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Amplification of gravitational waves in scalar-tensor theories of gravity

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The gravitational wave equation for a spatially flat Friedmann-Robertson-Walker universe is derived in the context of scalar-tensor theories of gravity, which have the Brans-Dicke theory as a particular case. This equation is solved for several cosmological scenarios, including the expansions governed by the Nariai as well as the Gurevich-Finkelstein-Ruban solutions of Brans-Dicke theory and for a new set of exact solutions of other scalar-tensor theories. The amplification of gravitational waves is studied in comparison to what happens in the general relativistic case. It is shown how the coupling with the scalar field changes the scales defining very large and very small wave numbers, and consequently the value of the amplification coefficient. It is found that very small values for the coupling parameter could lead to amplification of subhorizon waves. The creation of the corresponding high-frequency gravitons is explained as a response to the rapid time variation of the gravitational "constant," which can occur near the singularity in some models. It is also shown that there could be amplification of waves even in a radiation-dominated universe in some cases, because the wave equation is not conformally invariant, except for the case of Nariai's solution in the Brans-Dicke theory.

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I. INTRODUCTION

One of the most remarkable predictions of a metric theory of gravity, such as general relativity (GR), is that perturbations of a background spacetime generate gravity waves. The general results concerning gravitational waves have been investigated by numerous authors long ago [1–5]. In the context of GR cosmologies, Grishchuk has shown that the varying gravitational field of the expanding Universe would amplify zero-point fluctuations, and lead to the formation of a nonthermal, stochastic background of relic gravitons [6–9]. The advent of the inflationary models [10] (for an updated review see [11] and references therein) provides a new perspective on the cosmological features of gravitational radiation [12–14]. In fact, not only was an explanation for the origin of these perturbations put forward, but it also became possible to study their contribution to the quadrupole anisotropies in the microwave background [15–17]. Thus, the idea of using gravitational radiation as probe of the early Universe has turned into a realistic possibility. Recently, the auspicious results of the Cosmic Background Explorer (COBE) [18] seem to indicate that those expectations are now closer to being realized, and this fact has led to a reassessment of the predictions from inflation, so that a detailed understanding of the interplay between the data and possible theoretical models might be reached [19–27].

Among the various prescriptions for the early inflationary epoch, a framework was constructed which seems to provide a simple way of circumventing the "graceful exit" problem, reviving the original idea of a first-order phase transition. We refer to the La-Steinhardt proposal of extended inflation [28], which considers Brans-Dicke (BD) [29] scalar-tensor theory of gravity instead of GR as the underlying gravitational theory. Notwithstanding the problems faced by the original proposal of La and Steinhardt, mainly due to the difficulty in matching the value of the coupling parameter required by the nucleation process with its post-Newtonian limits [30,31] and the restrictions imposed by the COBE data [19], the general idea of their prescription is endowed with appealing features. For instance, as pointed out by several authors, a coupling between a scalar field and gravity seems to be a generic outcome of the low-energy limit of string theories [32], and this in itself justifies further consideration of scalar-tensor theories of gravity.

It is therefore a matter of great interest to address the question of cosmological gravitational waves within the context of the scalar-tensor theories characterized by the general action discussed by Bergmann [33], Wagoner [34], and Nordtvedt [35], which we write as found in Will [36]:

$$S=\frac{1}{16\pi}\int\sqrt{-g}\,d^4x\left[\phi\mathcal{R}-\frac{\omega(\phi)}{\phi}\phi_{,\mu}\phi^{,\mu}+2\phi U(\phi)\right]+S_m . \tag{1}$$

Here $\mathcal{R}$ is the Ricci curvature, ϕ the scalar field, $\omega(\phi)$ the coupling parameter, $U(\phi)$ can be interpreted as a potential associated with ϕ, and S_M represents the action for the matter fields. It is important to notice at this point that the usual condition

$$T^{b}_{a;b}=0 , \tag{2}$$

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where $T_{\mu\nu} \equiv \delta S_M / \delta g_{\mu\nu}$, must be independently imposed to guarantee that the equivalence principle remains valid.

The fundamental feature of this class of theories is that there is a scalar field ϕ coupled to the curvature. Its essential role in homogeneous cosmologies is to induce a time variation of the gravitational constant, for $G_{\rm eff} \propto 1/\phi$. Subsequently, any gravitational phenomena become affected by such variation.

Here, we aim to assess the modifications induced by the coupling of the scalar field to the mechanism of gravitational wave amplification. We derive the equation which governs the radiative modes of a small disturbance on the background of a homogeneous and isotropic space-time with flat spatial sections. We solve for different cosmological models. For the special situation where $\omega = \omega_0$ is constant and $U(\phi) \equiv 0$ (Brans-Dicke theory), we consider the power-law solutions obtained by Nariai [37] and the solutions given by Gurevich *et al.* [38]. This enables us to distinguish the cases where the expansion is dominated by the energy density of matter (Nariai's solutions) from the ones for which the solutions clearly deviate from the GR behavior when t is small, due to the domination of the scalar field energy. We study some exact solutions of the wave equation as well as the limiting cases characterized by very large and very small wave numbers which play an important role in determining the contribution of gravitational waves to the anisotropy of the microwave background.

Previous work on the subject of gravitational radiation within Brans-Dicke theory, carried out by Wagoner [34], was concerned with gravitational waves generated in the weak-field limit. Here, we consider only the cosmological aspects of the problem.

We shall also analyze solutions of the cosmological gravitational wave equation is some general scalar-tensor theories $[\omega(\phi) \neq \text{const}]$. This more general setting is of particular interest because the constraints placed on a constant coupling by post-Newtonian solar-system tests and by the evaluations of primordial nucleosynthesis seriously limit the applicability of the Brans-Dicke theory to model phenomena such as inflation which take place during the early Universe. An example is provided by a theory in which the coupling increases from a small value at early times towards a value in agreement with the constraints at the time of nucleosynthesis. Both in BD and in the $\omega(\phi)$ theories we investigate the possibility of having amplification of modes with wavelengths smaller than the Hubble length as a consequence of rapid variations of the scalar field.

An outline of this paper is as follows. In the next section, we derive the gravitational wave equation for the general class of scalar-tensor theories characterized by the action (1). In Sec. III we consider the particular case of Brans-Dicke theory, solving the equation for a number of cosmological behaviors of the background space-time. We first analyze the simpler case of a power-law expansion and then the situation where the expansion is governed by the solutions derived by Gurevich *et al.* [38]. In Sec. IV we extend our study to the case of more general theories where we allow the coupling parameter to run with ϕ. Some solutions are investigated. Finally, we conclude with a section devoted to an overall summary and discussion of the results.

II. DERIVATION OF THE GRAVITATIONAL WAVE EQUATION

We follow Weinberg's treatment of the analogous problem in GR [39]. That is, we consider a small disturbance $h_{\mu\nu}$ of the spatially flat Friedmann-Robertson-Walker (FRW) metric, and equate the perturbations induced on the field equations. A synchronous gauge is chosen ($h_{00} = 0 = h_{0i}$ with $i = 1,2,3$), and the computations are made to first order in the small quantities considered.

