

Relating black holes in two and three dimensions

Ana Achúcarro*

*Department of Mathematics, Tufts University, Medford, Massachusetts 02155
and Department of Theoretical Physics, University of the Basque Country, Bilbao, Spain*

Miguel E. Ortiz†

*Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics,
Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*

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The three-dimensional black hole solutions of Bañados, Teitelboim, and Zanelli (BTZ) are dimensionally reduced in various different ways. Solutions are obtained to the Jackiw-Teitelboim theory of two-dimensional gravity for spinless BTZ black holes, and to a simple extension with a nonzero dilaton potential for black holes of fixed spin. Similar reductions are given for charged black holes. The resulting two-dimensional solutions are themselves black holes, and are appropriate for investigating exact “*S*-wave” scattering in the BTZ metrics. Using a different dimensional reduction to the string-inspired model of two-dimensional gravity, the BTZ solutions are related to the familiar two-dimensional black hole and the linear dilaton vacuum.

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Although examples of black hole solutions abound in two and four dimensions, it was until recently believed that no such solutions exist in three spacetime dimensions [1]. However, in a recent paper, Bañados, Teitelboim, and Zanelli (BTZ) [2,3] found a vacuum solution to Einstein gravity with a negative cosmological constant which may be interpreted as a black hole. The solution has everywhere constant curvature, but the global topology is different from that of three-dimensional anti-de Sitter space (AdS₃). As a result, the causal structure of the solution is closer to that of the Schwarzschild solution. However, the singularity hidden behind the horizon is of a weaker form than that of Schwarzschild [3,4].

Below we discuss how the BTZ solutions may be dimensionally reduced to solutions of various two-dimensional theories of gravity. Our motivation is provided by the recent evidence that progress may be made in understanding black hole radiation and evaporation in the context of two dimensions [5]. The solutions we derive may all be interpreted as two-dimensional black holes, and some of the corresponding two-dimensional theories of gravity may in principle be used to exactly describe the scattering of rotationally symmetric matter (“*S* waves”) off the three-dimensional black holes. Since our present understanding of three-dimensional quantum gravity coupled to matter indicates that it is nonrenormalizable, these models may provide the only route for understanding the quantum behavior of the BTZ solutions.

First let us review the BTZ solutions [2–4]. They arise in a three-dimensional theory of gravity

$$S = \int d^3x \sqrt{-g} (R + 2\Lambda) \quad (1)$$

with a negative cosmological constant, i.e., $\Lambda > 0$. It is straightforward to check that the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

are solved by the metric

$$ds^2 = -N^2(r)dt^2 + \frac{dr^2}{N^2(r)} + r^2 [N^\theta(r)dt + d\theta]^2, \quad (2)$$

where

$$N^2(r) = \Lambda r^2 - M + \frac{J^2}{4r^2}, \quad N^\theta(r) = -\frac{J}{2r^2}.$$

For $J = 0$, this metric is similar to Schwarzschild. It has a single horizon at $r = \sqrt{M/\Lambda}$, and a singularity at $r = 0$. However there are two important differences. Firstly, (2) is not asymptotically flat—it is a constant curvature metric. Secondly, the singularity at $r = 0$ is much weaker than that of Schwarzschild spacetime. Whereas the singularity of Schwarzschild is manifested by the power-law divergence of curvature scalars at small r , the BTZ solution has at most a δ -function singularity at $r = 0$, since everywhere else the spacetime is of constant curvature [6]. An interesting special case occurs when $M = 0$. In this case the metric takes the form

$$ds^2 = -\Lambda r^2 dt^2 + \frac{dr^2}{\Lambda r^2} + r^2 d\theta^2.$$

This should be regarded as the extremal or vacuum solution of the $J = 0$ family. The spatial geometry of this vacuum solution is an infinite wormhole, whose radius shrinks to zero at $r = 0$ at an infinite spacelike proper distance from the asymptotic region. Geodesics reach the end of the wormhole at $r = 0$ within a finite proper

*Electronic address: aachucar@pearl.tufts.edu

†Electronic address: ortiz@mitlns.mit.edu

time, and it is unclear how they should be continued beyond $r = 0$. However, since this solution is extremal, the expectation is that a horizon develops before even the lightest test particle reaches this point.

