Nonsingular scalar-tensor cosmologies

John D. Barrow
Astronomy Centre, University of Sussex, Brighton BN1 9QH, Sussex, United Kingdom
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We present a class of scalar-tensor gravity theories which admit simple Friedmann cosmological models in the vacuum and radiation-dominated cases. A subclass of these theories yields a one-parameter family of cosmological models displaying either expansion from an initial singularity or a bounce following contraction from an infinitely extended initial state.

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I. SCALAR-TENSOR GRAVITATION THEORIES

Scalar-tensor gravity theories have been formulated in two different ways. Steinhardt and Accetta [1] express the Lagrangian of the theory in the form

$$L_\Phi = -f(\Phi)R + \tfrac{1}{2}\partial_a\Phi\partial^a\Phi + 16\pi L_m \ , \tag{1}$$

where Φ is a scalar field $f(\Phi)$ with coupling to the four-curvature and L_m is the Lagrangian of the remaining matter fields. If we define a new scalar field $\phi = f(\Phi)$ with a coupling

$$\omega(\phi) = \tfrac{1}{2} f(f'^2)^{-1} \ , \tag{2}$$

then (1) becomes

$$L_\phi = -\phi R + \phi^{-1}\omega(\phi)\partial_a\phi\partial^a\phi + 16\pi L_m \ . \tag{3}$$

The theory proposed by Brans and Dicke [2] arises in the special case that $\omega = \text{const}$ and $f(\Phi) \propto \Phi^2$. The relative merits of adopting (1), as do La and Steinhardt [3], or (3) as do Barrow and Maeda [4], have been discussed by Liddle and Wands [5].

By varying the action associated with (3) with respect to the space-time metric and the scalar field ϕ we obtain the generalized Einstein equations and the wave equation for ϕ as follows:

$$R_{ab} - \tfrac{1}{2}g_{ab}R = -8\pi\phi^{-1}T_{ab} - \omega(\phi)\phi^{-2} \times \{\phi_a\phi_b - \tfrac{1}{2}g_{ab}\phi_i\phi^i\} - \phi^{-1}\{\phi_{a;b} - g_{ab}\Box\phi\} \ , \tag{4}$$

$$\{3+2\omega(\phi)\}\Box\phi = 8\pi T - \omega'(\phi)\phi_i\phi^i \ , \tag{5}$$

$$T^{ab}_{;b} = 0 \ , \tag{6}$$

where T^{ab} is the energy-momentum tensor of the matter content of the theory.

Clearly, if T, the trace of the energy momentum tensor, vanishes, and ϕ is a constant, then (4)–(6) reduce to the standard Einstein equations with a gravitational constant $G = \phi^{-1}$. Hence, any exact solution of Einstein's equations with a trace-free matter source will also be a particular exact solution of the scalar-tensor theory with ϕ, and hence $\omega(\phi)$, constant. However, these particular solutions will not necessarily constitute the general solution for the prescribed matter content. We shall study a class of exact cosmological solutions in which ϕ is not constant, using the methods introduced in Ref. [6].

II. FRIEDMANN UNIVERSES

We shall confine our attention to zero-curvature Friedmann models with metric ($c \equiv 1$),

$$ds^2 = dt^2 - a^2(t)\{dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\} \ , \tag{7}$$

where $a(t)$ is the expansion scale factor. We shall assume that the material content of the universe is blackbody radiation with the equation of the state relating the pressure p to the density ρ as

$$3p = \rho \ . \tag{8}$$

Equations (6) and (7) then reduce to

$$\dot\rho + 3\dot a a^{-1}(\rho + p) = 0 \ . \tag{9}$$

Hence, with (8) we have

$$8\pi\rho = 3\Gamma a^{-4} \ , \tag{10}$$

where $\Gamma \geq 0$ is a constant. The case $\Gamma = 0$ will define the vacuum model in which $p = \rho = 0$.

The metric (7) reduces (4)–(6) to the two equations

$$\frac{\dot a^2}{a^2} = \frac{\Gamma}{\phi a^4} - \frac{\dot\phi}{\phi}\frac{\dot a}{a} + \frac{\omega(\phi)}{6}\frac{\dot\phi^2}{\phi^2} \ , \tag{11}$$

$$\ddot\phi + \frac{3\dot a}{a}\dot\phi = -\frac{\dot\phi^2\omega'(\phi)}{3+2\omega(\phi)} \ , \tag{12}$$

where an overdot denotes d/dt.