The essential difference between the GR derivation and the present one results from the fact that the field equations which one obtains from varying the action (1) are no longer

$$R_{\mu\nu} = 8\pi G \left[T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T \right] , \tag{3}$$

but

$$\begin{aligned} R_{\mu\nu} = {} & \frac{8\pi G}{\phi} \left[T_{\mu\nu} - \frac{1+\omega}{2\omega+3} g_{\mu\nu} T \right] + \frac{\omega}{\phi^2} \phi_{;\mu} \phi_{;\nu} + \frac{1}{\phi} \phi_{;\mu\nu} \\ & - \frac{1}{2} \frac{g_{\mu\nu}}{\phi} \frac{d\omega/d\phi}{2\omega+3} \phi_{;\sigma} \phi^{;\sigma} \\ & - g_{\mu\nu} \frac{\phi(dU/d\phi) + 2(\omega+1)U}{2\omega+3} , \end{aligned} \tag{4}$$

where ϕ satisfies

$$\begin{aligned} \Box\phi = {} & \frac{8\pi}{2\omega+3} T - \frac{d\omega/d\phi}{2\omega+3} \phi_{;\sigma} \phi^{;\sigma} \\ & = \frac{2[\phi^2(dU/d\phi) - \phi U]}{2\omega+3} . \end{aligned} \tag{5}$$

Therefore, when perturbing the field equations, we must consider some additional terms not present in GR. These are

$$\left[T_{\mu\nu} - \frac{1+\omega}{2\omega+3} g_{\mu\nu} T \right] \delta \left[\frac{8\pi}{\phi} \right] , \tag{6}$$

$$\delta \left[\frac{\omega}{\phi^2} \phi_{;\mu} \phi_{;\nu} \right] , \quad \delta \left[\frac{1}{\phi} \phi_{;\mu\nu} \right] ,$$

$$\delta \left[\frac{1}{2} \frac{g_{\mu\nu}}{\phi} \frac{d\omega/d\phi}{2\omega+3} \phi_{;\sigma} \phi^{;\sigma} \right] , \tag{7}$$

$$\delta \left[g_{\mu\nu} \frac{\phi(dU/d\phi) + 2(\omega+1)U}{2\omega+3} \right] ,$$

which arise from the presence of the scalar field in the equations.

The perturbations to the metric that represent weak gravitational waves can be expressed as

$$h_{ij}(t,\mathbf{x}) = \int d^3k \, h_{ij}^{(k)}(t,\mathbf{x}) , \tag{8}$$

$$h_{ij}^{(k)}(t,\mathbf{x}) = \frac{1}{a^2(t)} Y_k(t) \zeta_{ij}(\mathbf{k},\mathbf{x}) , \tag{9}$$

where the functions $Y_k(t)$ and $\zeta_{ij}(\mathbf{k},\mathbf{x})$ satisfy

$$(\nabla^2+k^2)\zeta_{ij}(\mathbf{k},\mathbf{x})=0\ , \tag{10}$$

$$\ddot{Y}_k+f(t)\dot{Y}_k+g(t)Y_k=0\ , \tag{11}$$

where

$$f(t)\equiv\frac{\dot\phi}{\phi}-\frac{\dot a}{a}\ , \tag{12}$$

$$g(t)\equiv\frac{k^2}{a^2}+4\frac{\dot a^2}{a^2}-\frac{8\pi}{\phi}\left[\frac{2(1+\omega)}{2\omega+3}\rho-\frac{2\omega}{2\omega+3}p\right]-\frac{d\omega/d\phi}{2\omega+3}\frac{\dot\phi^2}{\phi}+\frac{\phi dU/d\phi+2(\omega+1)U}{2\omega+3}\ . \tag{13}$$

In the above equations, $a(t)$ is the scale factor, $i,j=1,2,3,\nabla^2$ is the spatial Laplacian operator, an overdot indicates derivatives with respect to the synchronous cosmic time t, and $k\equiv|\mathbf{k}|$, with $\mathbf{k}$ being the comoving wave vector, i.e.,

$$k=\frac{2\pi a}{\lambda}\ . \tag{14}$$

The eigenfunctions $\zeta_{ij}(\mathbf{k},\mathbf{x})$ can be written in terms of plane-wave solutions $\exp(\pm i\mathbf{k}\cdot\mathbf{x})$ times a constant polarization tensor and using the field equations in the zero-curvature FRW model we can simplify (11) and recast it as

$$\ddot{Y}_k+\left[\frac{\dot\phi}{\phi}-\frac{\dot a}{a}\right]\dot{Y}_k+\left[\frac{k^2}{a^2}-2\frac{\ddot a}{a}-2\frac{\dot a}{a}\frac{\dot\phi}{\phi}-\frac{d\omega/d\phi}{2\omega+3}\frac{\dot\phi^2}{\phi}+\frac{\phi dU/d\phi+2(\omega+1)U}{2\omega+3}\right]Y_k=0\ . \tag{15}$$

III. SOLUTIONS OF THE GRAVITATIONAL WAVE EQUATION IN BRANS-DICKE THEORY

The Brans-Dicke scalar-tensor theory of gravity is characterized by the restriction that $\omega(\phi)$ be a constant, and $U=dU/d\phi=0$. We write $\omega\equiv\omega_0=$const. We introduce the conformal time η defined by

$$dt=a\,d\eta\ . \tag{16}$$

Each component $h_{ij}^{(k)}$ of the gravitational wave perturbations can be written as

$$h_{ij}^{(k)}(\eta,\mathbf{x})=\frac{1}{R(\eta)}\mu(k,\eta)\zeta_{ij}(\mathbf{k},\mathbf{x}) \tag{17}$$

where

$$R(\eta)\equiv a(\eta)\phi^{1/2}(\eta) \tag{18}$$

and

$$\mu''(k,\eta)+\left[k^2-\frac{R''}{R}\right]\mu(k,\eta)=0 \tag{19}$$

with the prime indicating derivatives with respect to η.

The solution given by (17) and (19) is formally identical to the corresponding one obtained in the general relativistic case [6–9], but with the quantity $R(\eta)$ defined by (18), replacing the scale factor $a(\eta)$. Equation (19) describes an oscillator with varying frequency or a Schrödinger equation with "potential" R''/R for a particle with energy k^2 and with the variable η playing the role of a spatial coordinate [6,9].

