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Stability of FRW cosmology in higher order gravity

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We analyze the behavior of radiation-filled, homogeneous, and isotropic cosmological solutions to a generalized higher order gravity theory which is derived from a gravitational Lagrangian that is an arbitrary function of the scalar spacetime curvature $f(R)$. We give necessary and sufficient conditions for the existence and stability of general relativistic $\sigma=\pm1,0$ FRW solutions within this general theory. We show that under some general conditions any homogeneous and isotropic solution of general relativity is also an exact solution of the $f(R)$ theory, and every radiation solution (not necessarily isotropic) in general relativity is an exact solution in higher order gravity provided there are no nonzero constants and the Einstein term is present in the gravitational Lagrangian of our theory. We then prove that nonflat FRW solutions of general relativity are generically unstable and so do not approach the corresponding ones in higher order gravity for large times. This may be interpreted as an indication that homogeneous and isotropic solutions of higher order gravity cannot be obtained from the corresponding nonflat FRW solutions of general relativity via perturbation theory. However, we find a stable regime for flat FRW solutions of general relativity in higher order gravity. In particular, under fairly general circumstances, flat FRW solutions of general relativity are stable against homogeneous and isotropic perturbations in higher order gravity and always approach their corresponding ones in the generalized theory at the large time limit. The requirements for stability of the flat FRW solutions in higher order gravity coincide with well-known constraints for the absence of tachyons and the existence of complex instanton solutions in the theory, and are exactly those needed to produce bouncing, regular solutions on approach to the singularity.

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I. INTRODUCTION

Hilbert [1] was the first to recognize the fact that the field equations of general relativity can be obtained from an action principle of the form (see Sec. II for notation)

$$S_{\mathrm{GR}}=-\tfrac{1}{2}\int L_{\mathrm{GR}}\sqrt{-g}\;d^4x\ , \tag{1}$$

where

$$L_{\mathrm{GR}}=R \tag{2}$$

and R is the Ricci scalar. Since then, there have been numerous attempts to generalize this action by considering action functionals that contain curvature invariants of higher than linear order in (2) (see, for instance, [1,2]). These works examined the effects of "quadratic Lagrangians" which involve combinations of the four possible second order curvature invariants R^2, $R_{ab}R^{ab}$, $R_{abcd}R^{abcd}$, and $\varepsilon^{iklm}R_{ikst}R^{st}_{lm}$, where ε^{iklm} is the completely antisymmetric tensor of rank 4. Moreover, on approach to spacetime singularities, one expects that curvature invariants of all polynomial orders should come into play and acquire significance. Motivated by such considerations, one is led [3–5] to consider the effects of an obviously generalized gravity theory which is obtained from a Lagrangian that is an analytic function of the scalar curvature alone, for example, a polynomial in R. We collectively call these theories higher order gravity theories. This choice is obviously not totally general since we exclude contributions from any curvature invariants other than R [to include them would greatly complicate matters because the number of curvature invariants of dimension exceeding $(\text{length})^{-2n}$ increases very rapidly with n (cf. Ref. [3])], but has triggered several studies which have contributed a lot to the uninvestigated area of higher order gravity (see Ref. [6] and references therein).

One reason for considering higher order gravity theories is closely linked with cosmology. Since there is no *a priori* physical reason to restrict attention to linear gravitational Lagrangians (which, of course, produce general relativity) (see, however, in this respect, Ref. [5]), one expects that the inclusion of quadratic or higher order terms in the gravitational action might produce better behavior of the theory near singularities where $R\rightarrow\infty$ and $t\rightarrow0$ and also recover the familiar Friedmann-Robertson-Walker (FRW) solutions of general relativity when higher order corrections become negligible in the large t limit. On a more fundamental level, we insist that such drastic alterations of the action for general relativity must appear ultimately in the gravitational action and this is intimately connected with quantum gravity. The principal hope here is that higher order gravity theories subject to constraints would create a first approximation to an as-yet unknown theory of quantum gravitation. Last but not least, higher order gravity theory can be viewed as a *stochastic* gauge theory of gravitation wherein the basis for the gravitational Lagrangian is a completely arbitrary function of the scalar curvature [7].

In 1970, Ruzmaikin and Ruzmaikina [2] analyzed the singularity behavior and stability of the $\sigma=0$, radiation-filled FRW solutions to quadratic Lagrangian theories of gravity. This was partly motivated by the well-known fact that these solutions in the framework of general relativity are geodesically incomplete to the past and future. Their results pointed to the fact that these solutions may avoid the initial singularity, but if they did, they failed to approach the FRW solution of general relativity, $a \sim t^{1/2}$, when the quadratic quantum corrections become negligible as $t \rightarrow \infty$. Also, in the case when their solutions approached the FRW solution of general relativity, they also had an initial singularity and, in addition, $R \rightarrow \infty$ as $t \rightarrow \infty$. These results, if true—see comments at the end of Sec. IV—show that, at least for the case of quadratic Lagrangians they considered, higher order theories of gravity are not completely satisfactory.