If we introduce a conformal time η through

$$a\,d\eta = dt \tag{13}$$

then, denoting $d/d\eta$ by a prime, (12) becomes

$$\phi'' + \frac{2a'}{a}\phi' = -\frac{\phi'^2\omega'(\phi)}{3+2\omega(\phi)} \ . \tag{14}$$

This integrates to give

$$\phi' a^2 = 3^{1/2}A(2\omega+3)^{-1/2} \ ; \quad A = \text{const} \ . \tag{15}$$

If we introduce the variable employed by Lorenz-Petzold [7] in the more specialized context of the Brans-Dicke theory (where $\omega=\text{const}$), then the conformal relationship between general relativity and scalar-tensor theories [4] is fully exploited:

$$y=\phi a^2 \ . \tag{16}$$

This then transforms the generalized Friedmann equation (11) into the form

$$y'^2=4\Gamma y+\phi'^2 a^4(3+2\omega) \ , \tag{17}$$

which, upon using the integral (15), becomes

$$y'^2=4\Gamma y + A^2 \ . \tag{18}$$

In the vacuum and radiation cases we have [6] (for $y'\neq 0$),

$$y(\eta)=A(\eta+\eta_0) \ , \quad \text{in vacuum} \ , \tag{19}$$

$$y(\eta)=\Gamma(\eta+\eta_0)^2-A^2/4\Gamma \ , \quad p=\rho/3 \ . \tag{20}$$

To complete the solution of the problem after the specification of $\omega(\phi)$, we integrate (18) to obtain $y(\eta)$, divide (15) by y to obtain ϕ'/ϕ, integrate to obtain $\phi(\eta)$, hence, $a(\eta)$ from (16) and $a(t)$ from (13). This procedure requires the solution of the following equation to determine $\phi(\eta)$:

$$\int \frac{(2\omega+3)^{1/2}}{\phi} d\phi=\sqrt{3}\ln(\eta+\eta_0) \ , \quad \text{in vacuum} \ , \tag{21}$$

$$\int \frac{(2\omega+3)^{1/2}}{\phi} d\phi=\sqrt{3}\ln\left|\frac{2\Gamma\eta+2\Gamma\eta_0-A}{2\Gamma\eta+2\Gamma\eta_0+A}\right| \ , \quad \text{when } p=\rho/3 \ . \tag{22}$$

We shall investigate solutions of scalar-tensor theories specified by choosing $\omega(\phi)$ to be of the form

$$2\omega(\phi)+3=2\beta(1-\phi/\phi_c)^{-\alpha} \tag{23}$$

where $\alpha, \beta>0$, and ϕ_c are constants. This representation has been introduced by Garcia-Bellido and Quiros [10]. The case $\alpha=0$ corresponds to Brans-Dicke theory. Barker's theory [9] is obtained when $\alpha=1$ and $\beta=-\frac{1}{2}$. For the class of theories defined by (23) the integral on the left of (21) and (22) reduces to

$$I_\alpha \equiv (2\beta)^{1/2}\int x^{-\alpha/2}(x-1)^{-1}dx \ , \tag{24}$$

where

$$x\equiv 1-\phi/\phi_c \ . \tag{25}$$

The family of integrals in (24) can be evaluated in a convenient closed form for a variety of values of α. We shall present solutions for the cases in which $\frac{1}{2}\alpha$ is integer valued or half-integer valued. If we put $\frac{1}{2}\alpha=\in\mathbb{Z}^+$, then we have

$$I_\alpha=(2\beta)^{1/2}\left[\sum_{k=1}^{m-1}(m-k)^{-1}x^{k-m}+\ln\left|\frac{x-1}{x}\right|\right] \ . \tag{26}$$

If $\frac{1}{2}\alpha=-m\in\mathbb{Z}^-$, then we have

$$I_\alpha=(2\beta)^{1/2}\left[\sum_{k=0}^{m-1}(m-k)^{-1}x^{m-k}+\ln|x-1|\right] \ , \tag{27}$$

and all the I_α with half-integral values of $\frac{1}{2}\alpha$ are easily derived using the recursion relations

$$I_{m+1/2}=2(2m-1)^{-1}x^{(1/2)-m}+I_{m-(1/2)} \ . \tag{28}$$

These relations allow us to express $x(\eta)$ explicitly in all the cases where $\frac{1}{2}\alpha$ is an integer or a half integer. Hence, knowing $y(\eta)$ from (19) and (20), it is possible to deduce a relation between a and η. However, in general, this relation is not invertible because of the mixture of logarithms and powers in the expressions for the I_α. There are three simple examples where this is not the case. One is the Brans-Dicke theory, which arises when $\alpha=0$, solutions of which were given in Ref. [8]. Another is when $\alpha=1$, as in the theory of Barker [9], solutions of which were given in Ref. [6]. The third is the case where $\alpha=2$. We shall study this previously unexplored case in detail because it is representative of the behavior of other cases with $\alpha\neq 2$ at early times as $x\to 1$ and $\eta+\eta_0\to 0$.

