreducible decomposition of torsion and curvature in exterior form notation, see Ref. [19].

In order to isolate the Riemannian part of our RC spacetime we decompose the RC connection into the Levi-Civita connection $\Gamma^{\{\}}_{\alpha\beta}$ and the contortion one-form $K_{\alpha\beta}=-K_{\beta\alpha}$:

$$\Gamma_{\alpha\beta}=\Gamma^{\{\}}_{\alpha\beta}-K_{\alpha\beta} . \tag{2.8}$$

Algebraically, the contortion is the equivalent of the torsion according to $T^\alpha=\vartheta^\beta\wedge K_\beta{}^\alpha$. Then the curvature decomposes into Riemannian and contortion pieces as

$$R_{\alpha\beta}=R^{\{\}}_{\alpha\beta}-DK_{\alpha\beta}+K_\alpha{}^\gamma\wedge K_{\gamma\beta} . \tag{2.9}$$

We have developed the general geometrical formalism for arbitrary n dimensions. However, as we can infer already from Table I, for the *two-dimensional RC space* we have two translation and one rotation generators. This allows us to introduce a Lie (or right) duality operation, that is, a duality with respect to the Lie algebra indices, which maps a vector into a covector and vice versa:

$$\psi^\star_\alpha:=\eta_{\alpha\beta}\psi^\beta \Longleftrightarrow \psi^\alpha=-(-1)^s\eta^{\alpha\beta}\psi^\star_\beta . \tag{2.10}$$

The complete antisymmetric tensor is defined by

$$\eta_{\alpha\beta}:=\sqrt{|\det o_{\mu\nu}|}\,\epsilon_{\alpha\beta} ,$$

where $\epsilon_{\alpha\beta}$ is the Levi-Civita symbol normalized to $\epsilon_{\hat{0}\hat{1}}=+1$; for details of the η basis, see Appendix A. For $\psi^\beta=\vartheta^\beta$ we get $\vartheta^\star_\alpha=\eta_\alpha={}^*\vartheta_\alpha$. In the case of a bivector-valued p-form $\psi^{\alpha\beta}=-\psi^{\beta\alpha}$, the Lie dual is defined by

$$\psi^\star:=\tfrac{1}{2}\eta_{\alpha\beta}\psi^{\alpha\beta}\Longleftrightarrow\psi^{\alpha\beta}=(-1)^s\eta^{\alpha\beta}\psi^\star . \tag{2.11}$$

In two dimensions we can appreciably compactify formulas according to the notation given in Table II.

For $n=2$ torsion is irreducible and contains only the vector piece (vector-valued zero form; see Appendix B for further details):

$$T^\alpha:=d\vartheta^\alpha+(-1)^s\eta^\alpha\wedge\Gamma^\star=(-1)^s t^\alpha\eta . \tag{2.12}$$

TABLE I. Gauge-field strengths, matter currents, and η basis.

	Valuedness	p-form	Components	$n=4$	$n=3$	$n=2$
T^α	Vector	2	$n^2(n-1)/2$	24	9	2
$R^{\alpha\beta}$	Bivector	2	$n^2(n-1)^2/4$	36	9	1
Σ_α	Vector	$n-1$	n^2	16	9	4
$\tau_{\alpha\beta}$	Bivector	$n-1$	$n^2(n-1)/2$	24	9	2
η_α	Vector	$n-1$	n^2	16	9	4

TABLE II. 2D geometrical objects.

$n=2$	Valuedness	p-form	Components
$\Gamma^\star := \frac{1}{2}\eta_{\alpha\beta}\Gamma^{\alpha\beta}$	Scalar	1	2
$t^\alpha := {}^*T^\alpha$	Vector	0	2
$T := e_\alpha \rfloor T^\alpha$	Scalar	1	2
$t^2 := o_{\alpha\beta}t^\alpha t^\beta$	Scalar	0	1
$R^\star = d\Gamma^\star$	Scalar	2	1
$R := e_\alpha \rfloor e_\beta \rfloor R^{\alpha\beta}$	Scalar	0	1

Since the curvature two-form has only one irreducible component, it can be expressed in terms of the curvature scalar:

$$R := e_\alpha \rfloor e_\beta \rfloor R^{\alpha\beta} \Longleftrightarrow R^{\alpha\beta} = -\tfrac{1}{2} R\, \vartheta^\alpha \wedge \vartheta^\beta\ . \tag{2.13}$$

Let us confine ourselves to the case $s=1$ up to the end of this section. The local Lorentz transformations are defined by the 2×2 matrices $\Lambda_\beta{}^\alpha(x)\in SO(1,1)$ and, for the basic gravitational variables, read

$$\vartheta'^\alpha = \Lambda_\beta^{-1\alpha}\vartheta^\beta, \quad \Gamma_\alpha{}'^\beta = \Lambda_\alpha{}^\gamma \Gamma_\gamma{}^\delta \Lambda_\delta^{-1\beta} - \Lambda_\alpha{}^\gamma d\Lambda_\gamma^{-1\beta}\ . \tag{2.14}$$

With respect to the parametrization

$$\Lambda_\alpha{}^\beta = \delta_\alpha{}^\beta \cosh\omega + \eta_\alpha{}^\beta \sinh\omega\ , \tag{2.15}$$

Eqs. (2.14) can be rewritten as

$$\vartheta'^\alpha = \vartheta^\alpha \cosh\omega - \eta^\alpha \sinh\omega\ , \tag{2.16}$$

$$\Gamma_\alpha{}'^\beta = \Gamma_\alpha{}^\beta + \eta_\alpha{}^\beta d\omega \quad \text{or} \quad \Gamma^{\star\prime} = \Gamma^\star - d\omega\ . \tag{2.17}$$

III. TELEPARALLEL 2D GRAVITY

We regard gravity as a Yang-Mills-type gauge theory of translations [7]. In this approach the coframe ϑ^α and the torsion T^α are the associated gauge potentials and gauge-field strengths, respectively. (The intricate details of such a (generalized) affine gauge approach are spelled out in Ref. [8]. There local translations are considered as a "hidden" gauge symmetry such that no need for a "central extension" [4] arises.)

In our new model, the two-dimensional Hilbert-Einstein Lagrangian is supplemented by a *kinetic term* for the coframe, a *cosmological* term, and a *boundary* term. Since two-forms are constructed solely from the translational gauge potential ϑ^α, conventional general relativity appears to be rather minimally modified. Thus we consider, instead of (1.1), the 2D Lagrangian

$$V_\infty = V_{\rm HE} + (-1)^s \tfrac{1}{2} T^\alpha \wedge {}^*T_\alpha + \Lambda\eta - R^{\alpha\beta}\wedge\lambda_{\alpha\beta} - (-1)^s d(\vartheta^\alpha \wedge {}^*T_\alpha)\ . \tag{3.1}$$

The fourth term, depending on the Lagrange multiplier zero-form $\lambda_{\alpha\beta}$, will enforce the constraint $R^{\alpha\beta}=0$ of vanishing Riemann-Cartan curvature on the residual Lorentz degrees of freedom. This corresponds to the *teleparallelism condition* and will replace the Teitelboim-Jackiw constraint of constant or, recently, vanishing Riemannian curvature $R^{\{\}\alpha\beta}$.

In order to fully recognize the Yang-Mills-type structure of our new Lagrangian, we employ a geometric identity (see Eq. (5.4) of Ref. [6]) which relates GR to its teleparallelism equivalent $GR_\parallel$ in $n\geq 2$ dimensions. Since the torsion two-form is already irreducible for $n=2$, this identity reduces rather drastically to

$$-\tfrac{1}{2}R^{\alpha\beta}\eta_{\alpha\beta} + \tfrac{1}{2}R^{\{\}\alpha\beta}\eta_{\alpha\beta} \equiv d(\vartheta^\alpha\, {}^*T_\alpha)\ . \tag{3.2}$$

Then, our new Lagrangian (3.1) can be rewritten such that the total Lagrangian reads

$$L = V_\infty + L_{\rm mat} = (-1)^s \tfrac{1}{2} T^\alpha\, {}^*T_\alpha + \Lambda\eta + (-1)^s \tfrac{1}{2} R^{\alpha\beta}\eta_{\alpha\beta} - R^{\alpha\beta}\lambda_{\alpha\beta} + L_{\rm mat}\ . \tag{3.3}$$

This presentation of the Lagrangian clearly exhibits the leading Yang-Mills term for the translational field strength, whereas

$$\Lambda\eta = (\Lambda/2)\eta_{\alpha\beta}\vartheta^\alpha \wedge \vartheta^\beta$$

formally corresponds to a mass term of the coframe. Observe that the Einstein-Cartan term $\frac{1}{2}R^{\alpha\beta}\eta_{\alpha\beta} = R^\star = d\Gamma^\star$ is a boundary term in two dimensions and, consequently, will not contribute to the field equations.

IV. FIELD EQUATIONS

The gravitational field equations resulting from varying (3.3) with respect to ϑ^α, $\Gamma^{\alpha\beta}$, and $\lambda_{\alpha\beta}$ are

$$D^*T_\alpha - \tfrac{1}{2}(e_\alpha \rfloor T^\beta)^* T_\beta + (-1)^s \Lambda \eta_\alpha = -(-1)^s \Sigma_\alpha \quad \text{(first)}\ , \tag{4.1}$$

$$D\lambda_{\alpha\beta} - (-1)^s \vartheta_{[\alpha}{}^* T_{\beta]} = \tau_{\alpha\beta} \quad \text{(second)}\ , \tag{4.2}$$

$$R^{\alpha\beta} = 0\ . \tag{4.3}$$

Because of $D\eta_{\alpha\beta}=0$ in two dimensions, the Einstein-Cartan piece in (3.3) does not give a contribution. By relaxing the teleparallelism constraint (4.3), one would obtain a more complicated model (the quadratic theory with Yang-Mills-type terms in the Riemann-Cartan curvature and torsion was analyzed in Ref. [16]). We will defer the analysis of the general theory to Sec. XII. The right-hand sides are the current one-forms Σ_α and $\tau_{\alpha\beta}$ of energy-momentum and spin, respectively, of hypothetical two-dimensional matter. Equations (4.1) and (4.2) represent *four* and *two* independent components, respectively.