In the years following 1970, several consequent directions were taken up by many researchers in the field as follows: It is possible to show [8] that the analytic choice $L_g = R + R^{4/3}$ leads to some bouncing FRW solutions which approach the general relativity FRW solutions at late times. However, one can prove [9] that the regularity of these solutions is unstable with respect to metric perturbations corresponding to rotational waves, except for some unphysical cases. Adding to L_g a term proportional to some hypergeometric function of R leads to some bouncing, regular solutions [10], but the question of stability with respect to metric perturbations is still an open one. Another attempt along these lines was that of Ref. [11] wherein "exotic" actions with terms proportional to $\ln R$ were considered. In this case the action becomes nonanalytic and one loses much of the aesthetic appeal of the theory.

More recently, a more complete analysis was undertaken by Barrow and Ottewill [3] who considered theories with a Lagrangian proportional to an analytic function of the scalar curvature, $f(R)$. Their results reinforced those of Ruzmaikina and Ruzmaikin and pointed to the fact that the pathologies unraveled in [2] are in a sense universal and hold also for this more general case.

In this paper we search for the existence and stability against homogeneous, isotropic perturbations of the $\sigma = 0, \pm 1$ FRW solutions to a higher order gravity theory that is described by a Lagrangian which is an analytic function of the scalar curvature (see below). Our main results are as follows. We first prove that nonflat, radiation-filled FRW solutions and also some homogeneous and anisotropic solutions exist in the generalized framework of higher order gravity theories. The FRW solutions are generically unstable with respect to perturbations in higher order gravity theory, but there is a sector of the theory which includes stable solutions with respect to homogeneous, isotropic perturbations at the large t limit. This last result was not appreciated before, and it is interesting that the stable solutions are also those that become nonsingular on approach to the singularity. This, we believe, sheds new light onto the question of the viability of higher order gravity.

The organization of the paper is as follows. In Sec. II we mainly establish notation. In Sec. III we discuss existence conditions for nonflat, radiation-filled FRW solutions and also of some anisotropic solutions in the framework of higher order gravity theory. In Sec. IV we analyze the stability of FRW solutions with respect to perturbations into higher order gravity. In Sec. V we discuss our results. Additional mathematical details and explanation of our asymptotic analysis are presented in the Appendix.

II. FIELD EQUATIONS

We consider a higher order gravity theory given by a generally covariant Lagrangian density of the form

$$L_{\rm HOG} = [f(R) + \kappa L_m](-g)^{1/2} , \tag{3}$$

where $f(R)$ is assumed to be an analytic (differentiable as many times as necessary) function of the scalar curvature and L_m represents possible matter couplings.

By varying $L_{\rm HOG}$ with respect to the metric tensor g_{ab}, one obtains the field equations

$$\delta L_{\rm HOG} = f'(R)(-g)^{1/2}\delta R + \tfrac{1}{2} f(R)(-g)^{1/2} g^{ab}\delta g_{ab} - \kappa \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\,L_m)}{\delta g_{ab}} (-g)^{1/2}\delta g_{ab} , \tag{4}$$

where δR can be expressed as

$$\delta R = P^a_{;a} - R^{ab}\delta g_{ab} , \tag{5}$$

with

$$P^a = (g^{ac}g^{bd} - g^{ad}g^{bc})\delta g_{cb;d} \tag{6}$$

(we use a semicolon and ∇ indistinguishably). Subtracting a total divergence, the field equations for our theory become [4–6]

$$f'R_{ab} - \tfrac{1}{2} f g_{ab} - \nabla_a \nabla_b f' + g_{ab}\Box f' + \kappa T_{ab} = 0 , \tag{7}$$

where $\Box = g_{ab}\nabla_a\nabla_b, \nabla_a$ is the usual covariant differential operator, $(') \equiv \partial/\partial R$, and we identify the stress-energy tensor T_{ab} with the variational derivative $(2/\sqrt{-g}\,)\delta(\sqrt{-g}\,L_m)/\delta g^{ab}$. Our analysis will be focused on $\sigma = \pm 1, 0$ FRW solutions to the field equation (7). The standard FRW metric in polar coordinates is

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\sigma r^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] , \tag{8}$$

where the values $\sigma = +1, 0$, and -1 correspond to closed, flat, open three-surfaces (and $\alpha, \beta = 1,2,3$), respectively. Below, we follow the sign conventions of [12].

For this metric,

$$R_{00} = 3\frac{\ddot{a}}{a} , \tag{9}$$

$$R_{\alpha\beta} = -\left[\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{\sigma}{a^2}\right] g_{\alpha\beta} , \tag{10}$$

$$R = -6\left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\sigma}{a^2}\right] . \tag{11}$$

Note that $\Box R = -[\ddot{R} + (3\dot{a}/a)\dot{R}]$.