If $J \neq 0$, (2) has two horizons, and its causal structure is similar to the Reissner-Nordström spacetime. When $M = \sqrt{\Lambda}|J|$, the two horizons coincide, and this should be regarded as the vacuum solution for fixed J . As before, the spacetimes are of constant curvature and have a singularity at $r = 0$ with distributional curvature and torsion [3,4]. Solutions with $M < \sqrt{\Lambda}|J|$ have no horizon. It has been conjectured in [2] that these should be discarded since they contain a naked singularity, except when $M = -1$, $J = 0$, when the spacetime is AdS_3 .

The thermodynamics of the BTZ black holes suggest that evaporation of the black holes should take place. Since the temperature decreases with decreasing mass, and is zero for the extremal solutions, these seem to be the natural end points of evaporation [2,7].

Let us now discuss the possible dimensional reductions of the BTZ black holes. We begin with the straightforward dimensional reduction from Einstein gravity with a cosmological constant in three dimensions to the Jackiw-Teitelboim (JT) theory [8] in two. Suppose that the gravitational field in three dimensions is independent of a single coordinate, which we shall call θ , and that the metric may be written in the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = h_{ij}(x^i)dx^i dx^j + \Phi^2(x^i)d\theta^2, \quad (3)$$

where $\mu, \nu = 0, 1, 2$ and $i, j = 0, 1$. Then it is simple to see that the action (1) reduces to

$$S = \int d^2x \sqrt{-h} \Phi (R + 2\Lambda) \quad (4)$$

which is precisely the JT action. It follows that any solution to the equations following from (1), of the form (3), yields a solution

$$h_{ij}(x^i), \quad \Phi(x^i),$$

of the equations following from (4). We shall henceforth refer to the field Φ as the dilaton.

The BTZ solutions with $J = 0$ are of the form (3), and they therefore yield a solution to the JT model [9]:

$$ds^2 = -(\Lambda r^2 - M)dt^2 + \frac{dr^2}{\Lambda r^2 - M}, \quad \Phi = r. \quad (5)$$

It is important to note that the dimensional reduction, although trivial in appearance, has radically changed the properties of the metric. In three dimensions, the point $r = 0$ is singular. The two-dimensional metric is perfectly well behaved at $r = 0$. This is a result of the weak nature of the singularity in the three-dimensional solution. The metric (5) may be analytically extended beyond $r = 0$, using the coordinate transformation

$$r = \sqrt{\frac{M}{\Lambda}} \cosh \rho \sin(\sqrt{\Lambda}\tau),$$

$$\tanh(\sqrt{\Lambda M}t) = \frac{\tanh \rho}{\cos(\sqrt{\Lambda}\tau)},$$

and the maximally extended spacetime is the whole of two-dimensional anti-de Sitter spacetime (AdS_2),

$$ds^2 = -\cosh^2 \rho d\tau^2 + \Lambda^{-1} d\rho^2, \quad (6)$$

$$-\infty < \tau, \rho < \infty,$$

which of course has no horizons. The dilaton in these coordinates takes the form

$$\Phi = \sqrt{\frac{M}{\Lambda}} \cosh \rho \sin(\sqrt{\Lambda}\tau),$$

which vanishes when $\sqrt{\Lambda}\tau = n\pi$. The embeddings of (5) into (6) are shown in Fig. 1. It is clear from the figure that t and r are Rindler type coordinates on AdS_2 .

In order to interpret the two-dimensional solution we must look at the behavior of the dilaton. Recall that the dilaton is the $\theta\theta$ component of the three-dimensional metric, and that the three-dimensional solution is singular where it vanishes. It is therefore natural to cut the two-dimensional spacetime off at this point, which we call the “strong coupling” region (although this name should not be taken too literally), if we wish to use the JT theory to model three-dimensional physics. In this case, (5) can represent a black hole, whose Penrose diagram, shown in Fig. 2, is identical to that of its three-dimensional counterpart.