The integrability condition for the second field equation is identically satisfied, because

$$DD\lambda_{\alpha\beta} = -2R_{[\alpha}{}^\gamma \lambda_{\gamma|\beta]} = 0 \tag{4.4}$$

in a teleparallel (Weitzenböck) spacetime, whereas

$$D[\tau_{\alpha\beta}+(-1)^s\vartheta_{[\alpha}{}^*T_{\beta]}]=0 \tag{4.5}$$

follows from the "weak" Noether identity (10.2) for matter and gravitational gauge fields, together with the first field equation. Thus, the second field equation determines (nonuniquely) the Lagrange multiplier $\lambda_{\alpha\beta}$.

In order to simplify the field equations we substitute ${}^*T^\alpha=t^\alpha$ into the field equations (4.1) and (4.2), respectively, and recall the formula ${}^*(\Phi\wedge\vartheta_\alpha)=e_\alpha\rfloor{}^*\Phi$, which is valid for any p-form Φ. Moreover, note that the torsion square piece in the Lagrangian is proportional to t^2:

$$T^\alpha{}^*T_\alpha=(-1)^s t^2\eta \ . \tag{4.6}$$

Then we find

$$Dt_\alpha-(-1)^s(\tfrac{1}{2}t^2-\Lambda)\eta_\alpha=-(-1)^s\Sigma_\alpha \ , \tag{4.7}$$

$$D\lambda_{\alpha\beta}-(-1)^s\vartheta_{[\alpha}t_{\beta]}=\tau_{\alpha\beta} \ . \tag{4.8}$$

Let us represent the Lagrange multiplier as $\lambda_{\alpha\beta}=(-1)^s(\lambda/2)\eta_{\alpha\beta}$, where $\lambda=\lambda^\star$, according to the notation in (2.11). Then, in the last equation, it is more economical to switch over to its Lie dual by multiplying it with $\eta^{\alpha\beta}$:

$$\tfrac{1}{2}d\lambda+(-1)^s\tfrac{1}{2}t_\beta\eta^\beta=\tau^\star \tag{4.9}$$

We do not lose any of its four components if we multiply (4.7) by ϑ^β from the right and employ the formula $\eta_\alpha\wedge\vartheta^\beta=-\delta_\alpha^\beta\eta$:

$$D(t_\alpha\vartheta^\beta)-(-1)^s[t_\alpha t^\beta-\tfrac{1}{2}\delta_\alpha^\beta(t^2-2\Lambda)]\eta =-(-1)^s\Sigma_\alpha\wedge\vartheta^\beta \ . \tag{4.10}$$

Thereby, the energy-momentum current of the gravitational field is nicely represented. The trace of (4.10), on substitution of (B5) of Appendix B, reads

$$(-1)^s d^*T+2\Lambda\eta=\Sigma_\alpha\wedge\vartheta^\alpha \ . \tag{4.11}$$

In a similar move we substitute (B4) into (4.9):

$$d\lambda+T=2\tau^\star \ . \tag{4.12}$$

A very useful condition for the torsion-square function t^2 can be derived by contracting (4.7) with t^α:

$$\tfrac{1}{2}dt^2-(\tfrac{1}{2}t^2-\Lambda)T=-(-1)^s t^\alpha\Sigma_\alpha \ . \tag{4.13}$$

We eliminate T by means of (4.12) and find

$$dt^2+(t^2-2\Lambda)d\lambda=2[-(-1)^s t^\alpha\Sigma_\alpha+(t^2-2\Lambda)\tau^\star] \ . \tag{4.14}$$

Let us now specialize to the *vacuum* field equations. They read

$$dt^2=-(t^2-2\Lambda)d\lambda \ , \tag{4.15}$$

$$d^*T=-(-1)^s 2\Lambda\eta \ , \tag{4.16}$$

$$d\lambda=-T \ , \tag{4.17}$$

$$R_{\alpha\beta}=0 \ . \tag{4.18}$$

In two dimensions, the volume two-form η equips the spacetime manifold M with a *symplectic structure. In vacuo,* the volume two-form turns out, via the field equation

$$\eta=-(-1)^s d^*T/(2\Lambda)=(-1)^s d(\vartheta^\alpha t_\alpha)/(2\Lambda) \ ,$$

to be an exact form, as was conjectured by Cangemi and Jackiw (Eq. (2.A7a) of Ref. [4(b)]). Since this volume two-form appears explicitly in the Lagrangian (3.1), the cosmological term in (3.1) turns out to be "weakly" equivalent to the boundary term $-(-1)^s d^*T/2$. Thus, to some extent, Machian ideas are realized: In fact, the total volume μ of our 2D "world" is, due to Stokes' theorem, given by the integral of the dual torsion one-form along the *boundary:*

$$\mu(M)=\frac{-(-1)^s}{2\Lambda}\int_{\partial M}{}^*T \ . \tag{4.19}$$

On the other hand, the cosmological term cannot completely compensate the explicit "topological" term

$$d(\vartheta^\alpha\wedge{}^*T_\alpha)=-d^*T$$

in (3.1). Observe also that, according to Ref. [4(b)], the one-form ${}^*T$ seems to be related to a gauge one-form a associated with the *central extension* of the 2D Poincaré algebra.

In vacuo, T is also an exact form. If it were chosen as one of the basis one-forms, it would be a natural basis one-form; that is, the Lagrange multiplier λ could be interpreted as a coordinate. Such a transmutation of λ from a "constraint force" to a generalized coordinate is known from mechanics [12] and quantum cosmology [20]. However, in our model, the vacuum field equations (4.16) and (4.17) impose the wave equation

$$\Box\lambda:=(-1)^s({}^*d{}^*d+d{}^*d{}^*)\lambda=(-1)^s 2\Lambda \tag{4.20}$$

on the "would-be" coordinate λ. Fortunately, it turns out (see the next section) that this is merely a condition on a metric function unspecified so far. Thus, Eq. (4.20) resembles the harmonic gauge condition in 4D general relativity.

Formally, Eq. (4.20) has the solution

$$\lambda=(-1)^s 2\Box^{-1}\Lambda \ , \tag{4.21}$$

such that the constraint part of the Lagrangian (3.1), (3.3) takes the form

$$(-1)^s R^\star\lambda=2R^\star\Box^{-1}\Lambda \ . \tag{4.22}$$

By imposing the additional constraint $R=\Lambda$, one can obtain the "weak" relation

$$(-1)^s R^\star\lambda\cong 2R^\star\Box^{-1}R=-(R\Box^{-1}R)\eta \ . \tag{4.23}$$

In Riemannian spacetime, this term is easily recognized as Polyakov's "string inspired" [21] Lagrangian.

V. BLACK-HOLE SOLUTION AND COMPLETE INTEGRABILITY

The general quadratic Poincaré gauge theory in two dimensions (in the absence of matter) is known to be completely integrable [16,17]. Usually this fact is established with the help of a convenient choice of coordinates, such as the light-cone or conformal ones. We will demonstrate that the model under consideration is also completely integrable. Again the choice of coordinates will be an essential step, but we will use an approach discussed by Solodukhin [22].

Before integrating the gravitational equations it is worth noticing that the flat (Minkowski) spacetime arises when both the Riemann-Cartan curvature and the torsion are zero. The former is described by (4.18), but it is clear that torsion cannot be zero in the case of a nontrivial cosmological term in (3.1), (3.3), (4.15), and (4.16). Hence the flat Minkowski spacetime is not a vacuum solution of the theory. It is also evident that t^2 in general is nonzero—again the cosmological constant prevents its identical vanishing.

The vacuum equation (4.15), i.e.,

$$\frac{dt^2}{d\lambda}=-t^2+2\Lambda\ , \tag{5.1}$$

can easily be solved for $t^2\neq 2\Lambda$. It yields the square of the torsion as a function of the Lagrange multiplier:

$$t^2=2\Lambda+(-1)^s 2M_0 e^{-\lambda}\ . \tag{5.2}$$

Here M_0 denotes an integration constant which, for Minkowskian signature $s=1$, will later be identified with the active gravitational mass of the configuration. Observe that we recover, for $M_0=0$, the special solution $t^2=2\Lambda$ which will be analyzed in the next section.

The field equation (4.17) suggests that we interpret the Lagrange multiplier λ as a coordinate, such that $T=-d\lambda$ is one leg ("bein") of an orthogonal coframe. Let us first construct the frame dual to the coframe. Define the vector field

$$\xi^\star=-(t^\alpha/t^2)e_\alpha\ , \tag{5.3}$$

which is dual to ${}^*T$, i.e.,

$$\xi^\star \rfloor {}^*T=1\ ; \tag{5.4}$$

cf. (B9) and (B10). In view of (B12), Eq. (4.17) yields the constancy of the λ variable along the vector field $\xi^\star$:

$$l_{\xi^\star}\lambda=\xi^\star \rfloor d\lambda=\xi^\star(\lambda)=0\ , \tag{5.5}$$

where $\ell_\xi=\xi\rfloor d+d\rfloor \xi$ is the Lie derivative. This fact is crucial, since (5.5) allows us to introduce a second coordinate, for example, ρ, defined by the integral lines of the vector field $\xi^\star$. In view of (5.5), the (λ,ρ) coordinate system is orthogonal. Thus the form ${}^*T$ should be proportional to $d\rho$, while T, in view of (4.17), is already proportional to $d\lambda$.

The leg orthogonal to T is ${}^*T$. Thus we introduce the orthogonal coordinate system (λ,ρ). Then

$${}^*T=B(\lambda,\rho)d\rho\ . \tag{5.6}$$

Because of the orthogonality, there enters no term proportional to $d\lambda$. We substitute the ansatz (5.6) into (4.16), use the explicit expression (B20) of the volume two-form, and find

$$\frac{\partial B(\lambda,\rho)}{\partial\lambda}=2\Lambda\frac{B(\lambda,\rho)}{t^2}\ . \tag{5.7}$$

The same relation could have been obtained from the wave equation (4.20) for λ. Upon integration we obtain

$$B(\lambda,\rho)=B_0(\rho)t^2e^\lambda\ . \tag{5.8}$$

In terms of the frame (B15) or (B19) of Appendix B, the metric explicitly reads

$$g=o_{\alpha\beta}\vartheta^\alpha\otimes\vartheta^\beta=(-1)^s\frac{d\lambda^2}{t^2}+\frac{B^2d\rho^2}{t^2}\ , \tag{5.9}$$

or, after substituting t^2 and absorbing $B_0(\rho)$ according to the coordinate transformation $d\tilde\rho:=B_0\,d\rho$,

$$g=(-1)^s e^{2\lambda}2[M_0e^{-\lambda}+(-1)^s\Lambda]d\tilde\rho^2 +\frac{d\lambda^2}{2[M_0e^{-\lambda}+(-1)^s\Lambda]}\ . \tag{5.10}$$

Remarkably, this metric has the form of the black hole in the two-dimensional dilaton (string motivated) gravitational theories, widely discussed in the literature (cf. [14,15,23–25]). We will see in Sec. XI that the integration constant M_0 is in fact related to the mass of this black hole.