In order to facilitate comparison with the BD case, we summarize the classical mechanism of gravitational wave amplification in GR described by Grishchuk [6–9] for a spatially flat FRW universe. In this context, the equation analogous to (19) (with a replacing R) becomes the ordinary wave equation in Minkowski space-time for those modes obeying the condition $k^2\gg|a''/a|$. Its solutions will be oscillatory and hence the amplitude of the perturbations, given by an expression analogous to (17), will decrease adiabatically with a^{-1}. This will not happen, however, if the above condition is not satisfied. In particular, in the opposite regime $k^2\ll|a''/a|$ we get two solutions of the form $\mu_1\propto a(\eta)$ and $\mu_2\propto a(\eta)\int^{\eta}a^{-2}(\eta')d\eta'$. For an expanding Universe, and after averaging over the initial phases, it is found that the dominant solution is μ_1. Therefore, as long as the wave stays under the potential $V\equiv|a''/a|$, its amplitude $h_{ij}^{(k)}$ will remain practically constant. (From now on, unless otherwise stated, we will reserve the word *potential* to indicate the modulus of a''/a or of R''/R.) If a wave exits from under the potential barrier and again satisfies $k^2\gg|a''/a|$, it will have a greater amplitude than it would following purely adiabatic behavior. Grishchuk has called this mechanism the *superadiabatic amplification of gravitational waves.* From a quantum-mechanical point of view, this will appear as creation of gravitons by the expanding Universe, since the gravitational wave equation is not conformally invariant [6].

If the background matter satisfies the equation of state $p=\alpha\rho,-1\le\alpha\le 1$, we will obtain different behaviors for the potential V, depending on the value of α. For $\alpha>-1/3$ ($\alpha\ne 1/3$), V will be arbitrarily large near the singularity and will decrease to zero as $\eta\to\infty$. The condition for amplification, $k^2\ll V$, will then apply near the singularity. For $\alpha=1/3$, a radiation-dominated universe, V will vanish identically and no amplification will occur, whereas $V=$const for $\alpha=-1/3$. In the most interesting cases, $\alpha<-1/3$, which will lead to power-law or exponential inflation, the potential barrier will have a bell shape, going to zero near the singularity as $\eta\to-\infty$ and as $\eta\to\infty$. A wave with a given k will then encounter the barrier at a certain η_i and be amplified until it reassumes its adiabatic behavior when it leaves the under-barrier region at a later time η_f. The amplification coefficient will scale as $a(\eta_f)/a(\eta_i)$ [9].

It is common to identify the condition for amplification $k^2\ll|a''/a|$ with that of having modes well outside the horizon (actually of having modes whose physical wavelengths λ are greater than the Hubble length $\lambda_H\equiv H^{-1}$). To be more precise, we wish to compare the comoving wave number k defined by (14) with the "Hubble comoving wave number" k_H given by

$$k_H=\frac{2\pi a}{\lambda_H}=2\pi(a'/a)\ . \tag{20}$$

Although in most situations the two requirement are approximately equivalent, this will not always be so [6]. For the above mentioned power-law or experimental expansions in GR, the ratio between V and k_H^2 can actually be written as

$$\frac{V}{k_H^2} = \pm \frac{1}{4\pi^2} \left[\frac{1-3\alpha}{2} \right] \tag{21}$$

with the plus sign holding if $\alpha \le 1/3$ and the minus sign holding if $\alpha \ge 1/3$. For $\alpha = 1/3$ this ratio will be zero. In general, as $\alpha \to 1/3, V \ll k_H^2$ and the conditions $k^2 \ll V$ and $k^2 \ll k_H^2$ will not be equivalent. Note that V/k_H^2 is constant in time and that, at a fixed $\eta, V(\alpha)$ will have a maximum at $\alpha = 1$. Moreover, V will always be less than k_H^2, so if a mode satisfies $k^2 \ll V$ it will automatically satisfy $k^2 \ll k_H^2$. The requirement $V < k_H^2$ agrees with the fact that, quantum mechanically, graviton production is exponentially suppressed for mode with frequencies higher than the expansion rate of the Universe ($k^2 > k_H^2$). This happens because as far as the high-frequency modes are concerned, the Universe approaches the limit of an infinitely slow expansion (the adiabatic regime) where no particle production is expected [12,40,41].

An intuitive understanding of the gravitational wave amplification and its dependence upon the equation of state parameter, α, in the GR case can be obtained by the following discussion. Suppose that the equation of state of the dominant form of material (excluded in the gravitons) has the form $p = \alpha\rho$ from the Planck time t_P until some later time $t_* \ll 1$ s. After t_* the material assumes the usual radiation-dominated equilibrium state ($\alpha = 1/3$) until $t \sim 10^{10}$ s. Initially, quantum processes will produce a graviton density $\rho_g \sim (Gt_P^2)^{-1}$ at t_P and the other material will be in equilibrium with a similar density $\rho \sim \rho_g$. Subsequently, $\rho_g \propto a^{-4}$ but $\rho \propto a^{-3(\alpha+1)}$. Hence, if $\alpha > 1/3$ the graviton density will fall off more slowly than the material density. After t_*, $\rho \propto \rho_g \propto a^{-4}$, ignoring minor injections of entropy into ρ from particle annihilations, and the present day value of $\rho_{\rm rad}/\rho_g$ will reflect the value of ρ/ρ_g at t_*:

$$\left[\frac{\rho_g}{\rho}\right]_{t_*} = \left[\frac{\rho_g}{\rho}\right]_{t_P} \left[\frac{a_*}{a_p}\right]^{3\alpha-1} \sim 0.01 \left[\frac{a_*}{a_P}\right]^{3\alpha-1} \tag{22}$$

where the 10^{-2} factor allows for pair annihilations in the standard model that couple to ρ but not to ρ_g. Hence, we see that if $\alpha = 1/3$ there is no relative increase or decrease of ρ_g with respect to the density of other matter. If $\alpha > 1/3$ then ρ_g exceeds ρ at t_* by a factor $(a_*/a_P)^{3\alpha-1}$ and this will lead to a present day graviton abundance that exceeds the microwave background density by this factor. Such an enhancement of ρ_g over the radiation density occurs when $t > t_*$ in all cases where the equation of state is hard ($\alpha > 1/3$) and is maximal when $\alpha = 1$. Clearly, this places a strong constraint on the possible behavior of the equation of state of matter at very high density during the very early stages of the Universe. We know that the accord between astronomical observations of helium-4 and the predictions of primordial nucleosynthesis mean that $\rho_g \le \rho$ at $t \sim 1$ s when neutron-proton freeze-out occurs otherwise helium will be overproduced. Hence $(a_*/a_P)^{3\alpha-1} \le 10^2$ when $\alpha > 1/3$. This means that there can be only a very brief period of post-Planck time evolution with an $\alpha > 1/3$ equation of state unless there is a subsequent period of inflation to dilute the graviton density relative to the density of matter reestablished at the end of inflation by reheating. However, we note that the essence of the argument leading to (22) is the $\rho(a)$ evolution of the matter when $\alpha > 1/3$. If the evolution of the Universe is dominated by some other stress (for example, anisotropy or a massless scalar field) in addition to the matter then, although these stresses may produce an $a(t)$ evolution for the scale factor that differs from $a \propto t^{1/2}$, they will not lead to a graviton enhancement if the equation of state of the sea of interacting matter still has $\alpha \le 1/3$.