The two-dimensional version of the extremal solution is also not geodesically complete unless it is extended

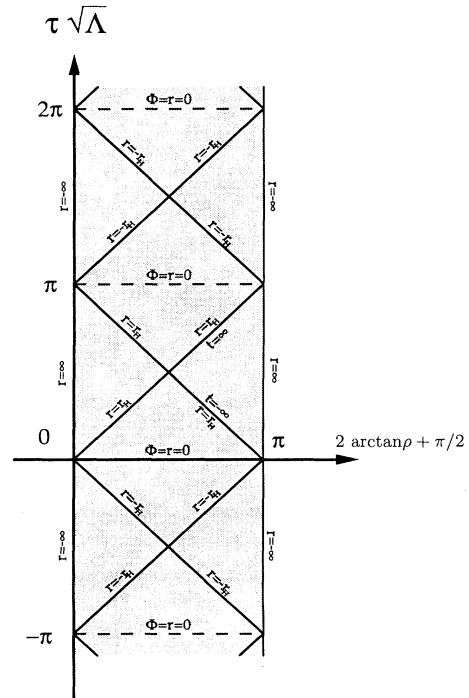


FIG. 1. Regions of the two-dimensional spacetime with $J=0$ covered by the coordinates r, t , embedded in anti-de Sitter space. The diagonal and dashed lines have no special significance. Here $r_H = \sqrt{M/\Lambda}$.

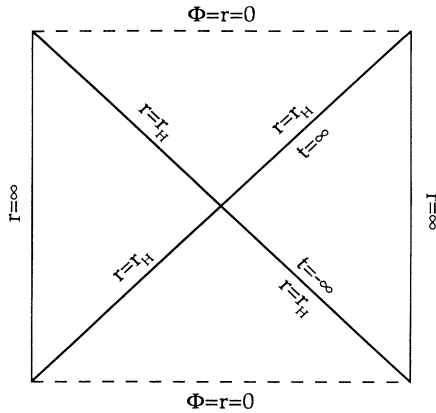


FIG. 2. The Penrose diagram for the region of the $J = 0$ spacetime for which $\Phi \geq 0$. The diagonal lines now represent event horizons, and the dashed line the “strong coupling” region.

(see also Ref. [10] for a discussion of this spacetime). The extended version of the spacetime is also identical to anti-de Sitter spacetime, but for our purposes, we should again restrict our attention to the region where the dilaton is greater than zero. In this extremal case the spacetime

$$ds^2 = -\Lambda r^2 dt^2 + \frac{dr^2}{\Lambda r^2}$$

has a “naked singularity” at $r = 0$ in the sense that the region where $\Phi = 0$ is not hidden behind a horizon. However, this is of no consequence: In both two and three-dimensions (as discussed above) we expect that no matter can probe this region of spacetime without a horizon developing. A Penrose diagram of the restricted extremal solution is shown in Fig. 3, which again is identical to the three-dimensional Penrose diagram. Incidentally, this region is the steady-state universe solution [11], but with timelike and spacelike directions interchanged.

The two-dimensional reduction outlined above yields a two-dimensional theory which can be used to model S -wave scattering off a spinless BTZ black hole. That solutions in two dimensions must be restricted by hand

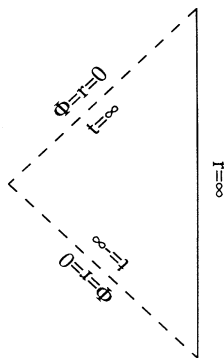


FIG. 3. The Penrose diagram for the region of the extremal two-dimensional spacetime with $M = J = 0$, for which $\Phi \geq 0$. The dashed line represents the “strong coupling” region.

to end where the dilaton vanishes is both a blessing and a curse. On the one hand, there is no singular region to worry about, but on the other we must specify boundary conditions at $r = 0$ in some fashion. In principle, however, by coupling matter to (4) in the natural way for a dimensionally reduced theory,

$$S = \int d^2x \sqrt{-h} \Phi (R + 2\Lambda + h^{ij} \partial_i f \partial_j f),$$

and imposing an appropriate boundary condition at $\Phi = 0$, it may be possible to model the evaporation of a spinless BTZ black hole.