Along with the metric (5.10) one can construct explicitly the coordinate components of the torsion

$$T^i=e^i{}_\alpha T^\alpha=\tfrac{1}{2}T_{jk}{}^i dx^j\wedge dx^k\ . \tag{5.11}$$

From (B19) one readily obtains the frame

$$e_\alpha=-\frac{t_\alpha}{t^2e^\lambda}\partial_{\tilde\rho}+(-1)^s t^\star_\alpha\partial_\lambda\ , \tag{5.12}$$

and thus the components of the torsion tensor read

$$T_{\tilde\rho\lambda}{}^\lambda=0,\quad T_{\tilde\rho\lambda}{}^{\tilde\rho}=1\ . \tag{5.13}$$

Using the definition of the torsion two-form (2.6) one can express the two-dimensional Lorentz connection (2.5), according to Table II, in the convenient dual form

$$\Gamma^\star=({}^*d\vartheta^\alpha)\vartheta_\alpha+{}^*T\ . \tag{5.14}$$

In order to complete the analysis of the integrability of the model under consideration, one should also study (4.18) and verify that it is satisfied by the solution described above. In general this is a nontrivial problem, especially in the presence of matter, see Sec. VII.

Note that the solution obtained above completely describes the behavior of torsion: Eq. (5.13) gives the torsion components with respect to the coordinates $(\tilde\rho,\lambda)$, while the torsion square was obtained explicitly in (5.2). Its Lorentz frame components seem to remain undetermined, but this is clearly related to the gauge freedom of the model, which means that a vector at any point can be arbitrarily rotated by means of the local Lorentz transformations (2.14). Let us demonstrate this explicitly. Since t^2 is a known function of λ one can assume the general ansatz for the frame components of the torsion:

$$t^{\hat 0}=t\sinh u,\quad t^{\hat 1}=t\cosh u\ , \tag{5.15}$$

where $t=\sqrt{t^2}$ and $u=u(\rho,\lambda)$ is some function of both space and time coordinates which is real for $s=1$ and purely imaginary for $s=0$. Substituting (5.15) into (B19) and differentiating it, one finds

$$d\vartheta^\alpha=\left\{t^\alpha\frac{t}{B}\left[\partial_\lambda\left[\frac{B}{t}\right]+\frac{1}{t}\partial_\rho u\right]-\eta^\alpha{}_\beta t^\beta\partial_\lambda u\right\}\eta \ , \qquad (5.16)$$

and hence the dual Lorentz connection (5.14) turns out to be

$$\begin{aligned}\Gamma^\star&=[B-t\partial_\lambda(B/t)]d\rho-\partial_\rho u\, d\rho-\partial_\lambda u\, d\lambda\\ &=\left[B-\partial_\lambda B+\frac{B}{2}\frac{\partial_\lambda t^2}{t^2}\right]d\rho-du \ . \end{aligned} \qquad (5.17)$$

Evidently the last term represents the local Lorentz transformation (2.17) and can be discarded by choosing the gauge $u=0$ in (5.15). While calculating the two-dimensional Riemann-Cartan curvature, one notes that the last term in (5.17) does not contribute to it. Therefore (4.18), i.e., the vanishing of the curvature two-form

$$R^\star=d\Gamma^\star=0 \ , \qquad (5.18)$$

reduces to the condition

$$\partial_\lambda\left[B-\partial_\lambda B+\frac{B}{2}\frac{\partial_\lambda t^2}{t^2}\right]=0 \ . \qquad (5.19)$$

Using (5.8) one finds

$$B-\partial_\lambda B+\frac{B}{2}\frac{\partial_\lambda t^2}{t^2}=-\frac{B}{2}\frac{\partial_\lambda t^2}{t^2}=(-1)^s M_0 B_0 \ . \qquad (5.20)$$

Since (5.19) holds for the solution (5.10) and (5.13), the proof of the integrability of the vacuum equations (4.15)–(4.18) is completed.

Let us investigate some of the properties of the metric of our black-hole solutions and, in particular, compare these with those of the dilaton gravity black holes. In a first step, one can try to find a new coordinate $\tilde{\lambda}$ such that $\theta^{\hat{0}}$ can be represented as a natural leg:

$$\theta^{\hat{0}}=d\tilde{\lambda}=d\lambda/\sqrt{2[M_0 e^{-\lambda}+(-1)^s\Lambda]} \ . \qquad (5.21)$$

Clearly, one has to distinguish different cases: $M_0e^{-\lambda}+(-1)^s\Lambda>0$, $=0$, or <0. Here we restrict ourselves to the first case. Then the coordinate transformation reads

$$\lambda=-\ln\left\{\frac{(-1)^s\Lambda}{M_0}\sinh^{-2}\left[\left[(-1)^s\frac{\Lambda}{2}\right]^{1/2}\tilde{\lambda}\right]\right\} . \qquad (5.22)$$

Substitution into the metric (5.10) yields

$$g=(M_0^2/2\Lambda)\sinh^2[\sqrt{(-1)^s 2\Lambda}\,\tilde{\lambda}]d\tilde{\rho}^2+d\tilde{\lambda}^2 \ . \qquad (5.23)$$

Then, the further coordinate transformation

$$\tilde{\lambda}=[2/\sqrt{(-1)^s 2\Lambda}]\operatorname{arctanh}\sqrt{\Lambda[(-1)^s x^2+y^2]/2} \ , \qquad (5.24)$$

$$\tilde{\rho}=\begin{cases}\dfrac{1}{M_0}\arctan(y/x) & \text{for } s=0 \ ,\\[2mm] \dfrac{1}{M_0}\operatorname{arctanh}(y/x) & \text{for } s=1\end{cases} \qquad (5.25)$$

converts the metric (5.23) into the explicit *conformally flat* form

$$g=\frac{dx^2+(-1)^s dy^2}{\{1-(\Lambda/2)[(-1)^s x^2+y^2]\}^2} \ . \qquad (5.26)$$

In the new coordinates (x,y) the coordinate components of the torsion read

$$\begin{aligned}T_{xy}{}^x&=\frac{2y}{[(-1)^s x^2+y^2]\left[1-\dfrac{\Lambda}{2}[(-1)^s x^2+y^2]\right]} \ ,\\ T_{xy}{}^y&=-\frac{(-1)^s 2x}{[(-1)^s x^2+y^2]\left[1-\dfrac{\Lambda}{2}[(-1)^s x^2+y^2]\right]} \ .\end{aligned} \qquad (5.27)$$

VI. GRAVITATIONAL WAVES

In order to exhibit the propagating degrees of freedom of our model we consider the vacuum field equations. For *nonvanishing* "cosmological" constant Λ we obtain from (4.1) and the constraint (4.3):

$$D^*\vartheta^\alpha=D\eta^\alpha=-d\ln(t^2-2\Lambda)\wedge\eta^\alpha \qquad (6.1)$$

and

$$\begin{aligned}\Box\vartheta^\alpha&:=(-1)^s({}^*D^*D+D^*D^*)\vartheta^\alpha\\ &=(-1)^s(t^2-2\Lambda)\vartheta^\alpha \ .\end{aligned} \qquad (6.2)$$

These gauge-covariant nonlinear *Proca-type equations* for the coframe are *exact* consequences of our "topological" gauge model (3.1) with teleparallelism.

For the special solution $t^2=2\Lambda$, which has been left out in Sec. V, these equations simplify to a wave equation for the coframe:

$$\Box\vartheta^\alpha=0 \ . \qquad (6.3)$$

We adopt the solution (5.10) of the previous section for $M_0=0$, except that we are using the coordinate freedom in order to set $B_0=e^{\pm\rho}$. Then we find the metric

$$g=[(-1)^s/2\Lambda]d\lambda^2+2\Lambda e^{2(\lambda\pm\rho)}d\rho^2 \qquad (6.4)$$

of a *left- or right-moving* wave solution. It is the analogue of the *plane fronted* gravitational wave solution

$$g=-d\lambda^2+L^2(\lambda-z)(e^{2\beta(\lambda-z)}dx^2+e^{-2\beta(\lambda-z)}dy^2)+dz^2 \qquad (6.5)$$

in 4D gravity (cf. [26], p. 975).

It can be shown that the Cauchy problem for (6.3) is well posed: In two dimensions, the coframe $\vartheta^\alpha=e_j{}^\alpha dx^j$ has $2\times2=4$ components. Two degrees of freedom get fixed by considering coframes in the conformal gauge $\vartheta^\alpha=\Omega\, dx^\alpha$. Moreover, the one local Lorentz degree of freedom $\Lambda_{\hat{1}}{}^{\hat{0}}$ in the transformation formula (2.14) has also to be subtracted out. Then for Minkowskian signature $s=1$, Eq. (6.3) constitutes a (hyperbolic) wave equation for the conformal factor Ω as the *only* remaining dynamical degree of freedom.

Thus our two-dimensional model contains only a massless "spin-2" mode, i.e., a "topological graviton." Quantization will be straightforward. Moreover, by relaxing the teleparallelism constraint, the extended model with a Yang-Mills-type curvature squared term appears to be renormalizable [27].

VII. CHARGED BLACK HOLES

Let us add to our gravitational Lagrangian (3.3) the standard Maxwell Lagrangian

$$L_M=(-1)^s\tfrac{1}{2}F\wedge {}^*F \ , \tag{7.1}$$

where $F=dA$ is the field strength of the Abelian gauge potential one-form A. In two dimensions there is no magnetic field: the only component of the field strength describes the electric field along the unique spatial direction. This is expressed by introducing the scalar f dual to the Maxwell field strength, i.e.,

$$f:={}^*F, \quad F=(-1)^s f\eta \ . \tag{7.2}$$

The energy-momentum current in (4.1) reads

$$\begin{aligned}\Sigma_\alpha &:=e_\alpha \rfloor L_M-(-1)^s(e_\alpha \rfloor F)\wedge {}^*F\\ &=-(-1)^s\tfrac{1}{2}(e_\alpha \rfloor F)\wedge {}^*F=-\tfrac{1}{2}f^2\eta_\alpha \ ,\end{aligned} \tag{7.3}$$

whereas the spin current vanishes, i.e., $\tau_{\alpha\beta}=0$, since the Lorentz connection does not couple to the Maxwell field. Observe that the energy-momentum trace, in contrast with four dimensions, does *not* vanish:

$$\vartheta^\alpha \wedge \Sigma_\alpha=-f^2\eta \ . \tag{7.4}$$

The inhomogeneous Maxwell equation is obtained from the variation of (7.1) with respect to A and in vacuum reads, as usual,

$$d^*F=df=0 \ . \tag{7.5}$$

In two dimensions this can be easily integrated to give $f=\text{const}=Q$. This constant is, indeed, the conserved total electric charge.