We turn now to investigate what happens in the BD case. As in GR we can distinguish two extreme regimes. For very large wave numbers or, more precisely, for waves such that

$$k^2 \gg V(\eta) , \tag{23}$$

where

$$V(\eta) \equiv |R''/R| , \tag{24}$$

(19) will have purely oscillatory solutions, and (17) takes the form

$$h_{ij}^{(k)}(\eta,\mathbf{x}) = \frac{1}{a(\eta)\phi^{1/2}(\eta)} \times [C_1 \epsilon^{ik(\eta-\bar\eta_0)} + C_2 \epsilon^{-ik(\eta-\bar\eta_0)}] \zeta_{ij}(\mathbf{k},\mathbf{x}) , \tag{25}$$

where C_1 and C_2 are arbitrary constants and the third constant $\bar\eta_0$ was introduced for future convenience. We see that the effect of the scalar field ϕ is for the scale of amplitude decrease to be set by the function $R(\eta)$ instead of by $a(\eta)$. As will be shown later, only when both $a(t)$ and $\phi(t)$ are assumed to have a power-law behavior in t *and* when the equation of state is $p = \rho/3$, will we get the adiabatic decrease with a^{-1}. In general it is possible to have the high-frequency regime (23) with amplitudes decreasing faster or slower than a^{-1}.

In the opposite regime of very small wave numbers, where

$$k^2 \ll V(\eta) , \tag{26}$$

we obtain

$$h_{ij}^{(k)}(\eta,\mathbf{x}) = \left[C_1 + C_2 \int^{\eta} R^{-2}(\eta')d\eta' \right] \zeta_{ij}(\mathbf{k},\mathbf{x}) . \tag{27}$$

As in GR, there is a mode which is constant in time. Note, however, that, for bell-shaped potentials, the amplification coefficient will scale as $R(\eta_f)/R(\eta_i)$ instead of $a(\eta_f)/a(\eta_i)$. Furthermore, the requirements $k^2 \ll V(\eta)$ and $k^2 \ll k_H^2$ may be quite different in some situations. In fact, it is not only possible to get a time-dependent ratio V/k_H^2, but also to have, at least in principle, $V > k_H^2$. Unless we conveniently restrict ω_0, this could lead to the existence of modes with $k^2 > k_H^2$ which would satisfy, at the same time, the amplification condition (26) (see below). This seems to violate the fast decline in particle production for modes whose frequencies are greater than the expansion rate. This new effect may be explained, however, if we realize that now the definition of an adiabatic vacuum should take into account not only the speed of the expansion of the Universe, but also the rate of variation of the scalar field ϕ. Actually, as it will be shown below, the condition $V > k_H^2$ can hold only near the singularity, where the energy of the free scalar field dominates, *and* for very small values of ω_0, when the gravitational "constant" varies very rapidly.

The general cosmological solutions for the zero-curvature FRW background space-time when matter satisfies the equation of state $p=\alpha\rho$, with $-1\le\alpha\le 1$ a constant, were derived by Gurevich *et al.* [38] (see below). An important feature of these solutions is that they distinguish two different limiting regimes associated with the asymptotic behaviors at early and late times. The first corresponds to an expansion dominated by the energy of the scalar field, and approaches the vacuum solution close to the singularity. The latter corresponds to an expansion dominated by the matter energy and tends to a power-law behavior. The asymptotes therefore coincide with the vacuum solutions derived by O'Hanlon and Tupper [42] and the power-law solutions derived by Nariai [37], respectively.

We start by considering these simpler cases, where the scale factor and the scalar field evolve as power-laws of the cosmic time and write

$$a(t)=a_0(t/t_0)^A , \tag{28}$$

$$\phi(t)=\phi_0(t/t_0)^B , \tag{29}$$

without yet determining the constants A and B.

For $A\neq 1$, the translation of (28) and (29) into the conformal time yields

$$a(\eta)=c_0(\eta-\bar{\eta}_0)^q , \tag{30}$$

$$\phi(\eta)=\sigma_0(\eta-\bar{\eta}_0)^r , \tag{31}$$

$$R(\eta)=C_0(\eta-\bar{\eta}_0)^s , \tag{32}$$

where

$$q\equiv\frac{A}{1-A} , \tag{33}$$

$$r\equiv\frac{B}{1-A} , \tag{34}$$

$$s\equiv q+\frac{r}{2}=\frac{2A+B}{2(1-A)} , \tag{35}$$

$$\bar{\eta}_0\equiv\eta_0-\frac{q}{a_0H_0} , \tag{36}$$

$$\eta_0\equiv\eta(t_0) , \tag{37}$$

$$H_0\equiv H(t_0) , \tag{38}$$

$$c_0\equiv a_0\left[\frac{a_0H_0}{q}\right]^q , \tag{39}$$

$$\sigma_0\equiv\phi_0\left[\frac{a_0H_0}{q}\right]^r , \tag{40}$$

$$C_0\equiv c_0\sigma_0^{1/2}=R_0\left[\frac{a_0H_0}{q}\right]^s , \tag{41}$$

$$R_0\equiv a_0\phi_0^{1/2} . \tag{42}$$

The wave equation takes the form

$$\mu''(k,\eta)+\left[k^2-\frac{s(s-1)}{(\eta-\bar{\eta}_0)^2}\right]\mu(k,\eta)=0 . \tag{43}$$

Its general solution can be written as

$$\mu(k,\eta)=k^{-1/2}X^{1/2}(\eta)[C_1H_m^{(1)}(X(\eta))+C_2H_m^{(2)}(X(\eta))] , \tag{44}$$

where

$$X(\eta)\equiv k(\eta-\bar{\eta}_0) , \tag{45}$$

$$m\equiv s-\frac{1}{2}=\frac{3A+B-1}{2(1-A)} . \tag{46}$$

The Hankel functions $H_m^{(1)},H_m^{(2)}$ can be replaced by any pair of linearly independent Bessel functions such as J_m, Y_m or J_m, J_{-m} (if m is noninteger). In order to define particle states it is important to have a solution consistent with the requirements for the definition of an adiabatic vacuum, hence it is useful to write the solution of (43) in terms of Hankel functions. The basic solution representing an adiabatic vacuum state is

$$\mu(k,\eta)=\frac{\sqrt{\pi}}{2}e^{i\theta}k^{-1/2}X^{1/2}(\eta)H_m^{(2)}(X(\eta)) , \tag{47}$$

where θ is an arbitrary constant phase and we have imposed the normalization condition on the Wronskian of the solutions [40]:

$$W(\mu,\mu^*)=\mu\mu^{*\prime}-\mu^*\mu'=i . \tag{48}$$

The difference between the solution (47) and the corresponding one in GR (see equations (57)–(59) of Ref. [41]) lies in the presence of the coefficient B in the parameters s and m. Setting $B=0$ we recover the GR solution as expected, since this corresponds to the requirement that ϕ, and hence G, remains constant.