Then the only necessary field equation is the (00) component of the field equation (7). This reads

$$f'R_{00}+\tfrac{1}{2}f+3f''\frac{\dot a}{a}\dot R+T_{00}=0 \tag{12}$$

or, using (9)–(11),

$$f''[a^2\dot a\dddot a+a\dot a^2\ddot a-2\dot a^4-2\dot a^2\sigma] + \tfrac{1}{6}f'a^3\ddot a+\tfrac{1}{36}fa^4+\tfrac{1}{18}a^4T_{00}=0 \ . \tag{13}$$

Also note that the trace of the field equation (7) is given by the equation

$$Rf'-2f+3f''\left[\ddot R+3\frac{\dot a}{a}\dot R\right]+3f'''\dot R^2+T=0 \ , \tag{14}$$

where $T=g^{ab}T_{ab}$ and we shall make the assumption that T_{ab} represents the stress-energy tensor of a perfect fluid with density ρ and pressure p. Thus $p=\gamma\rho=\gamma T_{00}$ and so $\rho=T_{00}=\rho_0 a^{-3(\gamma+1)}$, with ρ_0 constant.

III. EXISTENCE

We start by giving an existence condition for the FRW solutions within the framework of higher order gravity theories.

Proposition 1. Any homogeneous and isotropic solution of general relativity is also an exact solution of the $f(R)$ theory provided (a) the energy-momentum tensor associated with this solution is trace-free (i.e., it is a radiation solution), (b) $f(R=0)=0$ (i.e., no nonzero constants in L_{HOG}), (c) $f'(R=0)\neq 0$ (Einstein term present in L_{HOG}).

Proof. See Ref. [3].

The result is that the FRW solutions to the $f(R)$ radiation models differ from those of general relativity only in the definition of a numerical constant.

We can now generalize proposition (1) to include non-FRW solutions of general relativity, for example, homogeneous and anisotropic solutions. This is not difficult because conditions (b) and (c) above greatly simplify the structure of the (00) component of the field equation (7). Thus we have [6] the following generalized form of proposition (1).

Proposition 2. Any radiation solution of general relativity is also an exact solution of the $f(R)$ theory provided that conditions (b) and (c) in proposition 1 are valid.

Proof. From the Einstein equations we know that when $T=0$ the Ricci scalar also vanishes, $R=0$. Thus the (00) component of the $f(R)$ field equation (7) becomes

$$f'R_{00}+\tfrac{1}{2}f=-T_{00} \ . \tag{15}$$

In the case of radiation, $T_{00}=\rho_0 a^{-4}$; assuming conditions (b) and (c), this equation gives

$$R_{00}=-\rho_0 a^{-4}[f'(0)]^{-1} \ . \tag{16}$$

This equation is identical to the dynamical equation in general relativity for a general radiation-filled universe up to a numerical constant $[f'(0)]^{-1}$.

IV. STABILITY

We now begin our analysis of the solution space of the field equations of our theory described by (3) by writing down the general stability equation for solutions of equation (7). Let a_0 be a particular *exact* solution of (7). We then look for solutions of the form

$$a(t)=a_0(t)[1+\varepsilon(t)], \quad |\varepsilon(t)|\ll 1 \ . \tag{17}$$

By substituting the perturbed solution (17) in the (00) component of the field equations, Eq. (13), and linearizing about the exact solution a_0, we obtain a differential equation for the perturbation $\varepsilon(t)$ which now reads [3]

$$A\dddot\varepsilon+B\ddot\varepsilon+C\dot\varepsilon+D\varepsilon=0 \ , \tag{18}$$

where the coefficients A, B, C, and D are functions of a_0, $R_0\equiv R(a_0)$, $f_0\equiv f(R_0)$, $f_0'\equiv f'(R_0)$, etc. It is clear then that when we take $f(R)$ to be analytic we assume that $f(R)$ can be expanded in a Taylor series in $(R-R_0)$.

We next describe the necessary conditions for the FRW radiation solutions to be *stable* solutions of the $f(R)$ theory. To examine the behavior of perturbations to the Friedmann solutions within the context of higher order gravity theories, one simply substitutes the unperturbed Friedmann radiation solutions

$$a_0(t)=(t-\sigma t^2)^{1/2} \tag{19}$$

into Eq. (17). To simplify the calculation, we set [6]

$$\mu\equiv t-\sigma t^2 \tag{20}$$

and

$$\nu\equiv 1-2\sigma t \ . \tag{21}$$

Then, substituting the unperturbed FRW radiation solution (19) into the perturbation equation (18) and after some manipulation, we find the perturbation equation [6]

$$\dddot\varepsilon+\left[\frac{2\sigma}{\mu}\left[\frac{1}{\nu}+\frac{3\nu}{\mu}\right]+\frac{5}{2}\frac{\nu}{\mu}\right]\ddot\varepsilon - \left[\frac{4\sigma}{\mu}\left[1-\frac{3f_0'''}{f_0''}\frac{\nu^2}{\mu^2}\right]+\frac{f_0'}{3f_0''}+\frac{\nu^2}{\mu^2}\right]\dot\varepsilon - \left[2\sigma\left[\frac{f_0'}{3f_0''}\frac{1}{\nu}+\frac{\nu}{\mu^2}+\frac{\sigma}{\mu}\left\{\frac{2}{\nu}+\frac{9f_0'''}{f_0''}\frac{\nu}{\mu^2}\right\}\right] + \frac{4\rho_0}{9f_0''\mu\nu}\right]\varepsilon=0 \ . \tag{22}$$