As a result, the field equations (4.11)–(4.13) are *completely integrable* along the same line of reasoning, and the relevant charged black-hole solutions are simply obtained by the following shift of the cosmological constant:

$$\Lambda \to \tilde{\Lambda}=\Lambda-\tfrac{1}{2}Q^2 \ . \tag{7.6}$$

It is straightforward to see that this result is also valid for a Yang-Mills field with an arbitrary gauge group: After replacing F by the non-Abelian Lie-algebra-valued two-form F^A, and the Lagrangian (7.1) by

$$L_{\rm YM}=(-1)^s\tfrac{1}{2}F^A\wedge {}^*F_A \ , \tag{7.7}$$

one obtains, with the aid of (7.6), the same charged black holes except that

$$Q^2=f^Af_A, \quad f_A={}^*F_A \ . \tag{7.8}$$

Note that f_A is not a constant in view of the nonlinear nature of the Yang-Mills equations

$$D^*F_A=df_A+c_{ABC}A^Bf^C=0 \ , \tag{7.9}$$

but its square is conserved.

VIII. COUPLING TO A SCALAR FIELD

Let us now consider a gravitationally coupled scalar field ϕ for which $L_{\rm mat}$ in (3.3) is given by

$$L_\phi=(-1)^s\tfrac{1}{2}d\phi\wedge {}^*d\phi+U\eta \ . \tag{8.1}$$

The potential $U=U(\phi)$ may include the mass term $\frac{1}{2}m^2\phi^2$ as well as a nonlinear self-interaction of scalar matter. Introducing the notation

$$\begin{aligned}&\partial_\alpha\phi:=e_\alpha \rfloor d\phi \ ,\\ &(\partial\phi)^2=g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi \ ,\\ &\mathcal{L}:={}^*L_\phi=\tfrac{1}{2}(\partial\phi)^2+(-1)^sU \ ,\end{aligned} \tag{8.2}$$

where $\mathcal{L}$ is just the Lagrangian *function,* one finds for the sources of the gravitational field

$$\begin{aligned}&\Sigma_\alpha=(-1)^{s+1}[(\partial_\alpha\phi)(\partial^\beta\phi)-\delta^\beta_\alpha\mathcal{L}]\eta_\beta \ ,\\ &\tau_{\alpha\beta}=0 \ .\end{aligned} \tag{8.3}$$

The gravitational field equations (4.1)–(4.3) have to be supplemented by the field equation of the scalar matter:

$${}^*d{}^*d\phi-\frac{dU}{d\phi}=0 \ . \tag{8.4}$$

As a first step toward a solution of the highly nonlinear system (4.1)–(4.3) and (8.4) we will confine ourselves to the *static* case, such that

$$\partial_\rho\phi\sim t^\alpha\partial_\alpha\phi=0 \ . \tag{8.5}$$

Then, the vacuum equations (4.15), (4.16), and (4.17) are modified as

$$dt^2=\{-t^2+2[\Lambda+(-1)^s\mathcal{L}]\}d\lambda \ , \tag{8.6}$$

$$d^*T=-(-1)^s2(\Lambda+U)\eta \ , \tag{8.7}$$

$$d\lambda=-T \ . \tag{8.8}$$

Let us consider a configuration ϕ_0 for which the self-interaction potential has a nontrivial local extremum (usually a minimum), i.e.,

$$\frac{dU}{d\phi}=0 \ . \tag{8.9}$$

Evidently the constant configuration $\phi=\text{const}=\phi_0$ is then a solution of (8.4). The remaining gravitational field equations (8.5)–(8.7) are reduced to those of the "vacuum case," except that the cosmological constant is shifted to

$$\Lambda \to \bar{\Lambda}=\Lambda+U(\phi_0) \ . \tag{8.10}$$

The solutions are thus again represented by the black-hole metric (5.10).

In the general case we find the solution in the same way as in Sec. V: the coordinates (ρ,λ) are introduced such that (5.6) and (5.9) hold. However, (4.16) is replaced by (8.7), and this, for the function $B(\rho,\lambda)$, yields the condition

$$\frac{t^2}{B}\frac{\partial B}{\partial\lambda}=2(\Lambda+U)\ . \tag{8.11}$$

In these coordinates the wave equation (8.4) for the matter field reads explicitly

$$\partial_\lambda^2\phi+\frac{1}{B}(\partial_\lambda B)\partial_\lambda\phi-\frac{1}{t^2}\frac{dU}{d\phi}=0\ . \tag{8.12}$$

For $\Lambda=U=0$, similarly as in an exact Einstein-dilaton field solution in four dimensions [28], the term $\mathcal{L}=\frac{1}{2}C^2$ effectively replaces the cosmological constant. In two dimensions this leads to a conformally invariant model, for which the exact *dilaton solution*

$$\phi=C\lambda,\quad B_0=B_0(\rho)\ ,$$
$$g=\frac{d\lambda^2}{C^2-Ae^{-\lambda}}+(-1)^s\frac{B_0^2 d\rho^2}{C^2-Ae^{-\lambda}} \tag{8.13}$$

can be obtained. In general, the integration is more difficult mainly due to the necessity to satisfy the zero curvature constraint (4.18). Consequently, the system (8.5), (8.10), (8.11), and (5.18) may admit dilaton black-hole-type solutions only for specific potentials $U(\phi)$. Nevertheless, it is likely that generic two-dimensional black holes have no "scalar hair."

IX. COUPLING TO DIRAC MATTER

In this section and in Appendix A we consider only Minkowski spacetime ($s=1$). The theory of spinors in two dimensions can formally be constructed along the same lines as in n dimensions; see Ref. [29] and Appendixes A and C. However, there are certain peculiarities due to the Abelian nature of the two-dimensional Lorentz group. The most unusual feature is the absence of coupling of the 2D Dirac field to the local Lorentz connection.

Let $L_{\rm mat}$ in (3.3) now be the Dirac Lagrangian

$$L_\psi=(i/2)(\bar\psi\gamma\wedge{}^*D\psi+{}^*D\bar\psi\wedge\gamma\psi)-im\bar\psi\psi\eta\ , \tag{9.1}$$

where $\gamma:=\gamma_\alpha\vartheta^\alpha$ is the matrix-valued one-form of the Dirac algebra in 2D satisfying $\gamma\wedge\gamma=-2\eta\gamma_5$. (For the details on spinors and the realization of the Dirac algebra in two dimensions see Appendixes A and C.) The covariant exterior derivative is defined by

$$D\psi=d\psi+\Gamma\psi,\quad D\bar\psi=d\bar\psi-\bar\psi\Gamma\ , \tag{9.2}$$

where

$$\Gamma:=\frac{i}{4}\Gamma^{\alpha\beta}\sigma_{\alpha\beta}=\tfrac{1}{4}\eta_{\alpha\beta}\Gamma^{\alpha\beta}\gamma_5=\tfrac{1}{2}\gamma_5\Gamma^\star \tag{9.3}$$

is the $\overline{\rm SO}(1,1)$-valued connection.

In two dimensions, the connection is not only Abelian but also involves the γ_5 matrix. This implies that the spin current

$$\tau_{\alpha\beta}=\frac{\partial L_\psi}{\partial\Gamma^{\alpha\beta}}=\tfrac{1}{8}\bar\psi({}^*\gamma\sigma_{\alpha\beta}+\sigma_{\alpha\beta}{}^*\gamma)\psi$$
$$=-(i/8)\eta_{\alpha\beta}\bar\psi({}^*\gamma\gamma_5+\gamma_5{}^*\gamma)\psi=0 \tag{9.4}$$

vanishes identically on account of (C4) and (C5). Consequently, the Dirac Lagrangian (9.1) reduces to

$$L_\psi=(i/2)(\bar\psi\gamma\wedge{}^*d\psi+{}^*d\bar\psi\wedge\gamma\psi)+\Gamma^{\alpha\beta}\wedge\tau_{\alpha\beta}-im\bar\psi\psi\eta$$
$$=(i/2)(\bar\psi\gamma\wedge{}^*d\psi+{}^*d\bar\psi\wedge\gamma\psi)-im\bar\psi\psi\eta\ . \tag{9.5}$$

Variation of the Dirac Lagrangian (9.1) with respect to $\bar\psi$ yields the Dirac equation

$$\gamma\wedge{}^*D\psi+\tfrac{1}{2}\gamma^\alpha\eta_{\alpha\beta}T^\beta\psi-m\psi\eta=0\ . \tag{9.6}$$

Using (9.3) and (5.14) one can verify that (9.6) does not contain torsion: the apparent term is actually canceled by those hidden in the exterior covariant derivative. After defining $D_\alpha=e_\alpha\rfloor D$, Eq. (9.6) is equivalent to

$$\gamma^\alpha D_\alpha^{\{\}}\psi-m\psi=0\ . \tag{9.7}$$

This again proves the absence of any coupling of 2D Dirac spinors to the local Lorentz connection.

Variation of (9.5) with respect to the coframe yields the energy-momentum current

$$\Sigma_\alpha\cong-(i/2)(\bar\psi\gamma^\beta\partial_\alpha\psi-\partial_\alpha\bar\psi\gamma^\beta\psi)\eta_\beta\ , \tag{9.8}$$

where we took into account that the Dirac Lagrangian vanishes "weakly," i.e., $L_\psi\cong 0$, on account of the Dirac equation (9.6).

Similarly as in the case of the scalar matter we are not attempting to find the general solution, but restrict ourselves to the *static* case. Then we have

$$\partial_\rho\psi\sim t^\alpha\partial_\alpha\psi=0\ . \tag{9.9}$$

Thus

$$\Sigma_\alpha t^\alpha\cong 0,\quad \Sigma_\alpha\wedge\vartheta^\alpha\cong im\bar\psi\psi\eta\ , \tag{9.10}$$

where again $L_\psi\cong 0$ has been used.