For $m\neq 0$ ($s\neq 1/2$, $B\neq 1-3A$), the low frequency limit for the general solution is

$$\mu(k,\eta)=(\eta-\bar{\eta}_0)^{1/2}[C_1(\eta-\bar{\eta}_0)^m+C_2(\eta-\bar{\eta}_0)^{-m}] \tag{49}$$

whereas for $m=0$ ($s=1/2$, $B=1-3A$), we have, in this limit,

$$\mu(k,\eta)=(\eta-\bar{\eta}_0)^{1/2}[C_1+C_2\ln(\eta-\bar{\eta}_0)] . \tag{50}$$

For the very particular case $A=1$, a, ϕ, and R can be expressed as

$$a(\eta)=a_0e^{a_0H_0(\eta-\eta_0)} , \tag{51}$$

$$\phi(\eta)=\phi_0e^{Ba_0H_0(\eta-\eta_0)} , \tag{52}$$

$$R(\eta)=R_0e^{k_0(\eta-\eta_0)} , \tag{53}$$

where

$$k_0 \equiv \frac{2+B}{2} a_0 H_0 \ . \tag{54}$$

The wave equation (19) becomes

$$\mu''(k,\eta)+(k^2-k_0^2)\mu(k,\eta)=0 \ . \tag{55}$$

Its general solution is

$$\mu(k,\eta)=C_1 e^{i\Omega(\eta-\eta_0)}+C_2 e^{-i\Omega(\eta-\eta_0)} \tag{56}$$

with

$$\Omega \equiv \sqrt{k^2-k_0^2} \ . \tag{57}$$

An equation analogous to (55) is obtained in GR when, as in the present case, the scale factor grows linearly with the cosmic time t. The curvature term $-k_0^2$ enters in the same way as would a mass, albeit a negative one (corresponding to a tachyon). This situation is usually taken to mean that the underlying vacuum is unstable [40]. It is problematic to define an adiabatic vacuum state when $k^2 \le k_0^2$ since it would be impossible to satisfy the condition (48).

Near the singularity the expansion is dominated by the energy of the *free* scalar field which is *not* connected with the matter and approaches the vacuum solution derived in [42]. (We will be assuming $\omega_0 > 0$ in what follows.) The exponents A and B are then given by

$$A = \frac{\omega_0}{3\left[\omega_0+1 \pm \left[\frac{2\omega_0+3}{3}\right]^{1/2}\right]} \ , \quad B=1-3A \ . \tag{58}$$

The solution $\mu(k,\eta)$ will be expressed in terms of Bessel or Hankel functions of order zero. In GR the $m=0$ case occurs for matter with an equation of state $p=\rho$. This is not surprising since the free scalar field plays the role of an effective material source of the geometry analogous to an ideal fluid with the stiff equation of state $p=\rho$ [38,43]. As we are assuming $\omega_0>0$, $A \neq 1$, and a singularity will occur at $\bar{\eta}_0$. The potential V will be given by $1/4(\eta-\bar{\eta}_0)^2$, thus becoming arbitrarily large as $\eta \to \bar{\eta}_0$. On the other hand,

$$\frac{V}{k_H^2} = \frac{1}{16\pi^2 q^2} \tag{59}$$

and if we take the plus sign in (58), q would become arbitrarily small as $\omega_0 \to 0$. Therefore, we may have $V > k_H^2$ and, as it was stated above, at least in principle some modes with $k > k_H^2$ could obey the amplification condition (26), unless we set an appropriate lower bound to ω_0. The excitation of these high-frequency modes is a response to the rapid variation of ϕ near the singularity. As η grows larger than $\bar{\eta}_0$, V decreases and a few modes could have $k^2 > V$. It is then easily found that the amplitude of these modes would decrease slower than a^{-1} if the minus sign is taken in (58) and faster than a^{-1} if the plus sign is used. (This, of course, assumes that the vacuum solutions remain approximately valid as we move away from the singularity.)

The power-law matter solutions of Nariai [37] have the form (28) and (29) with

$$A \equiv \frac{2+2\omega_0(1-\alpha)}{4+3\omega_0(1-\alpha^2)} \quad B \equiv \frac{2(1-3\alpha)}{4+3\omega_0(1-\alpha^2)} \ , \tag{60}$$

$$\phi_0 \equiv \left[\frac{A}{H_0}\right]^B \frac{2\pi\rho_0[3\omega_0(1-\alpha^2)+4]^2}{3\{\omega_0^2(1-\alpha)^2+3\omega_0[(1-\alpha)^2-\frac{1}{18}(1-3\alpha)^2]+2-3\alpha\}} \ . \tag{61}$$

For $A \neq 1$, we can use the fact that $3(\alpha+1)A+B=2$ to recast the parameters s and m as

$$s = \frac{2-(1+3\alpha)A}{2(1-A)} = \frac{(1-\alpha)(2\omega_0+3)}{(1-\alpha)(1+3\alpha)\omega_0+2} \ , \tag{62}$$

$$m = \frac{1-3\alpha A}{2(1-A)} = \frac{3(1-\alpha)^2\omega_0+2(2-3\alpha)}{2[(1-\alpha)(1+3\alpha)\omega_0+2]} \ . \tag{63}$$

For $\alpha=1$, $(p=\rho)$ we get $s=0, m=-1/2$ which does not agree with the corresponding case in GR. Again, this is not surprising since, for $\alpha=1$, the expression for $a(t)$ itself does not approach the general relativistic case when $\omega_0 \to \infty$ (see, for instance, [44])]. In GR we would obtain $m=-1/2$ only if $|\alpha|$ could be allowed to be arbitrarily large, in which case we would get a constant scale factor [41].

For $\alpha=1/3$ $(p=-\rho/3)$, we obtain

$$s=\tfrac{2}{3}(2\omega_0+3) \ , \quad m=\frac{8\omega_0+9}{6} \ . \tag{64}$$

In GR, and $\alpha=-1/3$ case leads to $a(t) \propto t$ and therefore to the above-mentioned problems in defining an adiabatic vacuum. In BD this problem does not arise as long as ω_0 remains finite.

It is easily seen that for $\alpha \neq 1, -1/3$ the parameters s and m given by (62) and (63) agree with the GR results for $\omega_0 \to \infty$. In particular for $\alpha=-1$, 0, and 1/3 we obtain, in this limit, the general relativistic result $m=-3/2$, $+3/2$, and 1/2, respectively [41].