Equation (22) describes the behavior of homogeneous, isotropic perturbations to the radiation solution (19) in closed ($\sigma=+1$), flat ($\sigma=0$), and open ($\sigma=-1$) FRW universes in the context of higher order gravity theories derived from the gravitational Lagrangian (3). By solving this equation for the perturbation ε, we can decide whether or not the $\sigma=0,\pm 1$ FRW radiation solutions of general relativity are stable against homogeneous and isotropic perturbations in the context of the $f(R)$ theory.

We divide our analysis into three parts according to whether $\sigma=0$, $+1$, or -1.

Let us first assume that $\sigma=-1$ (open FRW model). Then Eq. (22) takes the form

$$\dddot{\varepsilon}+\left[-\frac{13}{t^3}+\frac{5}{t}\right]\ddot{\varepsilon}-\left[\frac{48f_0'''}{f_0''}\frac{1}{t^4}+\frac{f_0'}{3f_0''}\right]\dot{\varepsilon}+\left[\frac{f_0'}{3f_0''}\frac{1}{t}-\frac{1}{t^3}\left[-\frac{2\rho_0}{9f_0''}+\frac{36f_0'''}{f_0''}-2\right]\right]\varepsilon=0 . \quad (23)$$

We see that terms that involve cubic pieces of $f(R)$ are coupled to terms that die off like t^{-4} or t^{-3} for large t. Neglecting such terms asymptotically as $t\to\infty$, we get the equation

$$\dddot{\varepsilon}+\frac{5}{t}\ddot{\varepsilon}-\lambda^2\dot{\varepsilon}+\lambda^2\frac{1}{t}\varepsilon=0 , \quad (24)$$

where we have set $\lambda^2=f_0'/3f_0''$. We first note that a special solution of (24) is [6]

$$\varepsilon_{sp}=t . \quad (25)$$

We set

$$\varepsilon(t)=tu(t), \quad \dot{u}(t)=y(t) , \quad (26)$$

and the perturbation equation (24) becomes

$$t^2\ddot{y}+8t\dot{y}+(10-\lambda^2t^2)y=0 . \quad (27)$$

This is a Lommel equation (cf. Ref. [13]). Assuming $\lambda^2>0$, i.e., $\lambda>0$, without loss of generality, the general solution can be written in terms of two linearly independent cylinder functions (see the Appendix for the full derivation) as

$$y(t)=t^{-7/2}[c_1I_{3/2}(\lambda t)+c_2K_{3/2}(\lambda t)], \quad \lambda^2>0 , \quad (28)$$

where $I_{3/2}(\lambda t)$ and $K_{3/2}(\lambda t)$ are the modified Bessel functions of the first and second kind, respectively. Substituting (28) in (26), we obtain

$$\varepsilon(t)=c_1t\int t^{-7/2}I_{3/2}(\lambda t)dt+c_2t\int t^{-7/2}K_{3/2}(\lambda t)dt+c_3t, \quad \lambda^2>0 . \quad (29)$$

This is the general analytic solution for the perturbation ε with three (as required) arbitrary constants c_1,c_2,c_3.

Now, if $\lambda^2<0$, we set $k^2=-\lambda^2$, where $k>0$ without loss of generality, and the general solution for $\varepsilon(t)$ is altered. The general solution of the perturbation equation (27) becomes

$$y(t)=t^{-7/2}[c_1J_{3/2}(kt)+c_2J_{-3/2}(kt)] , \quad \lambda^2<0, \quad k>0 , \quad (30)$$

wherein $J_{\pm 3/2}(kt)$ denote the standard Bessel functions. Therefore the general solution for the perturbation $\varepsilon(t)$ now reads

$$\varepsilon(t)=c_1t\int t^{-7/2}J_{3/2}(kt)dt+c_2t\int t^{-7/2}J_{-3/2}(kt)dt+c_3t , \quad \lambda^2<0, \quad k>0 . \quad (31)$$

To find the asymptotic behavior of $\varepsilon(t)$ as $t\to\infty$, we consider solutions (29) and (31) separately (see the Appendix for the full derivation). Equation (29) yields the asymptotic form

$$\varepsilon(t)\sim e^{\lambda t}, \quad \lambda^2>0, \quad t\to\infty , \quad (32)$$

and (31) gives the form

$$\varepsilon(t)\sim t, \quad \lambda^2<0, \quad t\to\infty . \quad (33)$$

It is clearly seen from Eqs. (32) and (33) that the perturbation $\varepsilon(t)$ is unbounded as $t\to\infty$. This means that the corresponding homogeneous and isotropic FRW $\sigma=-1$ solutions are unstable in higher order gravity *irrespective of the sign of* λ^2.