Hence the gravitational field equations (4.1)–(4.3) are reduced to the system

$$dt^2=(2\Lambda-t^2)T\ , \tag{9.11}$$

$$d^*T=(2\Lambda-im\bar\psi\psi)\eta\ , \tag{9.12}$$

$$d\lambda=-T\ , \tag{9.13}$$

$$R_{\alpha\beta}=0\ , \tag{9.14}$$

In the massless case $m=0$, a static Dirac field completely decouples from the gravitational field equations. Hence, Eqs. (9.11)–(9.14) reduce to the vacuum case (4.15)–(4.18) and thus give rise to the same black-hole and wave solutions. The massive spinor case will be discussed elsewhere.

X. NOETHER IDENTITIES AND CONSERVED CURRENTS

The sources for the gravitational gauge fields are the material energy-momentum current $\Sigma_\alpha:=\delta L_{\rm mat}/\delta\vartheta^\alpha$ and the spin current $\tau_{\alpha\beta}:=\delta L_{\rm mat}/\delta\Gamma^{\alpha\beta}$, which are both $(n-1)$-forms in n dimensions. In fact, in two dimensions Σ_α represents stress—this is a well-known concept of a force distributed over a (one-dimensional spacelike) line element. In four dimensions, however, Σ_α describes energy-momentum distributed in a (three-dimensional) volume element. Accordingly, Σ_α corresponds to the intuitive notions of a line-stress and energy-momentum density in two and four dimensions, respectively. This convinces us of the correctness of the interpretation of the $(n-1)$-form Σ_α. An analogous consideration applies to $\tau_{\alpha\beta}$ as spin moment stress and spin angular momentum density, respectively.

From local Poincaré invariance [structure group: $R^n \circledS SO(s,n-s)$], one finds [5,13], for $n\geq 2$, the *1st* and the *2nd Noether identity*

$$D\Sigma_\alpha \cong (e_\alpha \rfloor T^\gamma)\wedge\Sigma_\gamma + (e_\alpha \rfloor R^{\gamma\delta})\wedge\tau_{\gamma\delta} \tag{10.1}$$

and

$$D\tau_{\alpha\beta} + \vartheta_{[\alpha}\wedge\Sigma_{\beta]} \cong 0 \ . \tag{10.2}$$

These equations having n and $n(n-1)/2$ independent components, respectively, hold only "weakly," denoted by $\cong$, i.e., provided the matter field equation $\delta L/\delta\psi=0$ is satisfied.

In two dimensions, the Noether identities can be rewritten as

$$D\Sigma_\alpha \cong (-1)^s \eta_\alpha \wedge (t^\gamma \Sigma_\gamma - R\tau^\star) \tag{10.3}$$

and

$$d\tau^\star - \tfrac{1}{2}\eta_\beta\wedge\Sigma^\beta \cong 0 \ . \tag{10.4}$$

Note that the right (or Lie) dual $\tau^\star$ of the spin current is also given by

$$\tau^\star = \frac{(-1)^s}{2}\frac{\delta L_{\rm mat}}{\delta\Gamma^\star} \ . \tag{10.5}$$

For spinless matter, the energy-momentum becomes symmetric and covariantly conserved with respect to the Riemannian connection [30]:

$$\vartheta_{[\alpha}\wedge\Sigma_{\beta]} \cong 0, \quad D\Sigma_\alpha^{\{\}} \cong 0 \ . \tag{10.6}$$

If a spacetime admits symmetries, from the Noether currents we can construct a set of *invariant* conserved quantities, one for each symmetry. We consider Killing symmetries, where the vector field $\zeta=\zeta^\alpha e_\alpha$ is a generator of a one-parameter group of diffeomorphisms. Then the generalized Killing equations

$$\begin{aligned} &\mathcal{L}_\zeta g = (\not{L}_\zeta g_{\alpha\beta} + 2g_{\gamma(\alpha}e_{\beta)} \rfloor \not{L}_\zeta \vartheta^\gamma)\vartheta^\alpha\otimes\vartheta^\beta = 0 \ , \\ &\mathcal{L}_\zeta \Gamma_\alpha^\beta = 0 \end{aligned} \tag{10.7}$$

hold, where $\mathcal{L}_\zeta$ is the usual Lie derivative and $\not{L}_\zeta := \zeta \rfloor D + D\zeta \rfloor$ is the gauge-covariant version for exterior forms.

As it was shown in Ref. [31], the current one-form

$$\varepsilon_{\rm RC} := \zeta^\alpha \Sigma_\alpha + (e_\beta \rfloor \widehat{D}\zeta^\gamma)\tau^\beta{}_\gamma \ , \tag{10.8}$$

involving the exterior covariant derivative $\widehat{D}$ with respect to the transposed connection

$$\widehat{\Gamma}_\alpha{}^\beta := \Gamma_\alpha{}^\beta + e_\alpha \rfloor T^\beta \ , \tag{10.9}$$

is such a weakly closed form:

$$d\varepsilon_{\rm RC} \cong 0 \ . \tag{10.10}$$

Thus, in the presence of spacetime symmetries $\varepsilon_{\rm RC}$ is a *globally conserved* energy-momentum current.

XI. MASS OF THE BLACK-HOLE SOLUTION

In order to apply these results to our exact black-hole solution, observe that the metric (5.10) is independent of the coordinate $\tilde{\rho}$. The corresponding timelike Killing vector field $\partial_{\tilde{\rho}}$ can be expanded in terms of the frame:

$$\xi^\star \rfloor {}^\star T = 0 \Longrightarrow \zeta = \partial_{\tilde{\rho}} = B\xi^\star = -e^\lambda t^\alpha e_\alpha \ . \tag{11.1}$$

The material spin current of our exact solution vanishes [32], i.e., $\tau_{\alpha\beta}=0$ such that (10.8) reduces to $\varepsilon_{\rm RC} := \zeta^\alpha\Sigma_\alpha$. This is, in fact, a general feature of all the matter sources considered in this paper: the Yang-Mills bosons, the dilaton, and even a Dirac field. On the other hand, for a nonzero mass M_0, the material energy-momentum current cannot vanish everywhere, but needs to have a δ-type concentration at the origin, i.e.,

$$\Sigma_\alpha \sim \delta(0)\eta_\alpha \ . \tag{11.2}$$

For the derivation of the corresponding weakly conserved current (10.8), we eliminate Σ_α in (10.8) by means of the field equation (4.7). Because of (4.9), i.e.,

$$t_\alpha \eta^\alpha = (-1)^s T = -(-1)^s d\lambda \ ,$$

we easily obtain

$$\varepsilon_{\rm RC} = \zeta^\alpha\Sigma_\alpha = (-1)^s \tfrac{1}{2} e^\lambda [dt^2 + (t^2 - 2\Lambda)d\lambda] \ . \tag{11.3}$$

Since this current can be derived, via

$$\varepsilon_{\rm RC} = dM \ , \tag{11.4}$$

from the superpotential

$$M = (-1)^s e^\lambda (\tfrac{1}{2}t^2 - \Lambda) \ , \tag{11.5}$$

the current $\varepsilon_{\rm RC}$ is conserved, indeed. On the "mass shell,"

$$M = M_0 \ , \tag{11.6}$$

we recover (5.2).

XII. INTEGRABILITY OF THE GENERAL PG EQUATIONS IN TWO DIMENSIONS

For the *quadratic* Poincaré gauge (PG) model in two dimensions the complete integrability in vacuum has been established earlier [16,17,22]. However, these proofs rely on certain choices of a gauge. In this section we will extend this result to the case of the *general* 2D Poincaré gauge theory *without* imposing any gauge condition.

In PG theory the total action of interacting matter and gravitational gauge fields reads

$$W=\int[L(\vartheta^\alpha,\Psi,D\Psi)+V(\vartheta^\alpha,T^\alpha,R^{\alpha\beta})] \ . \tag{12.1}$$

It is a functional of a minimally coupled matter field Ψ, which, in general, may be a p-form, and of the geometrical variables ϑ^α and $\Gamma^{\alpha\beta}=-\Gamma^{\beta\alpha}$. Their *independent* variations yield the *field equations*

$$\frac{\delta L}{\delta\Psi}=\frac{\partial L}{\partial\Psi}-(-1)^p D\frac{\partial L}{\partial D\Psi}=0, \quad \text{(matter)} \ , \tag{12.2}$$

$$DH_\alpha-E_\alpha=\Sigma_\alpha \quad \text{(first)} \ , \tag{12.3}$$

$$DH_{\alpha\beta}-E_{\alpha\beta}=\tau_{\alpha\beta} \quad \text{(second)} \ . \tag{12.4}$$

Observe that, in two dimensions, the *gauge-field momenta* are zero-forms:

$$H_\alpha:=-\frac{\partial V}{\partial d\,\vartheta^\alpha}=-\frac{\partial V}{\partial T^\alpha} \ , \qquad H_{\alpha\beta}:=-\frac{\partial V}{\partial d\,\Gamma^{\alpha\beta}}=-\frac{\partial V}{\partial R^{\alpha\beta}} \ . \tag{12.5}$$

The sources of these Yang-Mills-type field equations are the one-forms of material *energy-momentum* and *spin*, respectively:

$$\Sigma_\alpha:=\frac{\delta L}{\delta\vartheta^\alpha}, \quad \tau_{\alpha\beta}:=\frac{\delta L}{\delta\Gamma^{\alpha\beta}} \ . \tag{12.6}$$

Because of the universality of the gravitational interaction, the one-forms of gravitational energy-momentum

$$E_\alpha:=\frac{\partial V}{\partial\vartheta^\alpha} =e_\alpha\rfloor V+(e_\alpha\rfloor T^\beta)\wedge H_\beta+(e_\alpha\rfloor R^{\beta\gamma})\wedge H_{\beta\gamma} \tag{12.7}$$

and gravitational spin angular momentum

$$E_{\alpha\beta}:=-\vartheta_{[\alpha}\wedge H_{\beta]} \tag{12.8}$$

provide a self-coupling of the gravitational gauge field.