Consider first the $\alpha=2$ vacuum case. We have

$$(2\beta)^{1/2}\ln\{x^{-1}(1-x)\}=\sqrt{3}\ln\{\eta+\eta_0\} \ . \tag{29}$$

Hence,

$$\phi=\frac{\phi_c(\eta+\eta_0)^\lambda}{1+(\eta+\eta_0)^\lambda} \ , \tag{30}$$

where

$$\lambda=(3/2\beta)^{1/2} \ , \tag{31}$$

so

$$a^2=y\phi^{-1}=\frac{A(\eta+\eta_0)\{1+(\eta+\eta_0)^\lambda\}}{\phi_c(\eta+\eta_0)^\lambda} \ . \tag{32}$$

If we choose $\eta_0=0$, then $a(0)=0$ so long as $0<\lambda<1$ and the full solution has the form

$$a^2(\eta)=A\phi_c^{-1}\eta^{1-\lambda}\{1+\eta^\lambda\} \ , \quad 0<\lambda<1 \ , \tag{33}$$

$$\phi(\eta)=\phi_c\eta^\lambda\{1+\eta^\lambda\}^{-1} \ . \tag{34}$$

Hence, at large η, the asymptotic form is

$$a\propto \eta^{1/2}\propto t^{1/3} \ , \quad \phi\to\text{const} \ . \tag{35}$$

As $\eta\to 0$ we have

$$a\propto\eta^{(1-\lambda)/2}\propto t^{(1-\lambda)/(3-\lambda)} \ , \quad \phi\propto\eta^\lambda\propto t^{2\lambda/(3-\lambda)} \ . \tag{36}$$

The particular case $\lambda=1$ requires separate treatment. We can pick $\eta_0=-1$ so that $a=0$ at $\eta=0$. Thus, we have at all times

$$a\propto\eta\propto t^{1/3} \ , \tag{37}$$

and

$$\phi=\phi_c\eta^{-1}(\eta-1)=\phi_c\{1-(\tfrac{3}{2})^{2/3}(\phi_c/A)^{1/3}t^{2/3}\} \ . \tag{38}$$

Thus we see that $\phi(t)$ changes sign at $t=\frac{2}{3}(A/\phi_c)^{1/2}$ on approach to the singularity at $t=0$.

When $3>\lambda>1$, the large η behavior is the same but $a(\eta)$ has a minimum when

$$\eta=\eta_*=(\lambda-1)^{1/\lambda} \tag{39}$$

and

$$a^2 \equiv a_*{}^2 = A\lambda\phi_c^{-1}(\lambda-1)^{(1-\lambda)/\lambda} . \tag{40}$$

When $\eta<\eta_*$, we have

$$a \propto \eta^{(1-\lambda)/2} \propto t^{(1-\lambda)/(3-\lambda)} . \tag{41}$$

This is a decreasing function of time so long as $3<\lambda<1$. The form of the solution $a(\eta)$ is shown in Fig. 1.

When $\lambda=3$ there is no singularity. There is an expansion minimum at $\eta=2^{1/3}$; the small η time behavior is $a(\eta)\propto\eta^{-1}\propto\exp(-ct)$, and $\phi\propto\exp(3ct)$, with c a positive constant, as $t\rightarrow-\infty$.

For $\lambda>3$ we see that $a(\eta)\propto\eta^{-(\lambda-1)/2}$ and $t\propto-2(\lambda-3)^{-1}\eta^{(3-\lambda)/2}$ so $t\rightarrow-\infty$ as $\eta\rightarrow0$ and there is no singularity. As $t\rightarrow-\infty$, we have $\phi\propto\eta^{\lambda}\rightarrow0$.

This concludes our analysis of the vacuum solutions for all values of $\lambda>0$. We have found that there is an initial singularity for models with $0<\lambda\leq1$ but a "bounce" at an expansion minimum in all cases with $\lambda>1$, although none of the bouncing solutions are oscillatory in either η or in t time.

Now we consider the behavior of the radiation-dominated models when $\alpha=2$. We find