The analysis of the $\sigma=+1$ case is completely analogous to the $\sigma=-1$ case discussed above, and so we simply quote the results (see the Appendix). Equation (22) with $\sigma=+1$ yields, for $t\to\infty$,

$$\dddot{\varepsilon}-\frac{5}{t}\ddot{\varepsilon}+\lambda^2\dot{\varepsilon}-\lambda^2\frac{1}{t}\varepsilon=0 . \quad (34)$$

Under the transformation (26), Eq. (34) yields the Lommel-type equation

$$t^2\ddot{y}-2t\dot{y}+(\lambda^2t^2-10)y=0 . \quad (35)$$

Solving this, we obtain the asymptotic form for the perturbation as follows:

$$\varepsilon(t)\sim t^4, \quad \lambda^2>0, \quad t\to\infty . \quad (36)$$

The case with $\lambda^2<0$ gives

$$\varepsilon(t)\sim e^{kt}, \quad \lambda^2<0, \quad t\to\infty , \quad (37)$$

where we have again set $k^2=-\lambda^2$ $(k>0)$. The net result is that the $\sigma=+1$ solutions are unstable in our theory.

Last, we analyze the case of a flat FRW metric [$\sigma=0$ in Eq. (8)]. Here the perturbation equation (22) takes the form, for $t\to\infty$,

$$\dddot{\varepsilon}+\frac{5}{2t}\ddot{\varepsilon}-\left[\lambda^2+\frac{1}{t^2}\right]\dot{\varepsilon}-\frac{\lambda^2}{t}\varepsilon=0 . \quad (38)$$

In contrast with the previous cases, we note that this equation has a special solution of the form

$$\varepsilon_{sp}=\frac{1}{t} . \quad (39)$$

If we thus set

$$\varepsilon(t)=\frac{1}{t}u(t), \quad \dot{u}(t)=y(t) , \quad (40)$$

we obtain the Lommel equation

$$\ddot{y}(t)-\frac{1}{2t}\dot{y}(t)-\lambda^2y(t)=0 . \quad (41)$$

If $\lambda^2>0$, i.e., $\lambda>0$, without loss of generality, then this equation has the general solution

$$y(t)=t^{3/4}[AI_{3/4}(\lambda t)+BI_{-3/4}(\lambda t)] , \quad (42)$$

where $I_{\pm 3/4}(\lambda t)$ stands for the usual modified Bessel functions. On the other hand, if we take $\lambda^2<0$, we put

$\lambda^2=-k^2$, $k>0$, and the general solution reads

$$y(t)=t^{3/4}[AJ_{3/4}(kt)+BJ_{-3/4}(kt)]\ , \quad \lambda^2<0,\quad k>0\ , \tag{43}$$

and $J_{\pm 3/4}(kt)$ are again the standard Bessel functions.

The general solution for the perturbation is obtained by using the transformation (40). Thus from (40) and (42) we obtain the general exact solution for $\varepsilon(t)$ in the form

$$\varepsilon(t)=\frac{c}{t}+\frac{A}{t}\int t^{3/4}I_{3/4}(\lambda t)dt + \frac{B}{t}\int t^{3/4}I_{-3/4}(\lambda t)dt,\quad \lambda^2>0\ , \tag{44}$$

whereas (43) yields

$$\varepsilon(t)=\frac{c}{t}+\frac{A}{t}\int t^{3/4}J_{3/4}(kt)dt + \frac{B}{t}\int t^{3/4}J_{-3/4}(kt)dt,\quad \lambda^2<0,\ k>0\ . \tag{45}$$

Then, as $t\to\infty$, the asymptotic behavior of the perturbation $\varepsilon(t)$ obtained from the two exact solutions for $\varepsilon(t)$ in (44) and (45) (see the Appendix for the full derivation) is governed by the following forms. From (44) we have

$$\varepsilon(t)\sim\frac{c}{t}+\frac{\alpha_1 A+\beta_1 B}{t^{3/4}}e^{\lambda t},\quad \lambda^2>0\ , \tag{46}$$

whereas Eq. (45) gives

$$\varepsilon(t)\sim\frac{c}{t}+\frac{\alpha_2 A+\beta_2 B}{t^{3/4}},\quad \lambda^2<0\ , \tag{47}$$

with $\alpha_1,\beta_1,\alpha_2,\beta_2$ constants. The conclusion is that, if $\lambda^2>0$, the FRW solution of general relativity is unstable in higher order gravity, whereas, if $\lambda^2<0$, it is stable with respect to homogeneous, isotropic perturbations [14]. As an example to the $\sigma=0$ case, we consider the quadratic Lagrangian theory

$$L_2\equiv f(R)=R-\tfrac{1}{6}\alpha R^2,\quad \alpha=\text{const}\ . \tag{48}$$