The trace of the energy-momentum current (12.7), formed with the aid of the coframe ϑ^α, in general gives us back the gauge Lagrangian V amended by Yang-Mills-type terms according to

$$\vartheta^\alpha\wedge E_\alpha=2V+2T^\alpha\wedge H_\alpha+2R^{\beta\gamma}\wedge H_{\beta\gamma} \ . \tag{12.9}$$

In order to reduce the field equations we introduce the one-forms

$$H:=H_\alpha\vartheta^\alpha, \quad {}^*H=H_\alpha\eta^\alpha \ . \tag{12.10}$$

In the first field equation we do not lose any of its four components if we multiply (12.3) by ϑ^β from the right:

$$-D(\vartheta^\beta H_\alpha)+T^\beta H_\alpha-\vartheta^\beta\wedge E_\alpha=\vartheta^\beta\wedge\Sigma_\alpha \ . \tag{12.11}$$

The trace of (12.11), on substitution of (12.9) and (12.10), reads

$$-dH-T^\alpha H_\alpha-2V-2R^{\beta\gamma}\wedge H_{\beta\gamma}=\vartheta^\alpha\wedge\Sigma_\alpha \ . \tag{12.12}$$

In the second field equation it is more economical to switch over to its Lie dual by multiplying it with $\frac{1}{2}\eta^{\alpha\beta}$ and to introduce the notation

$$H^\star=\tfrac{1}{2}\eta^{\alpha\beta}H_{\alpha\beta}=:\tfrac{1}{2}\kappa \ . \tag{12.13}$$

Then we get

$$\tfrac{1}{2}\,d\kappa-\tfrac{1}{2}H_\alpha\eta^\alpha=\tau^\star \tag{12.14}$$

or, in view of (12.10),

$$d\kappa-{}^*H=2\tau^\star \ . \tag{12.15}$$

Observe that the gravitational energy-momentum current can be rewritten as

$$E_\alpha=(-1)^s(\mathcal{V}+t^\beta H_\beta-\tfrac{1}{2}R\kappa)\eta_\alpha =:(-1)^s\tilde{\mathcal{V}}\eta_\alpha \ , \tag{12.16}$$

where $\mathcal{V}:={}^*V$ is the Lagrangian function (zero-form). Then a very useful condition for the squared translational momentum

$$H^2:=o^{\alpha\beta}H_\alpha H_\beta \tag{12.17}$$

can be derived by contracting the first field equation (12.3) with H^α:

$$\tfrac{1}{2}dH^2-(-1)^s\tilde{\mathcal{V}}\,{}^*H=H^\alpha\Sigma_\alpha \ . \tag{12.18}$$

We eliminate ${}^*H$ by means of (12.15) and find

$$dH^2-(-1)^s2\tilde{\mathcal{V}}\,d\kappa=2[H^\alpha\Sigma_\alpha-(-1)^s2\tilde{\mathcal{V}}\tau^\star] \ . \tag{12.19}$$

Let us now specialize to the *vacuum* field equations. They read

$$dH^2=(-1)^s2\tilde{\mathcal{V}}\,d\kappa \ , \tag{12.20}$$

$$dH=(-1)^s(t^\alpha H_\alpha-2\tilde{\mathcal{V}})\eta \ , \tag{12.21}$$

$$d\kappa={}^*H \ . \tag{12.22}$$

Moreover, in our general PG model, we can derive from the vacuum field equations (12.21) and (12.20) the wave equation

$$\Box\kappa:=(-1)^s({}^*d{}^*d+d{}^*d{}^*)\kappa =(-1)^s(2\tilde{\mathcal{V}}-t^\alpha H_\alpha) \tag{12.23}$$

for the would-be coordinate κ.

In order to obtain the general solution one can proceed along the same line of reasoning as in Sec. V: We introduce a coordinate system (ρ,κ) which is related to the translational one-forms (12.10) via

$$H=B\,d\rho, \quad {}^*H=d\kappa \ , \tag{12.24}$$

with some function $B(\rho,\kappa)$. Similarly as in the teleparallel case, the volume two-form is, for $H^2\neq 0$, given by

$$\eta=-(B/H^2)d\kappa\wedge d\rho \ ; \tag{12.25}$$

cf. (B20) with the torsion one-form being replaced by ${}^*T\to(-1)^sH$.

Insertion of this ansatz into (12.21), together with (12.20), yields

$$\frac{\partial}{\partial\kappa}\ln B=(-1)^s(2\tilde{\mathcal{V}}-t^\alpha H_\alpha)\frac{1}{H^2}$$
$$=\frac{\partial}{\partial\kappa}\ln H^2-(-1)^s\frac{t^\alpha H_\alpha}{H^2}\ . \qquad (12.26)$$

A formal integration of (12.26) straightforwardly leads to the solution

$$B=B_0(\rho)H^2\exp\left[-(-1)^s\int d\kappa\frac{t^\alpha H_\alpha}{H^2}\right], \qquad (12.27)$$

where again $B_0(\rho)$ is an arbitrary function only of ρ.

Let us conclude with several remarks on the integration of the vacuum field equations (12.3), (12.4), or (12.20)–(12.22), respectively.

First of all, let us treat the case $H^2=0$. In two dimensions the curvature has only one nontrivial component, namely, the curvature scalar R. Thus, in view of (2.13), the general gravitational action (12.13) has the form

$$V(\vartheta^\alpha,T^\alpha,R^{\alpha\beta})=V(\vartheta^\alpha,T^\alpha,R)\ . \qquad (12.28)$$

The gravitational gauge field momentum (12.13), on account of (12.5), can then be rewritten as

$$\kappa=2\frac{\partial\mathcal{V}}{\partial R}\ . \qquad (12.29)$$

For $H^2=0$, Eq. (12.20) yields $\tilde{\mathcal{V}}=0$, and, by means of (12.16), $E_\alpha=0$. Hence in vacuum the first field equation (12.3) degenerates to $DH_\alpha=0$. The integrability condition of this equation is the vanishing of the curvature: $R_{\alpha\beta}=0$. If this is satisfied, we actually return to the teleparallel case, which was analyzed in detail in previous sections. However, in general, $R_{\alpha\beta}\neq0$. Then, in combination with the equation $\tilde{\mathcal{V}}=0$, we find

$$\mathcal{V}=\frac{R\kappa}{2}=R\frac{\partial\mathcal{V}}{\partial R}\ .$$

Summarizing, one is left with two *algebraic* equations for curvature and torsion:

$$H_\alpha=0,\quad \mathcal{V}-R\frac{\partial\mathcal{V}}{\partial R}=0\ , \qquad (12.30)$$

the roots of which yield *constant* values for R and T.

Let us now turn to the general case with $H^2\neq0$. Since R is a scalar, it is clear form (12.28) that, modulo boundary terms, the torsion can only appear in V in the form of the scalar t^2, i.e.,

$$V(\vartheta^\alpha,T^\alpha,R)=V(\vartheta^\alpha,t^2,R)=(-1)^s\mathcal{V}(t^2,R)\eta\ . \qquad (12.31)$$

Hence the relevant translational momentum reads

$$H_\alpha=-2\frac{\partial\mathcal{V}}{\partial t^2}t_\alpha:=P(t^2,R)t_\alpha\ . \qquad (12.32)$$

Together with $\kappa=\kappa(t^2,R)$, this function plays a decisive role in the *formal* integration of the system (12.20)–(12.22). Indeed, since $t^\alpha H_\alpha=Pt^2$ and $H^2=P^2t^2$, one recognizes (12.20) as a well-posed equation which involves one dependent (e.g., t^2) and one independent (e.g., R) variable. Provided V, and hence P, is smooth, the solution of this first-order ordinary differential equation always exists, thus completing our formal demonstration of the integrability of the general two-dimensional Poincaré gauge theory. Remarkably, the complete vacuum solution (if $H^2\neq0$) is again of the black-hole type with the metric

$$g=(-1)^s\frac{d\kappa^2}{H^2}+H^2\exp\left[-(-1)^s\int d\kappa\frac{t^\alpha H_\alpha}{H^2}\right]d\rho^2\ , \qquad (12.33)$$

even if it is more complicated than (5.10). In (12.33) we set $B_0=1$. Torsion and curvature for our solution are obtained by inverting the definitions in (12.5) of the gauge-field momenta, or equivalently, by inverting the relations

$$\left.\begin{matrix}\kappa=\kappa(t^2,R)\\ P=P(t^2,R)\end{matrix}\right\}\rightarrow\left\{\begin{matrix}t^2=t^2(\kappa,P)\\ R=R(\kappa,P)\end{matrix}\right.\ .$$

For the solution to be unique one must assume the relevant Hessian $(\partial^2\mathcal{V}/\partial R\,\partial R,\partial^2\mathcal{V}/\partial t^2\partial t^2)$ to be nondegenerate. It is straightforward to derive from (5.14) the curvature scalar of our general solution:

$$R=-(-1)^s\frac{H^2}{B}\frac{\partial}{\partial\kappa}\left[\frac{B}{H^2}\frac{\partial}{\partial\kappa}H^2\right]\ . \qquad (12.34)$$

XIII. COMPLETE INTEGRABILITY OF QUADRATIC PG LAGRANGIANS IN TWO DIMENSIONS

Let us apply these results of the general Lagrangian to a specific example, namely, to a Lagrangian with terms quadratic in torsion and curvature. Since both, torsion and curvature, possess only one irreducible piece, respectively, the most general quadratic (parity conserving) Lagrangian reads

$$L=(-1)^s\left[\frac{a}{2}T_\alpha{}^*T^\alpha+\frac{1}{2}R^{\alpha\beta}\eta_{\alpha\beta}+\frac{b}{2}R_{\alpha\beta}{}^*R^{\alpha\beta}\right]$$
$$+\Lambda\eta+L_{\rm mat}\ . \qquad (13.1)$$

Following the prescriptions (12.5), (12.7), and (12.8), respectively, we calculate from (13.1) the gauge-field momenta

$$H_\alpha=-(-1)^s at_\alpha,\quad H^2=a^2t^2\ , \qquad (13.2)$$

and

$$H_{\alpha\beta}=-\tfrac{1}{2}(-1)^s(1-bR)\eta_{\alpha\beta}\ , \qquad (13.3)$$

as well as the gravitational energy-momentum current

$$E_\alpha=-\left[\frac{a}{2}t^2+(-1)^s\frac{b}{4}R^2-\Lambda\right]\eta_\alpha\ , \qquad (13.4)$$

and the gravitational spin current

$$E_{\alpha\beta}=(-1)^s a\vartheta_{[\alpha}t_{\beta]}\ . \qquad (13.5)$$

Then the vacuum field equations (12.20)–(12.22) read

$$a^2 dt^2 = -[at^2 + (-1)^s (b/2) R^2 - 2\Lambda] d\kappa \,, \qquad (13.6)$$

$$a\, d^*T = [(b/2)R^2 - (-1)^s 2\Lambda]\eta \,, \qquad (13.7)$$

and

$$d\kappa = b\, dR = -aT \,. \qquad (13.8)$$

In the special case $t^2=0$ we are led to a space of constant Riemannian curvature, similarly as in the TJ model:

$$T^\alpha = 0, \quad R^2 = (-1)^s (4/b)\Lambda \,. \qquad (13.9)$$

For $t^2 \neq 0$ we find from (13.6) and (13.8) by integration

$$t^2 = (-1)^s \left[2M_0 e^{-bR/a} - \frac{b}{2a} R^2 + R + (-1)^s \frac{2\Lambda}{a} - \frac{a}{b} \right] . \qquad (13.10)$$

