Cosmological models with a variable cosmological term and bulk viscous models

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Recently, flat variable-Λ cosmological models have been discussed by Berman. It is shown that these models are equivalent to perfect fluid models with bulk viscosity which have been presented previously. The model of Berman, which is free of particle creation and which is claimed to solve the entropy problem, is shown to be the usual Einstein–de Sitter model. The nonflat variable-Λ solutions are obtained and their relationship to bulk viscous solutions is elucidated. Exponentially expanding solutions are also derived. The question of the stability of the models is briefly addressed.

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I. INTRODUCTION

One of the most important and outstanding problems in cosmology is the cosmological constant problem (for excellent reviews see [1,2], which attempt to explain the small value of the effective cosmological constant at present ($\Lambda_0 \leq 10^{-52}$ m^{-2})). The theory of the physics of elementary particles predicts that the vacuum energy contributions of the quantum fields must have been 10^{107} times larger in the past [3]. Among the various solutions proposed is the phenomenologically simple one of endowing the effective cosmological "constant" with a variable dynamical degree of freedom which allows it to relax to its present value in an expanding universe [4–11]. The cosmological term Λ is then small at the present epoch simply because the Universe is so old. The problem in this approach is then to try to determine the right dependence of Λ upon R or t. Motivated by dimensional grounds in keeping with quantum cosmology, Chen and Wu [10] considered Λ varying as R^{-2}. Such a dependence alleviates some problems in reconciling observational data with the inflationary Universe scenario.

A number of authors have instead argued in favor of the dependence $\Lambda \sim t^{-2}$. Berman, Som, and Gomide [12] found this relation in Brans-Dicke static models, Berman [13] and Bertolami [14,15] found it in Brans-Dicke theory as modified by Endo-Fukui and Berman and Som [16], and Berman [17] considered more general Brans-Dicke models. Lau [18], Abdel-Rahman [19], Sistero [20], Berman [21], and Kalligas, Wesson, and Everitt [22] augmented a variable Λ term with a simple variation of Newton's gravitational constant G. Kalligas, Wesson, and Everitt have pointed out that if Λ varies as t^{-2}, there is then no dimensional constant associated with Λ.

Recently Berman [23] has discussed the possibility for realizing the hypothesis $\Lambda \sim t^{-2}$ by adding an additional term to the usual energy-momentum tensor, resulting in a variable Λ-type term. The flat Friedmann-Lemaitre-Robertson-Walker (FLRW) models were studied in this approach by imposing the requirement of a constant deceleration parameter [24], and some other postulates. Carvalho, Lima, and Waga [11] have pointed out that, in general, variable Λ-type models involve particle creation. Berman [23] claimed to have found a variable Λ model which does not involve particle creation and which solves the entropy problem.

In this paper we first review the models of Berman [23] in Sec. II. We then show in Sec. III that only the requirement of a constant deceleration parameter is sufficient to ensure $\Lambda \sim t^{-2}$, and we show that no additional assumptions or postulates are necessary. We believe that this approach is better than that in [23]. Next, in Sec. IV we show that these models are equivalent to those with bulk viscosity which have been presented previously. Furthermore, the model of Berman [23], which claims to solve the entropy and particle creation problems, is shown to be the usual Einstein–de Sitter model of general relativity. As is well known this model has $\Lambda=0$ and does not solve the entropy problem. In Sec. V we extend our method of analysis to the nonflat case and discuss exponential solutions as well. Finally, we remark upon the structural stability of the models.

In our approach we assume a particular behavior for the scale factor by imposing the requirement of a constant deceleration parameter. We then solve for the time dependence of the energy density ρ and Λ. A truly physical approach actually requires the opposite point of view, viz., that physics provides the energy density and/or Λ and we than have to solve for the gravitational field consistent with the physics. Thus this work does not claim to be the complete physical solution to the cosmological constant problem.