In this case $\lambda^2=-\alpha^{-1}$ and so if $\alpha<0$ the Friedmann solution is unstable and will be approached by all solutions at late times. If $\alpha>0$, then $\lambda^2<0$ and the Friedmann solution is stable with respect to homogeneous, isotropic perturbations. We thus note the interesting circumstance, not appreciated before, that higher order gravity theories have the advantage of producing stable *and* nonsingular (i.e., bouncing) FRW solutions in the sector $\sigma=0$ and $\lambda^2<0$. (These bouncing solutions were first discussed in Ref. [2].) The behavior of the nonflat FRW solutions on approach to the singularity at $t=0$ will be considered in a future paper [15].

V. DISCUSSION

We have described the extent to which homogeneous and isotropic cosmological solutions of the higher order gravity theory which is derived from a Lagrangian that is an analytic function of the scalar curvature resemble those of general relativity, i.e., are stable with respect to homogeneous, isotropic perturbations. Of course, it is an open question whether the stable solutions found here remain stable with respect to any other of the infinite modes available to the system.

We found necessary and sufficient conditions for the existence of Friedmann cosmological models and also of some homogeneous but anisotropic solutions in the generalized framework of higher order gravity theory.

Our stability analysis reveals that nonflat radiation-filled FRW solutions which avoid an initial singularity are generically unstable with respect to perturbations in our generalized theory in the large t limit. However, in the case of a flat FRW universe, there is a region that includes well behaved, regular, bouncing solutions as $t\to 0$ which are also stable with respect to homogeneous, isotropic perturbations in higher order gravity theory. In the example mentioned above of a quadratic Lagrangian theory, we know [2] that, when $\alpha<0$, FRW solutions are bouncing at $t=0$. We have shown that in this case the solutions are also stable with respect to homogeneous, isotropic perturbations in this theory and this result was *not* realized before. Note that $\alpha<0$ is also needed as a nontachyonic constraint and also as an existence condition for complex instanton solutions (wormholes) in this theory (cf. Ref. [16]). We believe that these singularity-free well-behaved generalized solutions reflect some of the interesting features of higher order gravity theories especially in connection with the fact that these higher order curvature invariants present in $L_{\rm HOG}$ come up naturally as one considers the low-energy limits of some quantum gravity models [12] and also superstring theory [17].

Perhaps the interesting properties of the classical limit of some theories of gravity with quantum corrections discussed here will give rise to more detailed investigations into the field of classical and quantum structure of higher order gravity theories.

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APPENDIX

In this appendix we present all the necessary details for the rigorous derivation of the asymptotic behavior for the perturbation $\varepsilon(t)$ as $t\to\infty$ for the three FRW universes separately.

1. Case $\sigma=0$

Equation (41) is a special case of the general Lommel equation [13]

$$\ddot{y}(t)+\frac{1-2a}{t}\dot{y}(t)+\left[(bct^{c-1})^2+\frac{a^2-p^2c^2}{t^2}\right]y(t)=0\ , \quad t\in\mathbb{C},\ (a,b,c,p)\in\mathbb{C}\ , \tag{A1}$$

with the general solution

$$y(t)=t^{a}[AZ_p(bt^c)+BY_p(bt^c)] , \quad \text{(A2)}$$

where Z_p, Y_p are two linearly independent cylinder functions. Evidently, here we have $1-2a=-\frac{1}{2}$, $c=1$, $b^2=-\lambda^2$, $a^2-p^2=0$. Hence $a=\frac{3}{4}$, $c=1$, $p=\frac{3}{4}$, $b=k$, with $k^2=-\lambda^2$. Clearly, the general exact solution of (41) reads

$$y(t)=t^{3/4}[AJ_{3/4}(kt)+BJ_{-3/4}(kt)], \quad k\in\mathbb{C} , \quad \text{(A3)}$$

$J_{3/4}, J_{-3/4}$ being the standard Bessel functions.

a. $\lambda^2<0$

From (A3) and on taking $k>0$, obviously without loss of generality, Eq. (43) readily follows. Owing to the asymptotic behavior of $J_{\pm 3/4}(\tau)$ (cf. [18]), $\tau=kt$, and

$$J_{\pm 3/4}(\tau)\sim\left[\frac{2}{\pi\tau}\right]^{1/2}\cos\left[\tau\mp\frac{3\pi}{8}-\frac{\pi}{4}\right] ,$$

$$\tau\to\infty\,(\tau=kt,\ t\to\infty) ; \quad \text{(A4)}$$

both integrals in (45) diverge as $t\to\infty$ like t^m, $m>0$, but m is *not* equal to $\frac{5}{4}$ as a crude calculation would suggest [i.e., insertion of (A4) into (45)], thus leading to the erroneous result $\varepsilon(t)\to\infty$, $t\to\infty$.