It is also possible to construct an effective two-dimensional theory which arises from the dimensional reduction of the $J \neq 0$ solutions of BTZ, and which can be used to study S -wave scattering for the spinning black holes. We begin by considering the reduction of a metric of the form

$$ds^2 = h_{ij}(x^i) dx^i dx^j + \Phi^2(x^i) [d\theta + A_i(x^i) dx^i]^2. \quad (7)$$

The corresponding two-dimensional theory involves the three fields h_{ij} , A_i , and Φ . If we wish to consider spacetimes of fixed spin, however, we can use the identity

$$\frac{\Phi^3 \epsilon^{ij} \partial_i A_j}{\sqrt{-h}} = \text{const}, \quad (8)$$

which follows from the field equation for A_i . The constant is precisely the spin J of the metric (7), since it is equal to the charge corresponding to asymptotic rotational invariance [12]. Using this identity, the action (1) for spacetimes of spin J may be dimensionally reduced to

$$S = \int d^2x \sqrt{-h} \Phi \left(R + 2\Lambda - \frac{J}{2\Phi^4} \right). \quad (9)$$

This is an appropriate effective action for looking at S -wave scattering off a spinning BTZ black hole (interaction with rotationally symmetric matter will of course keep the black hole in the spin J sector). Again, any solution h_{ij} , Φ to (9) corresponds to a solution to (1) of spin J , of the form (7). In particular, the r, t section of the BTZ black hole of spin J ,

$$ds^2 = - \left(\Lambda r^2 - M + \frac{J^2}{4r^2} \right) dt^2 + \frac{dr^2}{\left(\Lambda r^2 - M + \frac{J^2}{4r^2} \right)}, \quad (10)$$

is a solution to (9), with $\Phi = r$. However, in this case, the equation for the scalar curvature is

$$R + 2\Lambda + \frac{3J}{2\Phi^4} = 0$$

so that R need not be constant. Indeed the two-dimensional spinning black hole (10) and extremal solution have power-law singularities in R at $r = 0$. The Penrose diagram for each of these spacetimes is identical to that of the three-dimensional metric, and is shown in Fig. 4.

In addition to the BTZ solutions described above, it was also shown in [2] that charged black hole solutions similar to (2) exist. These are solutions following from the action

$$\int d^3x \sqrt{-g} (R + 2\Lambda + 4\pi G F_{\mu\nu} F^{\mu\nu}) \quad (11)$$

and take the form (2), but with

$$N^2(r) = \Lambda r^2 - M - 8\pi G Q^2 \ln(r/r_0) + \frac{J^2}{4r^2}$$

and

$$F_{rt} = -\frac{Q}{r}.$$

These solutions have a power-law curvature singularity at $r = 0$, where $R \sim 8\pi G Q^2 / r^2$. They can have two, one, or no horizons, depending on the relative values of Λ , J , GQ^2 , and $M' = M - 8\pi G Q^2 \ln(\sqrt{\Lambda} r_0)$. In the simplest case $J = 0$, these possibilities depend on whether

$$M' - 4\pi G Q^2 (1 - \ln[4\pi G Q^2])$$

is greater than, equal to, or less than zero, respectively.

The action (11) may be dimensionally reduced in a similar way to that described above, in both the $J = 0$

and the $J \neq 0$ sectors, provided that we assume that F_{rt} is independent of θ and that the other two components $F_{\mu\theta}$ vanish. The resulting two-dimensional action is

$$\int d^2x \sqrt{-h} \Phi \left(R + 2\Lambda - \frac{J}{2\Phi^4} + 4\pi G F_{\mu\nu} F^{\mu\nu} \right).$$

The solution in two dimensions corresponding to the BTZ solution is the obvious analogue of (5) and (10) for $Q \neq 0$. The electromagnetic field has $F_{rt} = -Q/r$ as before. This two-dimensional spacetime has a curvature singularity at $r=0$, even for $J=0$, since $R = -2\Lambda - 8\pi G Q^2 / r^2 - 3J^2 / 2r^4$. As in three dimensions, this may be a naked singularity or may be shielded by one or two horizons.