II. SOLUTIONS OF BERMAN

In this section we review the solutions of Berman [23]. The usual energy-momentum tensor is modified by the addition of a term

$$T_{ab}^{(\text{vac})} = -\Lambda(t) g_{ab} , \tag{1}$$

where $\Lambda(t)$ is the cosmological term and g_{ab} is the metric tensor. Thus the new energy-momentum tensor is

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$$T_{ab}=(\rho+p)u_a u_b+pg_{ab}-\Lambda(t)g_{ab} \ , \quad (2)$$

where ρ and p are, respectively, the energy and pressure of the cosmic fluid, and u_a is the fluid four-velocity. The Robertson-Walker metric is

$$ds^2=-dt^2+R^2(t)\left[\frac{dr^2}{1-kr^2}+r^2(d\theta^2+\sin^2\theta\, d\varphi^2)\right] , \quad (3)$$

where units and being used such that $c=8\pi G=1$, and $k=0, \pm 1$ is the curvature constant.

For the energy-momentum tensor (2) and the Robertson-Walker metric (3), Einstein's field equations yield, for $k=0$, the Friedmann-type equation

$$3H^2=\rho+\Lambda \ , \quad (4)$$

where $H\equiv\dot{R}/R$ is the Hubble parameter, and an energy-conservation-type equation

$$\dot{\rho}+3H(\rho+p)=-\dot{\Lambda} \ , \quad (5)$$

where the overdot denotes a derivative with respect to time. The equation of state is taken to be of the usual form, viz.,

$$\alpha\rho=p \ , \quad (6)$$

where $-1\le\alpha\le 1$, $\alpha=$const. Equation (5) then becomes

$$\dot{\rho}+3H(1+\alpha)\rho=-\dot{\Lambda} \ . \quad (7)$$

Equations (4) and (7) will be the basic equations of the analysis. The deceleration parameter q is defined by

$$q=-\frac{\ddot{R}R}{\dot{R}^2} \ . \quad (8)$$

Berman [23] then assumes that q is constant, viz.,

$$q=\text{const}=m-1 \ , \quad (9)$$

which leads to [24]

$$R=(mDt)^{1/m}, \quad m\neq 0 \ , \quad (10)$$

where $D=$const. The following are then postulated [23]:

$$\rho=A/t^2 \ , \quad (11)$$

$$\Lambda=B/t^2 \ , \quad (12)$$

where $A,B=$const. Solutions are obtained by imposing [23]

$$\frac{1}{m^2}=\frac{1}{3}(A+B) \ , \quad (13)$$

$$2B=(1+3\alpha)A \ . \quad (14)$$

We note at this stage that Eq. (14), which is Eq. (15) of Berman [23], is incorrect as it does not follow from Eq. (7). The correct form will be given in the next section.

As pointed out in the Introduction, variable Λ models invariably involve creation of matter. For the present matter-dominated Universe, $\alpha=0$, and from Eqs. (10) and (11), this leads to

$$\rho R^3\sim t^{-2+3/m} \ . \quad (15)$$

Berman [23] then points out that in order to have no creation of particles, simply take

$$m=\tfrac{3}{2} \quad (\text{or } q=\tfrac{1}{2}) \ , \quad (16)$$

which yields the familiar result

$$R=(\tfrac{3}{2}Dt)^{2/3} \ . \quad (17)$$

This appears to be in contrast to the models of Ozer and Taha [4,5] and Chen and Wu [10], which involve the creation of matter.

The entropy problem which exists in general relativity is also claimed to be solved by the above-mentioned model [23]. The usual entropy law in general relativity reads

$$T\,dS\equiv d(\rho V)+p\,dV=0 \ . \quad (18)$$

In the present case, law (18) gets modified to

$$T\frac{dS}{dt}=-\gamma R^3\frac{d\Lambda}{dt}=2\gamma(mDt)^{3/m}\frac{B}{t^3} \ , \quad (19)$$

where

$$V=\gamma R^3, \quad \gamma=\text{const} \ . \quad (20)$$

Now

$$m\neq 1 \text{ and } B>0\Longrightarrow\frac{dS}{dt}>0 \quad (21)$$

and hence it is claimed that, since $m=\frac{3}{2}$ for the model under discussion, the entropy problem is solved.

III. BERMAN'S MODEL REVISITED

We note that Eqs. (4) and (7) contain three unknowns. Hence one more relation is necessary for a unique solution. Only the additional assumption (9) of a constant deceleration parameter is sufficient for a solution. We believe that this approach is better than that of Berman [23]. Equations (4), (7), and (9) lead to the unique solution

$$\rho=\frac{1}{(1+\alpha)m}\frac{1}{t^2} \ , \quad (22)$$

$$\Lambda=\frac{3+\alpha-2m}{(1+\alpha)m^2}\frac{1}{t^2} \ . \quad (23)$$

Hence assumptions (11) and (12) are unnecessary, but they are a consequence of Eqs. (4), (7), and (9).