Now, using [19],

$$\int_0^{\tau}x^{\mu}J_{\nu}(x)dx=\tau^{\mu}\frac{\Gamma([\nu+\mu+1]/2)}{\Gamma([\nu+\mu+1]/2)}\sum_{n=0}^{\infty}(\nu+2n+1)\frac{\Gamma([\nu-\mu+1]/2+n)}{\Gamma([\nu+\mu+3]/2+n)}J_{\nu+2n+1}(\tau) , \quad \text{(A5)}$$

where $\nu+\mu+1>0$ and $\Gamma(z)$, $z\in\mathbb{R}$, denotes the gamma function, and noting that

$$f_{\pm 3/4}(\tau)=\int\tau^{3/4}J_{\pm 3/4}(\tau)d\tau, \quad f_{\pm 3/4}(0)=0 , \quad \text{(A6)}$$

(45) becomes

$$\varepsilon(t)=\frac{c}{t}+\frac{A}{t}\int_0^{\tau}x^{3/4}J_{3/4}(x)dx$$

$$+\frac{B}{t}\int_0^{\tau}x^{3/4}J_{-3/4}(x)dx, \quad \tau=kt . \quad \text{(A7)}$$

Owing to (A5) and the asymptotic form

$$J_{\nu+2n+1}(\tau)\sim\left[\frac{2}{\pi\tau}\right]^{1/2}\cos\left[\tau-\frac{\pi}{2}(\nu+2n+1)-\frac{\pi}{4}\right] ,$$

$$\tau\to\infty , \quad \text{(A8)}$$

Eq. (A7) finally gives the result quoted in Eq. (47) in the main text, where

$$\alpha_2=\frac{\sqrt{2}k^{1/4}}{\pi}\Gamma(\tfrac{5}{4})\cos\left[kt-\frac{9\pi}{8}\right]$$

$$\times\sum_{n=0}^{\infty}(-1)^n\frac{(2n+\frac{7}{4})\Gamma(n+\frac{1}{2})}{\Gamma(n+\frac{9}{4})} , \quad \text{(A9)}$$

$$\beta_2=\frac{\sqrt{2}k^{1/4}}{\Gamma(-\frac{1}{4})}\cos\left[kt-\frac{3\pi}{8}\right]$$

$$\times\sum_{n=0}^{\infty}(-1)^n\frac{(2n+\frac{1}{4})\Gamma(n-\frac{1}{4})}{\Gamma(n+\frac{3}{2})} . \quad \text{(A10)}$$

It is central to our treatment to prove that both infinite sums in (A9) and (A10) converge. Indeed, the sum in (A9) can be written

$$\sum_{n=0}^{\infty}a_k=\sum_{n=0}^{N}a_n+\sum_{n>N}^{\infty}a_n ,$$

$$a_n=(-1)^n\frac{(2n+\frac{7}{4})\Gamma(n+\frac{1}{2})}{\Gamma(n+\frac{9}{4})} . \quad \text{(A11)}$$

For sufficiently large but *finite* N, we may replace the gamma functions in the infinite sum on the right-hand side of (A11) by their asymptotic form [20]

$$\Gamma(z)\sim\sqrt{2\pi}z^{z-1/2}e^{-z} . \quad \text{(A12)}$$

It is interesting to note that Eq. (A12) is valid even for small z [21]. Hence (A11) now reads

$$\sum_{n=0}^{\infty}a_n\simeq\sum_{n=0}^{N}a_n+\sum_{n>N}^{\infty}(-1)^n\frac{(2n+\frac{7}{4})(n+\frac{1}{2})^ne^{-n-1/2}}{(n+\frac{9}{4})^{n+7/4}e^{-n-9/4}} . \quad \text{(A13)}$$

For large but *finite* N, (A13) reduces to

$$\sum_{n=0}^{\infty}a_n\simeq\sum_{n=0}^{N}a_n+2\sum_{n>N}^{\infty}\frac{(-1)^n}{n^{3/4}} , \quad \text{(A14)}$$

from which it is evident that since

$$\sum_{n=1}^{\infty}(-1)^n/n^{3/4}=(2^{1/4}-1)\zeta(\tfrac{3}{4}) ,$$

$\zeta(\frac{3}{4})$ being the Riemann zeta function, the sum on the right-hand side in (A14) converges. A numerical calculation $\sum_{n=0}^{\infty}a_n\simeq 1.95$.

Similarly, one can show that the infinite sum appearing in β_2 in Eq. (A10) converges, while numerically we find that it is equal to -2.77.

b. $\lambda^2>0$

From (A3) with $k=i\lambda$ and since

$$J_{\pm 3/4}(i\lambda t)=i^{-(\pm 3/4)}I_{\pm 3/4}(\lambda t) ,$$

$I_{\pm 3/4}(\lambda t)$ denoting the standard modified Bessel functions, (42) immediately follows.