Finally, let us describe a third dimensional reduction of the uncharged BTZ black holes. This involves the reduction introduced in [13] from the three-dimensional action (1) to the string-inspired action for two-dimensional gravity [5,14],

$$\int d^2x \sqrt{-\hat{h}} e^{-2\phi} [\hat{R} + 4(\nabla\phi)^2 + 2\lambda],$$

or, rather, to the action [15]

$$\int d^2x \sqrt{-h} (\Phi R + 2\lambda), \quad (12)$$

which is obtained from the string-inspired action by means of the identification $\hat{h}_{ij} = h_{ij} e^{2\phi}$, $\Phi = e^{-2\phi}$. To get the action (12) from (1), consider the usual Kaluza-Klein dimensional reduction used above to obtain the JT action (4) [or the equivalent action obtained from spinning metrics (7)] followed by a shift of Φ by a constant:

$$\Phi \rightarrow \Phi + \frac{\lambda}{\Lambda} \quad (13)$$

(this procedure is in fact equivalent to implementing the shift (13) on the action (1) and then reducing [13]). It yields a two-parameter family of actions which include the string-inspired action in the limit $\Lambda \rightarrow 0$. Notice that the potentially divergent term

$$\int d^2x \frac{\lambda}{\Lambda} R$$

is proportional to the Euler characteristic of the two-dimensional manifold and, being a topological invariant, it does not affect the equations of motion. Also, note that the effect of the shift as seen from the three-dimensional point of view is to push all points in the ‘‘extra’’ θ dimension an infinite distance away, making the proper length of a θ orbit diverge.

Unlike the JT reductions, this procedure gives the same result irrespective of the value of J . This is best understood in the first-order formalism, where the dreibein e^a and the spin connection ω^a are the dynamical variables (the action (1) is then replaced by the Chern-Simons action for the anti-de Sitter group $SO(2,2)$ [16]). Inserting the condition that all fields be independent of θ into the vacuum equations of motion, we obtain two first integrals:

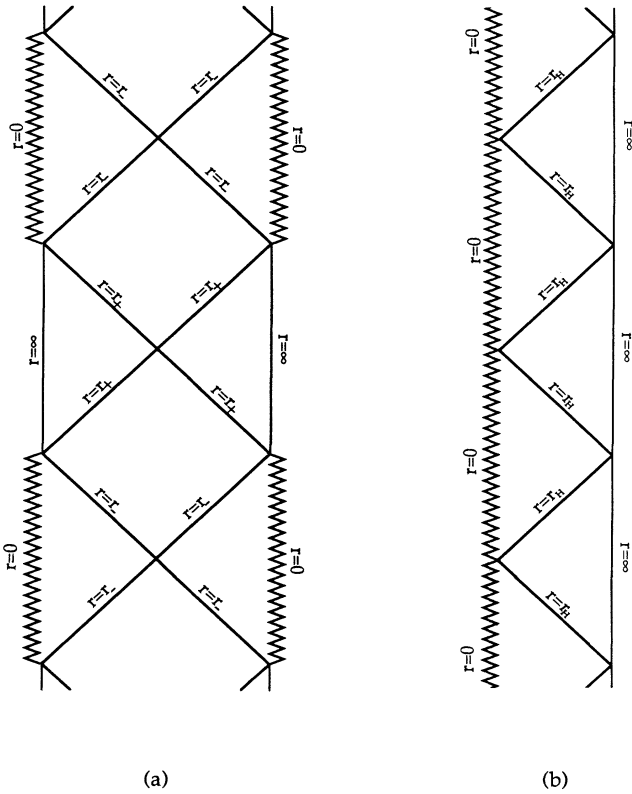


FIG. 4. The Penrose diagram for the two-dimensional spacetimes with $J \neq 0$. (a) has $M \geq \sqrt{\Lambda} |J|$ and $r_{\pm}^2 = (M \pm \sqrt{M^2 - \Lambda J^2}) / 2\Lambda$ and (b) is the extremal solution with $M = \sqrt{\Lambda} |J|$ and $r_H = \sqrt{M/2\Lambda}$.

$$2e^a{}_\theta\omega_{a\theta} \quad \text{and} \quad \Lambda e^a{}_\theta e_{a\theta} + \omega^a{}_\theta\omega_{a\theta}$$

($a = 0, 1, 2$). The first of these is precisely (8), and is equal to the angular momentum of the spacetime. The second quantity is the first integral of the tt component of Einstein's equations, and may be identified with the mass of the spacetime [17]. A metric such as (7) has $e^2{}_\theta = \Phi$ and, in the gauge $e^0{}_\theta = e^1{}_\theta = 0$,

$$2\Phi\omega_{2\theta} = J, \quad \Lambda\Phi^2 + \omega^a{}_\theta\omega_{a\theta} = M. \quad (14)$$