$$\phi=\phi_c f^{\lambda}(1+f^{\lambda})^{-1} , \tag{42}$$

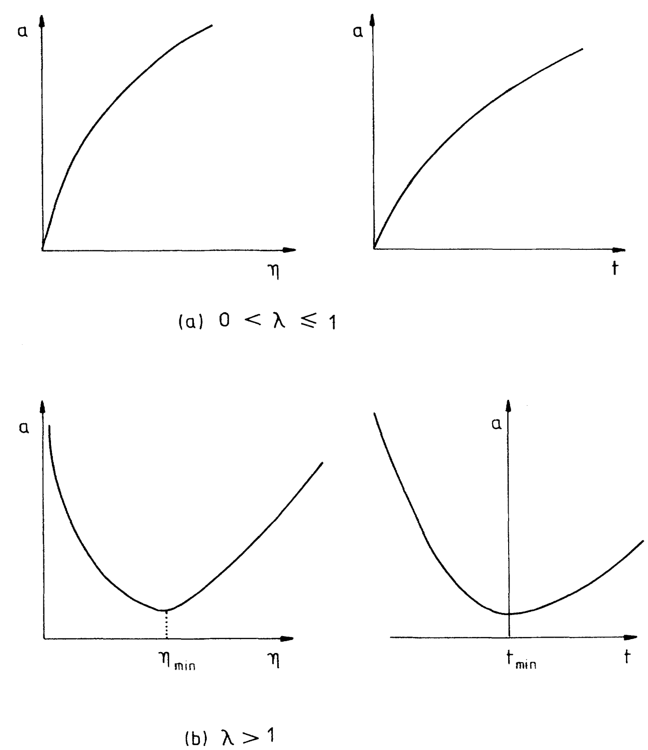

FIG. 1. The variation of $a(\eta)$ and $a(t)$ for the flat vacuum models in the singular and nonsingular cases where $0<\lambda\leq1$ and $\lambda>1$, respectively. (a) $0<\lambda\leq1$: the expansion begins at a singularity located at $\eta=t=0$. (b) $\lambda>1$: there is an expansion minimum at $\eta=\eta_{\min}$ corresponding to $t=t_{\min}$. As $t\rightarrow-\infty$ the conformal time $\eta\rightarrow0$ and $a\rightarrow\infty$.

where $f(\eta)$ is given by

$$f=\left[\frac{\eta+\eta_0-A/2\Gamma}{\eta+\eta_0+A/2\Gamma}\right] . \tag{43}$$

Hence,

$$a^2=\Gamma\phi_c^{-1}\{(\eta+\eta_0)^2-A^2/4\Gamma^2\}\{1+f^{\lambda}\}f^{-\lambda} . \tag{44}$$

As in the vacuum case, $a=0$ at finite η if $0<\lambda<1$. We pick $\eta_0=\frac{1}{2}A\Gamma^{-1}$ so that $a=0$ at $\eta=0$ and

$$a^2=\Gamma\phi_c^{-1}\{\eta^2+A\Gamma^{-1}\eta\}\{1+f^{-\lambda}\} , \tag{45}$$

with

$$f=\left[\frac{\eta}{\eta+A/\Gamma}\right] . \tag{46}$$

As $\eta\rightarrow\infty$ we have

$$a\propto\eta\propto t^{1/2} ; \quad \phi\rightarrow\text{const} . \tag{47}$$

Thus the solution approaches the usual radiation-dominated Friedmann model of general relativity. This is to be expected since $\phi\rightarrow$const ensures that $\omega(\phi)\rightarrow$const.

As $\eta\rightarrow0$, $f\rightarrow\Gamma A^{-1}\eta$, and $(0<\lambda<1)$, we have the solutions

$$a\propto\eta^{(1-\lambda)/2}\propto t^{(1-\lambda)/(3-\lambda)} , \tag{48}$$

$$\phi\propto\eta^{\lambda}\propto t^{2\lambda/(3-\lambda)} . \tag{49}$$

For all values of $\lambda>0$, the behavior of the radiation-dominated solution approaches that of the vacuum solution studies above for $\eta\rightarrow0$.

We see that all the solutions in this theory approach the general relativistic Friedmann radiation solution at late times since ϕ tends to a constant value as $t\rightarrow\infty$. This ensures $\omega(\phi)\rightarrow\infty$, but this is insufficient to guarantee that the scalar-tensor gravity theory specified by $\omega(\phi)$ will be in accord with solar-system tests of gravitation. This requires, as $\omega\rightarrow\infty$, that $\omega'(\phi)\omega^{-3}\rightarrow0$ in this limit also [11,12]. For the class of theories specified by (23) we see that at large ω, when $\phi\rightarrow\phi_c$, this condition requires

$$(1-\phi/\phi_c)^{2\alpha-1}\rightarrow0 . \tag{50}$$

Hence, at late times there will be an approach to the observational requirements of general relativity if $\alpha>\frac{1}{2}$, and this includes the $\alpha=2$ theory we have studied here.

III. SUMMARY

In this paper we have found exact Friedmann cosmological solutions to a class of scalar-tensor theories of gravitation. This class contains some previously studied theories of gravitation as well as providing a simple exact description of a range of behaviors to be found in more complicated scalar-tensor theories. A subclass of the cosmological models provides simple examples, parametrized by a single constant λ of vacuum, and radiation-filled cosmological models which can reach a singularity or bounce in the past according to the value of λ. These theories produce solar-system observations that converge upon those predicted by general relativity. This collection of simple solutions provides a testing ground for a variety of ideas and physical processes in the very early Universe. They allow us to investigate whether the presence of an expansion minimum leaves any trace in the subsequent structure of the Universe.

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