Because of the asymptotic behavior of $I_{\pm 3/4}(\tau)$ [22],

$$I_{\pm 3/4}(\tau) \sim \frac{e^{\tau}}{\sqrt{2\pi\tau}}, \quad \tau \to \infty \ (\tau=\lambda t,\ t \to \infty), \tag{A15}$$

it is immediately clear that both integrals in (44) diverge as $t \to \infty$ essentially like $e^{\lambda t}$, thus yielding

$$\varepsilon(t) \sim e^{\lambda t}, \quad t \to \infty . \tag{A16}$$

The precise analytic form of $\varepsilon(t)$ given by (44) for large t can be found by applying a formula similar to (A5) (cf. Ref. [19]) and proceeding precisely along the lines of the $\lambda^2<0$ case discussed above. The result is Eq. (46) with α_1, β_1 now given by

$$\alpha_1 = \frac{\lambda^{1/4}\Gamma(\frac{5}{4})}{\pi\sqrt{2}} \sum_{n=0}^{\infty} (-1)^n \frac{(2n+\frac{7}{4})\Gamma(n+\frac{1}{2})}{\Gamma(n+\frac{9}{4})}, \tag{A17}$$

$$\beta_1 = \frac{\lambda^{1/4}}{\sqrt{2}\Gamma(-\frac{1}{4})} \sum_{n=0}^{\infty} (-1)^n \frac{(2n+\frac{1}{4})\Gamma(n-\frac{1}{4})}{\Gamma(n+\frac{3}{2})} . \tag{A18}$$

We note that since the dominant part of $I_{\pm 3/4}(\tau)$, $\tau=\lambda t$, $\lambda>0$, is e^{τ} and *not* $e^{-\tau}$ as $\tau \to \infty$ ($t \to \infty$), Eq. (4.13) of Ref. [3] is incorrect. This led the authors of Ref. [3] to the wrong conclusion that stable solutions are possible in the $\lambda^2>0$ regime, whereas, if $\lambda^2<0$, FRW solutions are unstable in higher order gravity. For the sake of completeness, we note that the correct full asymptotic form of the modified Bessel function $I_\nu(\tau)$ can be found in [22].

2. Case $\sigma=-1$

a. $\lambda^2<0$

A mere comparison of (27) with (A1) yields $a=-\frac{7}{2}$, $c=1$, $p=\frac{3}{2}$, $b^2=-\lambda^2$. Hence, from Eq. (A2) with $\lambda^2=-k^2$, $k>0$, we obtain Eq. (30).

For the rigorous asymptotic treatment of Eq. (31), we have used repeatedly the formulas (see Ref. [23])

$$\int x^{m+1} Z_l(x)dx = (l^2-m^2)\int x^{m-1}Z_l(x)dx + x^{m+1}Z_{l+1}(x) + (m-l)x^m Z_l(x) , \tag{A19}$$

$Z_l(x)$ being a cylinder function. By means of Eq. (A19), we may reduce the integrals appearing in (31) to $\int t^{-3/2}J_{\pm 3/2}(kt)dt$, thus making them amenable to asymptotic investigation. As in the $\sigma=0$ case, after some manipulation we get

$$\varepsilon(t) \sim t(c_1 + c_2 \sin kt + c_3), \quad t \to \infty , \tag{A20}$$

which is essentially Eq. (33) appearing in the main text.

b. $\lambda^2>0$

Since now $b=i\lambda$, $\lambda>0$, Eq. (28) follows immediately from (A2). From computational reasons relevant to the asymptotic analysis of Eq. (29), we may choose $K_{3/2}(\lambda t)$ instead of $I_{-3/2}(\lambda t)$. Then the asymptotic analysis of (29) runs essentially along the same lines as in the $\lambda^2<0$ case above and yields

$$\varepsilon(t) \sim c_1 t e^{\lambda t}\left[1 - \frac{\sqrt{\pi}}{4}\sum_{n=0}^{\infty}(-1)^n \frac{\Gamma(n+1)(\frac{5}{2}+2n)}{\Gamma(\frac{5}{2}+n)} + c_2 t e^{-\lambda t} + c_3 t\right], \quad t \to \infty , \tag{A21}$$

where the infinite sum can be shown to be convergent as in the case $\sigma=0$ above.

3. Case $\sigma=+1$

This case is completely analogous to the $\sigma=0,-1$ cases considered previously. We simply quote the results: If $\lambda^2<0$,

$$\varepsilon(t) \sim c_1 t^2 e^{kt}\alpha_3 + c_2 t^4 e^{kt}(1+\beta_3) + c_3 t, \quad t \to \infty ; \tag{A22}$$

α_3, β_3 are defined by relations similar to (A9) and (A10). Last, if $\lambda^2>0$, the result is

$$\varepsilon(t) \sim c_1 \alpha_4 t^2 + c_2 t^4(\beta_4 - \sin\lambda t) + c_3 t, \quad t \to \infty , \tag{A23}$$

where again α_4, β_4 are similar to α_3, β_3, respectively.

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