According to (13.8), the torsion one-form T is again an exact form:

$$T = d[-(b/a)R] \,. \qquad (13.11)$$

Thus we can repeat the reasoning of Sec. V and regard dR as one natural leg. Then R is the associate coordinate such that $^*T$ is orthogonal to T, implying again the ansatz

$$^*T = B(\rho, R)\, d\rho \,. \qquad (13.12)$$

Following the steps performed in (12.25) and (12.26) with $^*H = -aT$ and $H = (-1)^s a\, ^*T$, the unknown function B in (13.12) turns out to be

$$B(\rho, R) = B_0(\rho) t^2 e^{bR/a} \,. \qquad (13.13)$$

For the black-hole solution we can set $B_0 = 1$ without loss of generality and, eventually, obtain the following new orthonormal one-form basis [cf. (B15) of Appendix B]:

$$\vartheta^{\hat 0} := \frac{T}{\sqrt{t^2}} = -\frac{b}{a}\frac{dR}{\sqrt{t^2}} \,, \qquad \vartheta^{\hat 1} := \frac{^*T}{\sqrt{t^2}} = e^{bR/a}\sqrt{t^2}\, d\rho \,, \qquad (13.14)$$

with the square of the torsion components given by (13.10). Accordingly, the anholonomic torsion components satisfy the relation [cf. (B21)]

$$T^{\hat 0} = 0, \quad T^{\hat 1} = -\sqrt{t^2}\, \vartheta^{\hat 0} \wedge \vartheta^{\hat 1} \,. \qquad (13.15)$$

Since the metric is

$$g = e^{2bR/a} t^2 d\rho^2 + (-1)^s \frac{b^2}{a^2} \frac{dR^2}{t^2} \,, \qquad (13.16)$$

our solution is given by (13.14)–(13.16), together with (13.10), and the proof of the integrability of the general quadratic 2D PG model is *formally* completed.

This was first done in Ref. [22], but the following essential point was not explicitly demonstrated: Compared to the teleparallelism model, where the Lagrange multiplier is an *independent* field which can "transmute" freely to a coordinate, there seems to be a catch in the case of the general theory. The scalar curvature R, regarded as a coordinate, may still keep alive the memory of its origin as a derivative of the Riemann-Cartan connection. For our exact solution, the connection one-form $\Gamma^\star$ contains the leg R in its expansion. Thus we have to face a highly *implicit* interrelation between the curvature $R^\star = d\Gamma^\star$ and the *formal* coordinate R. Fortunately, one can show the self-consistency of our scheme: inserting (13.2), (13.10), and (13.13) into (12.34) one can verify that the scalar curvature $R(\rho, R)$ is indeed equal to R regarded as the coordinate. We also checked this with the aid of the EXCALC package of the computer algebra system REDUCE [33]. This finally concludes the proof of *complete integrability* of the $R + T^2 + R^2$ model.

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APPENDIX A: (ANTI-)SELF-DUAL BASIS FOR EXTERIOR FORMS IN TWO DIMENSIONS

The symbol $\wedge$ denotes the exterior product of forms, the symbol $\rfloor$ denotes the interior product of a vector with a form, and the asterisk denotes the Hodge star (or left dual), which maps a p-form $\Phi^{(p)}$ into a $(2-p)$-form. It has the property that

$$^{**}\Phi^{(p)} = (-1)^{p(2-p)+s}\Phi^{(p)} \,, \qquad (A1)$$

The volume two-form is defined by

$$\eta := \tfrac{1}{2}\eta_{\alpha\beta}\vartheta^\alpha \wedge \vartheta^\beta \,, \qquad (A2)$$

where $\eta_{\alpha\beta} := \sqrt{|\det o_{\mu\nu}|}\,\epsilon_{\alpha\beta}$, and $\epsilon_{\alpha\beta}$ is the Levi-Civita symbol normalized to $\epsilon_{\hat 0 \hat 1} = +1$. Together with η, the following forms span a basis for the algebra of arbitrary p-forms in two dimensions:

$$\eta_\alpha := e_\alpha \rfloor \eta = {}^*\vartheta_\alpha, \quad \eta_{\alpha\beta} := e_\beta \rfloor \eta_\alpha = {}^*(\vartheta_\alpha \wedge \vartheta_\beta) \,. \qquad (A3)$$

We will call the forms

$$\{\eta, \eta_\alpha, \eta_{\alpha\beta}\} \qquad (A4)$$

the η basis of the two-dimensional space. In two dimensions $\eta_{\alpha\beta}$ is a zero-form which we took for the definition of the Lie dual in (2.10). For the inversion of the Lie dual, we have to use the two-dimensional relation

$$\eta_{\alpha\gamma}\eta^{\beta\gamma} = (-1)^s \delta_\alpha^\beta \,. \qquad (A5)$$

From Table I we recognize that two-forms are of central importance in two-dimensional gravity. This is also true for η_α, which is Lie dual to the one-form ϑ^β. Indeed,

$$\eta_\alpha = \eta_{\alpha\beta}\vartheta^\beta = {}^*\vartheta_\alpha \,. \qquad (A6)$$

The exterior product of the coframe with the η basis satisfies the relations

$$\vartheta^\gamma \wedge \eta_\alpha = \delta^\gamma_\alpha \eta, \quad \vartheta^\gamma \eta_{\alpha\beta} = -\delta^\gamma_\alpha \eta_\beta + \delta^\gamma_\beta \eta_\alpha \ , \tag{A7}$$

which imply, in particular, that

$$\eta = \tfrac{1}{2}\vartheta^\beta \wedge \eta_\beta \ . \tag{A8}$$

Differentiating the η's yields

$$D\eta_\alpha = T^\gamma \wedge \eta_{\alpha\gamma}, \quad D\eta_{\alpha\beta} = 0 \ . \tag{A9}$$

For $s=1$, the Lorentz transformation (2.15) suggests that we introduce one-forms which are irreducible with respect to the connected component of the Lorentz group:

$$\overset{(\pm)}{\sigma}{}^\alpha = \vartheta^\alpha \pm \eta^\alpha \ . \tag{A10}$$

These forms are self- and anti-self-dual,

$$*\overset{(\pm)}{\sigma}{}^\alpha = \pm \overset{(\pm)}{\sigma}{}^\alpha \ , \tag{A11}$$

and satisfy the relations

$$\begin{aligned} &\overset{(\pm)}{\sigma}{}^\alpha \wedge \overset{(\pm)}{\sigma}{}^\beta = 0 \ , \\ &\overset{(\pm)}{\sigma}{}^\alpha \wedge \overset{(\mp)}{\sigma}{}^\beta = 2\eta(\mp o^{\alpha\beta} - \eta^{\alpha\beta}) \ . \end{aligned} \tag{A12}$$

Under the $SO_0(1,1)$ transformations (2.15), these objects simply transform as

$$\overset{(\pm)}{\sigma}{}'^\alpha = e^{\pm\omega} \overset{(\pm)}{\sigma}{}^\alpha \ . \tag{A13}$$

This became manifest in the theory of spinors in two dimensions (see Sec. IX). Equations (A11) show that each σ-form actually has only one independent component, which can be denoted as

$$\overset{(\pm)}{\sigma} = \overset{(\pm)}{\sigma}{}^{\hat{0}} \ . \tag{A14}$$

For 2D spinors, these are the generalized Pauli matrices.

APPENDIX B: THE MANY FACES OF TWO-DIMENSIONAL TORSION

In two dimensions, the torsion has two independent components. And so has its Hodge dual, i.e.,

$$t^\alpha := {}^*T^\alpha \quad \text{with} \quad T^\alpha = (-1)^s\, {}^*t^\alpha, \tag{B1}$$

according to (A1). Thus, instead of T^α, we can equivalently express the field equations in terms of t^α. This is more convenient, since a zero-form can be handled more easily than a two-form. The two-form T^α can also be developed with respect to the volume two-form η:

$$\begin{aligned} T^\alpha &= (-1)^s\, {}^*t^\alpha = (-1)^s\, {}^*t^\alpha 1 \\ &= (-1)^s t^\alpha\, {}^*1 = (-1)^s t^\alpha \eta \ . \end{aligned} \tag{B2}$$

The torsion two-form T^α is not only fully contained in the zero-form t^α, but also in a one-form T. This comes about as follows. In n dimensions, the torsion can be decomposed into three irreducible pieces: a tensor, a vector, and an axial-vector piece; see [19]. In two dimensions the torsion is irreducible and only the (co)vector piece survives:

$$^{(2)}T^\alpha := \vartheta^\alpha \wedge (e_\beta \rfloor T^\beta) = \vartheta^\alpha \wedge T \quad \text{with} \quad T := e_\beta \rfloor T^\beta \ . \tag{B3}$$

The one-form T can be expressed in terms of t_α as

$$T = e_\beta \rfloor T^\beta = (-1)^s t^\beta e_\beta \rfloor \eta = (-1)^s t_\beta \eta^\beta \ . \tag{B4}$$

Because of $e_\alpha \rfloor {}^*\Phi = {}^*(\Phi \wedge \vartheta_\alpha)$, its dual reads

$$\begin{aligned} {}^*T &= {}^*(e_\beta \rfloor T^\beta) = (-1)^s\, {}^*(e_\beta \rfloor {}^*t^\beta) \\ &= (-1)^s\, {}^{**}(t^\beta \vartheta_\beta) = -t_\beta \vartheta^\beta \ . \end{aligned} \tag{B5}$$

Now it is easy to show that the vector piece of the torsion coincides with the total torsion:

$$^{(2)}T^\alpha = \vartheta^\alpha \wedge T = (-1)^s \vartheta^\alpha \wedge t_\beta \eta^\beta = (-1)^s t_\beta \vartheta^\alpha \wedge \eta^\beta = (-1)^s t^\alpha \eta = T^\alpha \ . \tag{B6}$$

Accordingly, we recognize that the torsion alternatively can be presented by the zero-form t^α, the two one-forms T or ${}^*T$, or by the standard two-form T^α:

$$t^\alpha := {}^*T^\alpha, \quad T := e_\alpha \rfloor T^\alpha, \quad {}^*T = -\vartheta^\beta\, {}^*T_\beta, \quad T^\alpha \ , \tag{B7}$$

with

$$T^\alpha = (-1)^s\, {}^*t^\alpha = (-1)^s t^\alpha \eta = \vartheta^\alpha \wedge T = (-1)^s \eta^\alpha \wedge {}^*T \ . \tag{B8}$$

The set $\{t^\alpha, T, {}^*T, T^\alpha\}$ of equivalent torsion forms turned out to be very useful.