[If desired we may also proceed from the reverse direction, i.e., starting from Eqs. (4) and (7) we impose

$$\Lambda=B/t^2 \ . \quad (12)$$

We may derive a Raychaudhuri-type equation

$$2\dot{H}+(1+\alpha)3H^2-(1+\alpha)\frac{B}{t^2}=0 \ . \quad (24)$$

This Riccati-type equation admits only power-law solutions for R of the type (10). This time dependence of the energy density turns out to be of the type (11).]

We already indicated that Eq. (14), which is Eq. (15) of Berman [23], is incorrect. The correct form of this relation, which can easily be derived from Eqs. (22) and (23), is

$$2B=A[-2+(3/m)(1+\alpha)] , \tag{25}$$

where

$$A=\frac{2}{(1+\alpha)m} , \tag{26}$$

$$B=\frac{3+3\alpha-2m}{(1+\alpha)m^2} . \tag{27}$$

Berman's no-creation model has $m=\frac{3}{2}$ and $\alpha=0$. If we substitute these into Eq. (27) we find, surprisingly, that

$$B=0 , \tag{28}$$

which means, from Eq. (23), that

$$\Lambda=0 . \tag{29}$$

Thus the no-creation model of Berman [23] is simply the usual Einstein–de Sitter model of general relativity with zero cosmological term.

It is well known that in standard general relativity the entropy is constant. Indeed from relations (29) and (19) we see that

$$\frac{dS}{dt}=0 , \tag{30}$$

which implies that

$$S=\text{const} . \tag{31}$$

IV. RELATIONSHIP TO BULK VISCOUS SOLUTIONS

The role of viscosity in cosmology has been studied by a number of authors (for an excellent review article we refer to Gron [25]). It was initially hoped that neutrino viscosity could smooth out initial anisotropies and lead to the isotropic Universe that we observe today [26–33]. Suggestions have been made that heating by viscous effects could explain the observed high photon to baryon ratio [34–38]. Bulk viscosity associated with the grand unified theory (GUT) phase transition can lead to an inflationary universe scenario [3,39,40]. It is known that the introduction of bulk viscosity [45] can avoid the big-bang singularity [34,41,42]. Bulk viscosity can provide a phenomenological description of quantum particle creation in a strong gravitational field. The back-reaction effects of string creation can be modeled by means of a bulk viscous model [43]. Further interesting models that arise when bulk viscosity is introduced are eternally oscillating models, in which the entropy increases with each cycle [44], and closed models, which do not always collapse to a big crunch [43]. Other workers have studied formation of galaxies [33].

The equations with bulk viscosity can be obtained from the general relativistic field equations as follows. If we let μ denote the energy density, then the general relativistic Friedmann and conservation equations are, respectively,

$$3H^2=\mu , \tag{32}$$

$$\dot{\mu}+3H(\mu+p)=0 , \tag{33}$$

where we have used the above symbols for reasons that will become clear as we proceed. To obtain the equations with bulk viscosity [45], we replace p by

$$p'=p-3\eta H , \tag{34}$$

where p is now the perfect fluid contribution and η is the coefficient of viscosity, usually taken by most authors to have a power-law form

$$\eta=\eta_0\mu^n, \quad \eta_0\geq 0, \quad \text{const} . \tag{35}$$

Hence (34) becomes

$$p'=\alpha\mu-3\eta_0\mu^n H . \tag{36}$$

Substituting Eq. (36) into Eq. (33) we obtain

$$\dot{\mu}+3H(1+\alpha)\mu=9\eta_0\mu^n H^2 . \tag{37}$$

We now see a resemblance between the equations with bulk viscosity and those with constant deceleration parameter. Equations (4) and (7) are similar to Eqs. (32) and (37). To make this similarity more transparent, and guided by Eqs. (4) and (32), we let

$$\mu=\rho+\Lambda . \tag{38}$$

Equation (7) then becomes

$$\dot{\mu}+3H(1+\alpha)\mu=3(1+\alpha)\Lambda H . \tag{39}$$

Comparing Eqs. (37) and (39) we see that they will be the same if we can find some n such that