Note that $q>0$ and so $0<A<1$ for $\alpha_1<\alpha \le 1$ but $q<0$ and so $A>1$ if $\alpha<\alpha_1$, where

$$\alpha_1=\frac{1-\sqrt{4+6/\omega_0}}{3} . \quad (65)$$

Therefore, there will be a singularity at $\eta=\bar{\eta}_0$ for $\alpha_1<\alpha\leq 1$ and at $\eta\to-\infty$ for $\alpha<\alpha_1$. This last condition corresponds to an inflationary expansion ($A>1$). In the $\omega_0\to\infty$ limit, α_1 approaches the general relativistic value $-1/3$. (In order to have $\alpha_1\geq-1$, it is necessary to require $\omega_0\geq 1/2$.) Furthermore, the potential $V(\eta)$ and the square of the Hubble wave number will be expressed, respectively, by

$$V=\frac{|s(s-1)|}{(\eta-\bar{\eta}_0)^2} , \quad (66)$$

$$k_H^2=4\pi^2\frac{q^2}{(\eta-\bar{\eta}_0)^2} . \quad (67)$$

Hence, V and k_H^2 will go to zero as $\eta\to\infty$. Nevertheless, in BD, the potential (but not k_H^2) will vanish identically not only for a radiation-dominated universe ($\alpha=1/3, s=1$), but also if the equation of state is that of stiff matter ($\alpha=1, s=0$). In this latter case, however, the amplitude of the fluctuations will remain constant in time, since $s=0$ implies $R=$const [see Eq. (32)]. For $\alpha=1/3$ we have $\phi=\phi_0=$const, and the amplitude will decrease as a^{-1}. At the singularity k_H^2 will go to infinity in both cases.

For $\alpha_1<\alpha<1$, both V (except for $\alpha=1/3$) and k_H^2 will be arbitrarily large at the singularity, whereas for $\alpha<\alpha_1$ they will approach zero as $\eta\to-\infty$. Hence, as in GR, we get a bell-shaped potential in the case of inflationary expansion. The ratio V/k_H^2 is again constant, but will approach zero whenever $\alpha\to 1/3$ or $\alpha\to 1$.

Since $R\propto t^{A+B/2}$, the amplitude of the high-frequency modes, $k^2\gg|R''/R|$, will decrease slower than a^{-1} if $\alpha>1/3$ and faster than a^{-1} if $\alpha<1/3$.

For $\alpha=\alpha_1$, we will have $A=1$, the solutions given by Eqs. (51)–(57) will apply, and

$$V=\left[1+\frac{3}{2\omega_0}\right](a_0H_0)^2 , \quad (68)$$

$$k_H^2=4\pi^2(a_0H_0)^2 . \quad (69)$$

The potential V is larger than k_H^2 for $\omega_0\leq 0.04$. The amplitude of the high-frequency modes will decrease faster than a^{-1} since $R\propto t^{\sqrt{1+3/2\omega_0}}$.

We now consider the more general background behavior given by the solutions derived by Gurevich *et al.* [38] forth zero-curvature FRW models in BD theory. For a radiation fluid ($p=\rho/3$) we have

$$a(\eta)=a_0(\eta-\eta_a)^M(\eta-\eta_b)^{(1-M)} , \quad (70)$$

$$\phi(\eta)=\phi_0\left[\frac{\eta-\eta_a}{\eta-\eta_b}\right]^N , \quad (71)$$

where $\eta_a>\eta_b$ are two constants of integration (only one of which is removable by a choice of the zero of time), and M,N are given by

$$M\equiv\frac{1}{2}\pm\frac{1}{2\sqrt{1+\frac{2}{3}\omega_0}} , \quad (72)$$

$$N\equiv\mp\frac{1}{\sqrt{1+\frac{2}{3}\omega_0}} . \quad (73)$$

The wave equation will then be expressed as

$$\mu''(k,\eta)+\left[k^2+\frac{(\eta_a-\eta_b)^2}{4(\eta-\eta_a)^2(\eta-\eta_b)^2}\right]\mu(k,\eta)=0 . \quad (74)$$

Note that for $\eta_a=\eta_b$ we get back the case of power-law behavior analyzed above in which there should be no production of gravitons in a radiation-dominated universe, since the wave equation is conformally invariant [6]. However, for $\eta_a\neq\eta_b$, Eq. (74) is *not* conformally invariant and we should expect graviton production *even* when the equation of state is $p=\rho/3$. In the low wave-number limit we obtain

$$\mu=(\eta-\eta_a)^{1/2}(\eta-\eta_b)^{1/2}\left[C_1+\frac{C_2}{(\eta_b-\eta_a)}\ln\left|\frac{\eta-\eta_b}{\eta-\eta_a}\right|\right] . \quad (75)$$

There will be a singularity at η_a [38] and

$$V=\frac{(\eta_a-\eta_b)^2}{4(\eta-\eta_a)^2(\eta-\eta_b)^2} , \quad (76)$$

$$k_H^2=4\pi^2\frac{(\eta-\eta_c)^2}{(\eta-\eta_a)^2(\eta-\eta_b)^2} , \quad (77)$$

where

$$\eta_c\equiv\eta_a-M(\eta_a-\eta_b) . \quad (78)$$

Both V and k_H^2 will go to infinity at the singularity and to zero as $\eta\to\infty$, but at different rates, that is, the ratio V/k_H^2 will be time dependent, decreasing as $1/(\eta-\eta_c)^2$. We may have $V>k_H^2$ for

$$(\eta-\eta_a)<(\eta_a-\eta_b)\left[\frac{1}{4\pi}-M\right] . \quad (79)$$

Obviously, the right-hand side of the above inequality will be positive only if $\omega_0\leq 0.6$. Note that as ω_0 grows we approach the GR case where $\phi\to$const and the rate of expansion alone will determine the definition of particle states.

For the other values of α, the solutions are

$$a(\tau)=a_0(\tau-\tau_a)^{P_1}(\tau-\tau_b)^{Q_1} , \quad (80)$$

$$\phi(\tau)=\phi_0(\tau-\tau_a)^{P_2}(\tau-\tau_b)^{Q_2} , \quad (81)$$

with $dt=a^{3\alpha}d\tau$, and

$$P_1\equiv\frac{1}{3[\sigma+1\mp(1+\frac{2}{3}\omega_0)^{1/2}]} , \quad (82)$$

$$Q_1\equiv\frac{1}{3[\sigma+1\pm(1+\frac{2}{3}\omega_0)^{1/2}]} , \quad (83)$$

$$P_2 \equiv \frac{1 \mp (1+\frac{2}{3}\omega_0)^{1/2}}{3[\sigma+1 \mp (1+\frac{2}{3}\omega_0)^{1/2}]} , \quad (84)$$

$$Q_2 \equiv \frac{1 \pm (1+\frac{2}{3}\omega_0)^{1/2}}{3[\sigma+1 \pm (1+\frac{2}{3}\omega_0)^{1/2}]} , \quad (85)$$

where $\sigma \equiv (1-\alpha)\omega_0+1$. Note that for $\alpha=0$, $\tau=t$.