The BTZ black hole (2) has

$$e^0 = Ndt, \quad e^1 = \frac{dr}{N}, \quad e^2 = \Phi(N^\theta dt + d\theta), \quad (15)$$

$$\omega^0 = \left[-\frac{1}{2}\Phi NN^\theta{}_{,r} - \Phi_{,r} NN^\theta\right] dt - \Phi_{,r} N d\theta,$$

$$\omega^1 = \frac{-\Phi N^\theta{}_{,r} dr}{2N},$$

$$\omega^2 = \left[NN_{,r} + \frac{1}{2}\Phi^2 N^\theta N^\theta{}_{,r}\right] dt + \frac{1}{2}\Phi^2 N^\theta{}_{,r} d\theta \quad (16)$$

and in that case, given that $\Phi = r$, the form of the functions N and N^θ follow from Eq. (14):

$$N^\theta{}_{,r} = \frac{J}{\Phi^3} = \frac{J}{r^3},$$

$$-N^2 = \frac{M - \Lambda\Phi^2 - J^2/4\Phi^2}{(\Phi_{,r})^2} = M - \Lambda r^2 - J^2/4r^2,$$

up to an irrelevant integration constant.

The effect of the shift (13) on N and N^θ may be computed immediately from (14) in a similar way, by replacing Φ by $\Phi + \lambda/\Lambda$, taking the limit $\Lambda \rightarrow 0$ and then setting $\Phi = r$ [18]. It follows that

$$N^\theta{}_{,r} = \lim_{\Lambda \rightarrow 0} \left[\frac{J}{\left(r + \frac{\lambda}{\Lambda}\right)^3} \right] = 0,$$

$$-N^2 = \lim_{\Lambda \rightarrow 0} \left[M - \Lambda \left(r + \frac{\lambda}{\Lambda}\right)^2 - \frac{J^2}{4\left(r + \frac{\lambda}{\Lambda}\right)^4} \right]$$

$$= \left(M - \frac{\lambda^2}{\Lambda}\right) - 2\lambda r = m - 2\lambda r.$$

The dependence on J disappears when the limit is taken, and the solutions depend only on the parameter m .

Following Ref. [13], the two-dimensional metric, determined in this case only by N , is $ds^2 = -N^2 dt^2 + dr^2/N^2$. This is flat, since $dr/N = dN/\lambda$. The familiar black hole solution, with a mass $m = M - \lambda^2/\Lambda$, appears when we consider the "string" metric $\Phi^{-1}h_{ij}$,

$$ds^2 = \frac{2\lambda}{m + N^2} \left[-N^2 dt^2 + \frac{dN^2}{\lambda^2} \right], \quad (17)$$

and can be brought to a more familiar form by the coordinate change $\lambda T = N \sinh(\lambda t)$, $\lambda X = N \cosh(\lambda t)$. Note that the shift relates two-dimensional black holes with finite mass m to three-dimensional ones with infinite mass M , for which the horizon has been pushed to infinity and all that remains is the black hole interior, and vice versa. (17) has been extensively studied in [5,14,15] and we shall not discuss it further, except to say that the linear dilaton vacuum solution occurs when $m = 0$ or $M = \lambda^2/\Lambda$.