$$3\eta_0\mu^n H=(1+\alpha)\Lambda . \tag{40}$$

Now the solutions to Eqs. (32) and (37) have been studied by several authors [42,43,46]. In particular, for $n=\frac{1}{2}$ it is found that

$$R\sim t^{2[3(1+\alpha-\eta_0\sqrt{3})]} . \tag{41}$$

From this solution we can find μ via Eq. (32) and thereby Λ via Eq. (40). It turns out that the solutions of Berman [23] are equivalent to those in [41]. To see this, all we have to do is to let

$$2m=3(1+\alpha-\eta_0\sqrt{3}) . \tag{42}$$

It may readily be identified from Eq. (40) that Λ has the t^{-2} dependence. We have thus shown that the constant q solutions of Berman [23] can equally well be bulk viscous solutions for $n=\frac{1}{2}$ which have been given previously [42,43,46].

There is also another parallel we can draw with bulk viscous solutions. In addition to the linear dependence of the viscosity coefficient upon the expansion $3H$ as in Eq. (34), some workers [47–50] have considered a quadratic dependence upon the expansion, i.e., instead of (34) we now have

$$p'=p-9\zeta H^2 . \tag{43}$$

The $\zeta=$const models analyzed by Romero [50] are equivalent to the models of Berman [23], as may easily be verified.

V. NONFLAT AND EXPONENTIALLY EXPANDING SOLUTIONS

Berman [23] only investigated the FLRW models with $k=0$ and $m\neq0$. We first investigate the existence of power-law solutions of the type (10) for the nonflat cases $k=\pm1$. The relevant equations are now

$$3H^2=\rho+\Lambda-3\frac{k}{R^2}\ , \tag{44}$$

$$\dot{\rho}+3H(1+\alpha)\rho=-\dot{\Lambda}\ , \tag{7}$$

$$R=(mDt)^{1/m},\quad m\neq0\ . \tag{10}$$

Substituting R and its derivative from Eq. (10) into Eqs. (44) and (7), we find the solution

$$\rho=\frac{2}{(1+\alpha)m}\frac{1}{t^2}+\frac{2k}{(1+\alpha)(mD)^{2/m}}\frac{1}{t^{2/m}}\ , \tag{45}$$

$$\Lambda=\frac{3+3\alpha-2m}{(1+\alpha)m^2}\frac{1}{t^2}+\frac{k(1+3\alpha)}{(1+\alpha)(mD)^{2/m}}\frac{1}{t^{2/m}}\ . \tag{46}$$

An obvious requirement that must be imposed is $\rho\geq0$, which can be achieved by taking $k=0,+1$.

These solutions have some interesting consequences. Only for $m=1$ can we have a $\Lambda\sim t^{-2}$ dependence. Second, if we calculate ρR^3, we get

$$\rho R^3\sim t^{(3-2m)/m}+t^{1/m}\ , \tag{47}$$

which implies particle creation for all m. Finally, if we compare with the bulk viscous solutions, we may again identify the two sets of solutions. However, only for some special values of m can we write Λ in the form (40) for some n. For example, the case $m=1$ corresponds to solutions studied by Barrow [43]. An alternative viewpoint would be to write the pressure in the form of Eq. (34), assume some form for R or H, and see whether η obeys certain general conditions, without requiring it to be of the form of (35) (see, for instance, Ref. [51]). If we accept the latter viewpoint, then all our solutions are equally well bulk viscous solutions.

In solving Eqs. (8) and (9), Berman [23] only considered $m\neq0$. However, if $m=0$, then

$$q=-1 \tag{48}$$

and we get

$$H=\text{const}=D\ , \tag{49}$$

which means exponentially expanding solutions of the type

$$R=Ce^{Dt}\ , \tag{50}$$

where $C,D=$const. Substituting Eqs. (49) and (50) into Eqs. (44) and (7) we obtain, for ρ and Λ,

$$\rho=\frac{2k}{(1+\alpha)C^2}e^{-2Dt}\ , \tag{51}$$

$$\Lambda=3E^2+\frac{k(1+3\alpha)}{(1+\alpha)C^2}e^{-2Dt}\ . \tag{52}$$

In order for $\rho\geq0$ we require $k=0,+1$.