The equation for the gravitational waves is

$$\frac{d^2 Y_k}{d\tau^2} + \left[-(3\alpha+1)\frac{a'}{a} + \frac{\phi'}{\phi} \right] \frac{dY_k}{d\tau} + \left[k^2 a^{2(3\alpha-1)} - 2\frac{a''}{a} + 6\alpha\frac{a'^2}{a^2} - 2\frac{a'}{a}\frac{\phi'}{\phi} \right] Y_k = 0 . \quad (86)$$

For the very small wave-numbers limit, we are able to derive a solution in the form

$$Y_k(\tau) = C_1(\tau-\tau_a)^{2P_1}(\tau-\tau_b)^{2Q_1} + C_2(\tau-\tau_a)^{(3\alpha-1)P_1+1-P_2}(\tau-\tau_b)^{(3\alpha-1)Q_1+1-Q_2} , \quad (87)$$

where C_1 and C_2 are again constants of integration. Unfortunately, the relation between τ and t is too complicated for us to be able to express $Y_k(t)$ explicitly.

IV. THE EQUATION FOR THE RADIATIVE MODES IN MORE GENERAL SCALAR-TENSOR THEORIES

In this section we extend the class of scalar-tensor theories to consider the possibility of having $\omega=\omega(\phi)$. The enables us to consider more flexible cosmological scenarios which permit ω to take very large values today (in accord with solar system observations), but take small values in the very early Universe when graviton production occurs. Although no definitive form for $\omega(\phi)$ has been proposed, some interesting cases are found in the literature, which address some generic features it would be possible to associate with ω. See for instance [45–52].

From Eq. (15) with $U=dU/d\phi=0$ and using the variables previously defined in Sec. II, we get

$$\mu''(k,\eta) + \left[k^2 - \frac{R''}{R} - \frac{d\omega/d\phi}{2\omega+3}\frac{\phi'^2}{\phi} \right] \mu(k,\eta) = 0 . \quad (88)$$

The presence, in the wave equation, of the additional term arising from the dependence of ω with ϕ will make it very difficult to solve, in general. It will also alter the scale which determines what is meant by small or large wave numbers and, hence, the conditions for amplification of the gravitational waves. Even a solution for $R(\eta)$ is not easily available for some common choices for $\omega(\phi)$, except in vacuum and radiation-dominated universes.

As an example, let us consider the vacuum solutions found by Barrow [52]. It was found in this reference that the behavior of $R(\eta)$ is independent of the choice for $\omega(\phi)$, though the behavior of ϕ is not. We consider first the ansatz [52]

$$2\omega(\phi)+3 = \varepsilon^2\phi^{2l} , \quad (89)$$

where $\varepsilon>0$ is a constant and l is real number (for $l=0$, $\omega=$const and we recover the BD theory). We have, in this case,

$$R(\eta) = C(\eta-\tilde{\eta}_0)^{1/2} , \quad (90)$$

$$\int \frac{[2\omega(\phi)+3]^{1/2}}{\phi} d\phi = \sqrt{3}\ln(\eta-\tilde{\eta}_0) , \quad (91)$$

where C and $\tilde{\eta}_0$ are arbitrary constants. The wave equation becomes

$$\mu'' + \left\{ k^2 + \frac{1}{4(\eta-\tilde{\eta}_0)^2} - \frac{3l}{(\eta-\tilde{\eta}_0)^2[\varepsilon\tilde{\phi}_0^l + l\sqrt{3}\ln(\eta-\tilde{\eta}_0)]^2} \right\} \mu = 0 , \quad (92)$$

where $\tilde{\phi}_0$ is a constant of integration. It is possible to show that a singularity will occur at $\tilde{\eta}_0$, and therefore near this singularity (and also for $\eta\to\infty$) the term $1/4(\eta-\tilde{\eta}_0)^2$ will dominant over the term originating from $\omega(\phi)$ and we would get the BD vacuum solution $\mu(\eta) \propto (\eta-\tilde{\eta}_0)^{1/2}H_0^{(2)}[k(\eta-\tilde{\eta}_0)]$.

The ratio $\tilde{V}/k_H^2$ will be time dependent and, in principle, could be grater than one, and the new potential is

$$\tilde{V}(\eta) \equiv \left| \frac{R''}{R} + \text{term from } \omega(\phi) \right| . \quad (93)$$

The condition for amplification, $k^2 \ll \tilde{V}(\eta)$, will be satisfied as $\eta\to\tilde{\eta}_0$ and, in this limit, the solution for (92) is

$$\mu(\eta) = (\eta-\tilde{\eta}_0)^{1/2}\{D_1[\varepsilon\tilde{\phi}_0^l + l\sqrt{3}\ln(\eta-\tilde{\eta}_0)]^{\beta_1} + D_2[\varepsilon\tilde{\phi}_0^l + l\sqrt{3}\ln(\eta-\tilde{\eta}_0)]^{\beta_2}\} , \quad (94)$$

where D_1 and D_2 are arbitrary constants and

$$\beta_1 = \tfrac{1}{2}\left[1+\sqrt{1+1/l}\right] , \quad (95)$$

$$\beta_2 = \tfrac{1}{2}\left[1-\sqrt{1+1/l}\right] . \quad (96)$$

The interesting point here is that, contrary to what happens in most situations of GR and BD, the dominant solution as η grows (exponent β_1) in this small wave number limit *is not constant in time,* but grows as $(\ln\eta)^{\beta_1}$. (See, however, [53].)

Similar comments apply in the case of Barker's theory of gravitation [45,52], except that the term originating from $\omega(\phi)$ cannot be neglected in comparison to the one arising from R''/R since the wave equation takes the form

$$\mu'' + \left\{ k^2 + \frac{1}{2(\eta-\tilde{\eta}_0)^2} \times \left[\frac{1}{2} + \frac{3}{\cos^2[\frac{1}{2}\sqrt{3}\ln(\eta-\tilde{\eta}_0)]} \right] \right\} \mu = 0 . \quad (97)$$

A nice example which enables us to solve the gravitational wave equation exactly is given by the scalar-tensor theory defined by the ansatz

$$2\omega(\phi)+3=\frac{3\delta^2}{(\ln\phi)^2}\ , \tag{98}$$

where δ is a constant. Then, for vacuum, (91) still holds and we find, using the methods of [52] that

$$\phi(\eta)=\exp(\eta^{1/\delta})\ , \tag{99}$$

$$a(\eta)=C_0\eta^{1/2}\exp\left[-\frac{\eta^{1/\delta}}{2}\right]\ , \tag{100}$$

where C_0 is a positive constant and we have set $a=0$ when $\eta=0$. The wave equation becomes

$$\mu''+\left[k^2+\frac{1}{4\eta^2}+\frac{1}{\delta^2\eta^{2-(1/\delta)}}\right]\mu=0\ . \tag{101}$$