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- [1] Solutions of pure Einstein gravity have been classified by S. Deser, R. Jackiw, and G. 't Hooft, *Ann. Phys. (N.Y.)* **152**, 220 (1984), and no black hole solutions were found. It is straightforward, if tedious, to see that no uncharged analogues exist in three dimensions of the dilatonic black hole solutions of G. Gibbons and K. Maeda, *Nucl. Phys.* **B298**, 741 (1988), and D. Garfinkle, G. Horowitz, and A. Strominger, *Phys. Rev. D* **43**, 3140 (1991); **45**, 3888(E) (1992). Charged solutions in three dimensions with an horizon are known, but they do not conform to the usual notion of a black hole; see the discussion by B. Reznik, *Phys. Rev. D* **45**, 2151 (1992).
- [2] M. Bañados, C. Teitelboim, and J. Zanelli, *Phys. Rev. Lett.* **69**, 1849 (1992).
- [3] M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, *Phys. Rev. D* **48**, 1506 (1993).
- [4] D. Cangemi, M. Leblanc, and R. Mann, *Phys. Rev. D* **48**, 3606 (1993), the following paper.
- [5] See, for example, J. A. Harvey and A. Strominger, "Quantum aspects of black holes," University of Chicago Report No. EFI-92-41, hep-th/9209055, 1992 (unpublished), and references therein.
- [6] The singularity at $r = 0$ arises because (2) is the covering space of AdS_3 , identified by a discrete subgroup of the isometry group. It is due to a singularity in the causal structure, and should not be regarded as a conical singularity in the usual sense [3,4].
- [7] For a discussion of quantum fields in the spinless BTZ black hole background, see G. Lifschytz and M. Ortiz (unpublished).
- [8] R. Jackiw, in *Quantum Theory of Gravity*, edited by S. Christensen (Hilger, Bristol, 1984); C. Teitelboim, *ibid.*
- [9] These metrics, together with other cosmological solutions of the $R = \Lambda$ model, were first studied by R. Mann, A. Sheikh, and L. Tarasov, *Nucl. Phys.* **B341**, 134 (1990). See also D. Christensen and R. Mann, *Class. Quantum Grav.* **9**, 1769 (1992); and R. Mann, *Gen. Relativ. Gravit.* **24**, 433 (1992). They are related to the general solutions given by R. Jackiw, "Gauge theories for gravity on a line," MIT Report No. CTP 2105, hep-th/9206093, 1992 (unpublished). The parameter M corresponds to the parameter α_0 of his equation (20), and the other two pa-

rameters α_a correspond to a shift of origin not explicit in our solution. Although these parameters can be absorbed into the metric by a change in coordinates, they cannot be made to simultaneously disappear from both the metric and the dilaton.

- [10] M. Crescimanno, Phys. Rev. D **47**, 4483 (1993).
- [11] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973).
- [12] The charge is given by J. D. Brown and M. Henneaux, Commun. Math. Phys. **104**, 207 (1986), as $2\pi J[g_{\alpha\beta}, \pi^{\alpha\beta}] = \lim_{r \rightarrow \infty} \oint dS_\alpha 2\xi^\beta \pi^\alpha{}_\beta$, where $\xi = \partial/\partial\theta$, $\pi^{\alpha\beta}$ is the usual momentum conjugate to $g_{\alpha\beta}$ in canonical gravity, and the integral is taken over the boundary of a spacelike hypersurface. For a metric of the form (7), the integrand over a circle at infinity is precisely the constant expression in (8). Note also that (8) says that the integrand of the surface term is constant over the entire spacelike hypersurface of constant x^0 ; by Stokes's theorem it appears at first sight that J should therefore be zero. However, the contribution giving rise to J comes from a distributional source for angular momentum at the singularity of the three-dimensional spacetime, as could be expected.
- [13] A. Achúcarro, Phys. Rev. Lett. **70**, 1037 (1993).
- [14] E. Witten, Phys. Rev. D **44**, 314 (1991); G. Mandal, A. Sengupta, and S. Wadia, Mod. Phys. Lett. A **6**, 1685 (1991).
- [15] H. Verlinde, in *Sixth Marcel Grossmann Meeting on General Relativity*, Kyoto, Japan, 1991, edited by H. Sato (World Scientific, Singapore, 1992).
- [16] A. Achúcarro and P. K. Townsend, Phys. Lett. B **180**, 89 (1986); **229**, 383 (1989); E. Witten, Nucl. Phys. **B311**, 46 (1988); **B323**, 113 (1989).
- [17] The anti-de Sitter group SO(2,2) is a direct product of two SO(2,1) factors, each of which gives rise to a conserved charge $M \pm \sqrt{\Lambda}J$. It is interesting to note that the extremal black holes are solutions for which one of these charges is zero.
- [18] Note that the angular momentum constraint is replaced by the condition $\omega_{2\theta}=0$, irrespective of the value of J . This explains why the action (12) does not include contributions from the nondiagonal terms in the metric (7), since such contributions appear multiplied by $\omega_{2\theta}$, and therefore vanish in the $\Lambda = 0$ limit.