For the presentation of exact 2D solutions it is rather convenient to introduce, instead of ϑ^α and e_β, quite generally the new coframe $\{T, {}^*T\}$ together with its dual vectors $\{\xi, \xi^\star\}$. By duality we have

$$\begin{aligned} &\xi \rfloor T = 1 \\ &\quad \Longrightarrow (-1)^s \xi^\alpha e_\alpha \rfloor (t_\beta \eta^\beta) = (-1)^s \xi_\alpha t_\beta \eta^{\beta\alpha} = 1 \ , \end{aligned} \tag{B9}$$

$$\xi^\star \rfloor {}^*T = 1 \Longrightarrow -\xi^{\star\alpha} e_\alpha \rfloor (t_\beta \vartheta^\beta) = -\xi^{\star\alpha} t_\alpha = 1 \ . \tag{B10}$$

Apart from singular points, the condition $t^2 \neq 0$ holds as a result of the field equation. Then we find

$$\xi = -\frac{\eta^{\alpha\beta} t_\beta}{t^2} e_\alpha, \quad \xi^\star = -\frac{t^\alpha}{t^2} e_\alpha \ . \tag{B11}$$

Furthermore, one can show that

$$\xi \rfloor {}^*T = 0 \quad \text{and} \quad \xi^\star \rfloor T = 0 \ . \tag{B12}$$

As a final proof that T and ${}^*T$ formally span a coframe we display the orthogonality of ξ and $\xi^\star$ with the help of the metric g. Since the e_α's are orthonormal we have

$$g(\xi,\xi^\star)=-\frac{\eta^{\alpha\beta}t_\beta t^\gamma}{t^4}g(e_\alpha,e_\gamma)=-\frac{\eta^{\alpha\beta}t_\alpha t_\beta}{t^4}=0 \ . \tag{B13}$$

The vectors $\{\xi,\xi^\star\}$ are not of unit length, rather

$$\xi^2:=g(\xi,\xi)=(-1)^s\xi^{\star 2}$$
$$:=(-1)^s g(\xi^\star,\xi^\star)=\frac{(-1)^s}{t^2} \ . \tag{B14}$$

Consequently, the new coframe

$$\theta^\alpha=\{\theta^{\hat 0},\theta^{\hat 1}\}:=\left\{\frac{T}{\sqrt{t^2}},\frac{{}^*T}{\sqrt{t^2}}\right\}$$
$$=\left\{(-1)^s\frac{t_\beta}{\sqrt{t^2}}{}^*\vartheta^\beta,-\frac{t_\beta}{\sqrt{t^2}}\vartheta^\beta\right\} , \tag{B15}$$

is *orthonormal* with respect to $o_{\alpha\beta}$, in contrast with the system $\{T,{}^*T\}$ which is only orthogonal. The dual frame reads

$$\mathscr{E}_\alpha=\{\mathscr{E}_{\hat 0},\mathscr{E}_{\hat 1}\}:=\left\{\frac{t^{\beta\star}}{\sqrt{t^2}}e_\beta,-\frac{t^\beta}{\sqrt{t^2}}e_\beta\right\} , \tag{B16}$$

that is,

$$\mathscr{E}_\alpha \rfloor \theta^\beta=\delta_\alpha^\beta \quad\text{and}\quad g(\mathscr{E}_\alpha,\mathscr{E}_\beta)=o_{\alpha\beta} \tag{B17}$$

with

$$\mathscr{E}_\alpha=\mathscr{E}^i{}_\alpha\frac{\partial}{\partial X^i} \quad\text{and}\quad \theta^\beta=\mathscr{E}_i{}^\beta dX^i , \tag{B18}$$

where X^i are some (holonomic) coordinates.

A two-dimensional Lorentz transformation depends only on one parameter. From (B15) we can read off its inverse:

$$\vartheta^\alpha=-\frac{t^{\alpha\star}}{\sqrt{t^2}}\left[\frac{T}{\sqrt{t^2}}\right]-\frac{t^\alpha}{\sqrt{t^2}}\left[\frac{{}^*T}{\sqrt{t^2}}\right] . \tag{B19}$$

The volume two-form can also be expressed in terms of the coframe (B15):

$$\eta=\theta^{\hat 0}\wedge\theta^{\hat 1}=(1/t^2)T\wedge{}^*T \ . \tag{B20}$$

Moreover, from (B6) we find for the torsion components with respect to the new coframe (B15) that

$$T^{\hat 0}=0,\quad T^{\hat 1}=-\sqrt{t^2}\eta \ . \tag{B21}$$

APPENDIX C: SPINORS IN TWO DIMENSIONS

Dirac spinors in two dimensions have two (complex) components,

$$\psi=\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} , \tag{C1}$$

and, as usual, the spinor space at any point of the spacetime manifold is related to the tangent space at this point via the spin-tensor objects: the Dirac and the Pauli matrices.

The Dirac matrices γ^α satisfy the standard relations

$$\gamma^\alpha\gamma^\beta+\gamma^\beta\gamma^\alpha=2o^{\alpha\beta} \tag{C2}$$

and in 2D these 2×2 matrices can be chosen to be real. Further elements of the 2D Clifford algebra are the γ_5 matrix and the $\overline{\mathrm{SO}}(1,1)$ generator $\sigma_{\alpha\beta}$ which are defined by

$$\gamma_5:=\tfrac{1}{2}\eta_{\alpha\beta}\gamma^\alpha\gamma^\beta,\quad \sigma_{\alpha\beta}:=(i/2)[\gamma_\alpha,\gamma_\beta] \ . \tag{C3}$$

From (C2) and (C3) one can derive the useful relations

$$\gamma_\alpha\gamma^\beta\gamma^\alpha=0,\quad \gamma_5\gamma^\alpha+\gamma^\alpha\gamma_5=0,\quad (\gamma_5)^2=1 \tag{C4}$$

and

$$\gamma^\alpha\gamma_5=\eta^{\alpha\beta}\gamma_\beta,\quad [\gamma_\alpha,\gamma_\beta]=-2\eta_{\alpha\beta}\gamma_5 \ . \tag{C5}$$

If we introduce the matrix-valued one-form

$$\gamma=\gamma_\alpha\vartheta^\alpha , \tag{C6}$$

Eqs. (C2)–(C5) can be rewritten in Clifford-algebra-valued exterior forms as

$$\gamma\otimes\gamma=g,\quad \gamma\wedge\gamma=-2\gamma_5\eta,\quad {}^*\gamma=\gamma_5\gamma \ . \tag{C7}$$

The action of the gauge (local Lorentz) group on spinors is given by

$$\psi\to\psi'=S\psi,\quad \bar\psi\to\bar\psi'=\bar\psi S^{-1} , \tag{C8}$$

where the Dirac adjoint is defined as

$$\bar\psi:=\psi^\dagger\gamma^{\hat 0} \ . \tag{C9}$$

Equations (C2) and (C7) relate the metric structure on a spacetime to the spinor space. Then the Lorentz transformation is converted, via the covering homomorphism $\overline{\mathrm{SO}}(1,1)\approx\mathrm{SO}(1,1)$, to the similarity transformation

$$\gamma'=S\gamma S^{-1} \tag{C10}$$

of the γ matrices, where $\gamma'=\gamma_\alpha\vartheta'^\alpha$. Substituting (2.14) and using the explicit form of the local Lorentz rotations (2.15) one finds

$$S=\exp\left[\frac{\omega}{2}\gamma_5\right]=\cosh\left[\frac{\omega}{2}\right]+\gamma_5\sinh\left[\frac{\omega}{2}\right] . \tag{C11}$$

This completes the definition of a *spinor algebra* on a 2D manifold. The next step is to develop the spinor *analysis,* and the central point is the notion of the so-called spinor covariant derivative D. The formal definition of the spinor covariant derivative is given by (9.2), where the connection one-form, due to the covering homomorphism, has the usual transformation law

$$\Gamma\to\Gamma'=S\Gamma S^{-1}+S\,dS^{-1} \ . \tag{C12}$$

The explicit form of the connection (9.3) is obtained from the natural assumption that spinor bilinears behave covariantly, that is, $\bar{\psi}\psi$, $\bar{\psi}\gamma\psi$, and $\bar{\psi}\gamma\wedge\gamma\psi$ are the zero-, one-, and two-forms, respectively, on the spacetime manifold. This is equivalent to the condition

$$\mathcal{D}\gamma^{\alpha}=d\gamma^{\alpha}+\Gamma_{\beta}{}^{\alpha}\gamma^{\beta}+[\Gamma,\gamma^{\alpha}]=0\ , \tag{C13}$$

for which the explicit solution is just (9.3).

The concrete realization of the Clifford algebra (C2) and (C7) on a 2D manifold is easily achieved in terms of the 1×1 Pauli matrices given in (A14). According to (A12) they satisfy

$$\begin{aligned}\overset{(+)}{\sigma}\wedge\overset{(+)}{\sigma}&=\overset{(-)}{\sigma}\wedge\overset{(-)}{\sigma}=0\ ,\\ \overset{(+)}{\sigma}\wedge\overset{(-)}{\sigma}&=-\overset{(-)}{\sigma}\wedge\overset{(+)}{\sigma}=2\eta\ .\end{aligned} \tag{C14}$$

Then one can easily prove that

$$\gamma:=\begin{bmatrix}0 & -\overset{(+)}{\sigma}\\ \overset{(-)}{\sigma} & 0\end{bmatrix} \tag{C15}$$

is, indeed, the matrix-valued one-form (C6) in two dimensions. Using (A14), one can read off from (C15) the explicit realization in terms of the Dirac matrices

$$\gamma^{\hat{0}}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},\ \ \gamma^{\hat{1}}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\ \ \gamma_5=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}. \tag{C16}$$

In view of (C10) it is clear that each of the two components of the Dirac spinor (C1), ψ_1 and ψ_2, represent the irreducible spinor fields with the simple transformation laws

$$\psi_1'=e^{\omega/2}\psi_1,\ \ \psi_2'=e^{-\omega/2}\psi_2\ ; \tag{C17}$$

compare with (A13).

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