For $\delta=1$ its general solution can be expressed as a linear combination of the two linearly independent Whittaker functions $W_{\chi,0}(z), W_{-\chi,0}(-z)$ [54]:

$$\mu[z(\eta)]=C_1W_{\chi,0}(z)+C_2W_{-\chi,0}(-z)\ , \tag{102}$$

where

$$z(\eta)=i2k\eta\ , \tag{103}$$

$$\chi=\frac{-i}{2k}\ . \tag{104}$$

Note that the condition for amplification, $k^2\ll\tilde{V}(\eta)$, [see Eq. (93)] will hold near the singularity, and that although only the product $k\eta$ appears in the argument z, the parameter χ is a function of k alone. Therefore, for a fixed wave number, the behavior of the above solution as $\eta\to0$ can be obtained by using the leading term in the expressions for the Whittaker functions near the origin [54]. We find

$$\mu(\eta)\approx(2k\eta)^{1/2}[D_1+D_2\ln(2k\eta)^{1/2}]\ . \tag{105}$$

The new constants $D_{1,2}$ are related to $C_{1,2}$ by

$$D_1=-e^{i(\pi/4)}\frac{\pi}{2}\left[\frac{iC_1}{\Gamma(\frac{1}{2}-\chi)}-\frac{3C_2}{\Gamma(\frac{1}{2}+\chi)}\right]\ , \tag{106}$$

$$D_2=-e^{i(\pi/4)}\left[\frac{C_1}{\Gamma(\frac{1}{2}-\chi)}+\frac{iC_2}{\Gamma(\frac{1}{2}+\chi)}\right]\ , \tag{107}$$

where Γ is the gamma function. Alternatively, we can neglect, for $\eta\to0$, the third term in the square brackets of Eq. (101) in comparison to the second. The solution of the resulting equation is simply

$$\mu(\eta)=\eta^{1/2}[c_1H_0^{(1)}(k\eta)+c_2H_0^{(2)}(k\eta)]\ . \tag{108}$$

By using the small argument approximation for the Hankel functions [55] we recover an expression of the type (105).

For a fixed η it is not straightforward to analyze the limit $k\to0$, since in this case $|\chi|\to\infty$ due to (104), whereas $z=z(\chi)\to0$ [54]. It is easier to study the extreme regime $k^2\ll\tilde{V}(\eta)$ and simply neglect the term k^2 in (101). The resulting equation has the solution

$$\mu(\eta)=\eta^{1/2}[c_1H_0^{(1)}(-2\eta^{1/2})+c_2H_0^{(2)}(-2\eta^{1/2})]\ . \tag{109}$$

For completeness we also present the asymptotic behavior valid for large η and/or k. For fixed k the limit of (102) for $\eta\to\infty$ will lead to

$$\mu[z(\eta)]\approx c_1z^{\chi}e^{-z/2}f_N^{(+)}(z)+c_2(-z)^{\chi}e^{z/2}f_N^{(-)}(z)\ , \tag{110}$$

where

$$f_N^{(\pm)}(z)=\sum_{j=0}^{N}\frac{[(\frac{1}{2}\mp\chi)_j]^2}{j!(\mp z)^j}+O[(\mp z)^{-N-1}]\ , \tag{111}$$

$$(x)_0=1\quad (x)_j=x(x+1)\cdots(x+j-1)\ . \tag{112}$$

For large k, $|\chi|\to0$ and the Whittaker functions $W_{0,0}(\pm z)$ can be written in terms of Hankel functions with argument $(iz)/2$ whose asymptotic form will lead to the expected result

$$\mu(\eta)\approx c_1e^{-ik\eta}+c_2e^{ik\eta}\ . \tag{113}$$

It is interesting to note that, in analogy to what happened with Eq. (43), we could have chosen other pairs of linearly independent functions to express the solution of (101) [54]. However, the choice of $W_{\pm\chi,0}(\pm z)$ is specially suited to write its solution in a Hermitian form since $W^*_{\chi,0}(z)=W_{\chi^*,0}(z^*)$ [54] and hence, in the present case, $W_{-\chi,0}(-z)=W^*_{\chi,0}(0)$. Moreover, unlike for other pairs of independent solutions of (101), the Wronskian of $W_{\pm\chi,0}(\pm z)$ never vanishes (it is equal to $e^{-i\pi\chi}$), and so enables us to impose the normalization condition (48). In the quantized theory this condition is necessary in order to guarantee that the commutation relations between the field $h_{ij}(\eta,\mathbf{x})$ and its conjugate momentum lead directly to the corresponding commutation relations between the creation and annihilation operators. The above analysis of the asymptotic behavior for $k\to\infty$ then shows that the basic solution representing an adiabatic vacuum state is

$$\mu(k,\eta)=\frac{\exp(\pi/4k)}{\sqrt{2k}}e^{i\theta}W_{\chi,0}(i2k\eta)\ , \tag{114}$$

where θ is an arbitrary constant phase. This model will be studied in more detail in a subsequent paper.

V. SUMMARY AND CONCLUSIONS

We have derived the gravitational wave equation for a general class of scalar-tensor theories of gravity, assuming a spatially flat Friedmann-Robertson-Walker universe. In the particular case of the BD theory we have solved this equation in several cosmological models, including the power-law expansion given by Nariai's solutions [37] and the more general expansion governed by the solutions presented by Gurevich *et al.* [38]. We have analyzed the limits of very large and very small wave numbers and have found that the coupling with the scalar field ϕ will change the determination of these limits in comparison with GR. As a result, the scale of amplitude decrease of the high-frequency modes and that of the amplification coefficient will be set by $R \equiv a\phi^{1/2}$ rather than by the scale factor $a(t)$ alone. Moreover, there is the possibility, for very small values of the coupling parameter ω_0, of significant amplification of subhorizon waves. The corresponding creation of high-frequency gravitons is seen as a new effect arising in response to the rapid time variation of ϕ. It was also shown that, unlike in GR, there can be amplification of gravitational waves in a radiation-dominated universe. These new features are also present in more general scalar-tensor theories. In this context we have presented an example in which the dominant solution for the wave equation in the small wave-number regime is not "frozen," as usually happens in GR and in BD, but can grow with time [53]. We have also introduced a particular $\omega(\phi)$ for which the gravitational wave equation can be exactly solved and have studied its solutions.

In a subsequent paper we will use the results derived here in order to obtain the spectrum of the relic gravitons for some specific models. This will enable us to constrain some of the parameters appearing in the scalar-tensor theories. In particular, it seems possible that an overproduction of gravitons due to the effects described above could impose a further restriction on the value of the coupling parameter at early times. A small enough $\omega(\phi)$ is required to have satisfactory nucleation processes in some inflationary cosmologies.

When this work was nearing completion we became aware of the paper by Gasperini and Giovannini [56] who have derived the gravitational wave equation in the context of a higher-dimensional Brans-Dicke theory. By considering a parametrization for the scale factor and for the scalar field which is equivalent to the Nariai's solution, they have obtained the low-frequency band of the spectrum for a three-stage model in which an inflationary phase is followed by the usual radiation- and matter-dominated epochs.

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