We notice that for $k=0$, we recover the familiar empty de Sitter model of general relativity. Also we have exponentially expanding solutions without ρ being constant for $k=+1$. Finally,

$$\rho R^3=\frac{2kC}{1+\alpha}e^{Dt}\ , \tag{53}$$

which implies particle creation for $k=+1$.

Other exponentially expanding solutions may also be found, for instance, if we search for constant energy density solutions for $k=0$. If we set $\rho=\rho_0=$const in Eqs. (4) and (7) we get

$$3H^2=\rho_0+\Lambda\ , \tag{54}$$

$$3H(1+\alpha)\rho_0+\dot{\Lambda}=0\ . \tag{55}$$

The solution is then

$$R=Fe^{-(1+\alpha)\rho_0t^2/4+Gt}\ , \tag{56}$$

where F and G are const. Depending upon the constant G we can have a solution that initially expands, attains a maximum, and then contracts asymptotically to 0.

VI. CONCLUSIONS

We have rederived the models of Berman [23] by allowing for a variable cosmological term and imposing only the requirement of a constant deceleration parameter. Remarkably, the Λ term was found to vary as the inverse square of time which matches its natural units, and means that there is then no dimensional constant associated with Λ. It was shown that the Λ term leads to matter creation. The only model of this kind without matter creation is just the Einstein–de Sitter model. The variable-Λ solutions were shown to be equivalent to bulk viscous solutions with a viscosity coefficient $\eta\sim\mu^{1/2}$. It is interesting to note that Golda, Heller, and Szydlowski [42] have examined the stability of the bulk viscous solutions in which $\eta\sim\mu^n$. They found that only the $n=\frac{1}{2}$ solutions were structurally stable, which means that the models which we have been examining are endowed with a special property.

The models of Berman [23] in which $\Lambda\sim t^{-2}$ and those of Chen and Wu in which $\Lambda\sim R^{-2}$ have nonempty intersections. An example belonging to both is the $m=1$ [see Eq. (10)] model, as $R\sim t$ and $\Lambda\sim t^{-2}$ implies $\Lambda\sim R^{-2}$.

We have extended the solutions of Berman [23] to the nonflat case and found that Λ does not always vary with the inverse square of time in contrast with the flat ones. Furthermore, these nonflat solutions all involve particle creation. Exponentially expanding solutions were found for nonconstant energy density for $k=+1$. In the $k=0$ case, a constant energy density solution was found.

If we allow more general forms of behavior for $\Lambda(t)$ then we notice that a much richer class of cosmological models may be obtained. In this connection we wish to draw attention to the very interesting papers by Jones [52] and Belinskii and Khalatnikov [36]. Jones examined qualitatively the set of Eqs. (7) and (44) without imposing any specific form for the proposed modification, except that $\Lambda=\Lambda(H,\rho)$. The resulting plane autonomous system exhibits all the features one expects from such a system. In addition to the familiar expanding FLRW solutions, periodic solutions, and solutions without an initial singularity are also possible. Belinskii and Khalatnikov [36] analyzed the corresponding bulk viscous system.

A variable G can also be incorporated into a simple framework in which Λ varies as well, while still retaining the usual energy conservation law [18–22]. A power-law variation for G leads naturally to a $\Lambda \sim t^{-2}$ dependence [22]. Interesting solutions can be found, for example, one in which the vacuum energy density is exponentially suppressed and which has zero total energy initially [22].

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[1] Y. J. Ng, Int. J. Mod. Phys. D **1**, 145 (1992).
[2] S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989).
[3] P. Langacker, Phys. Rep. **72**, 185 (1981).
[4] M. Ozer and M. O. Taha, Phys. Lett. B **171**, 363 (1986).
[5] M. Ozer and M. O. Taha, Nucl. Phys. **B287**, 776 (1987).
[6] K. Freese, F. C. Adams, J. A. Frieman, and E. Mottola, Nucl. Phys. **B287**, 797 (1987).
[7] M. Reuter and C. Wetterich, Phys. Lett. B **188**, 38 (1987).
[8] P. J. E. Peebles and B. Ratra, Astrophys. J. **325**, L17 (1988).
[9] B. Ratra and P. J. E. Peebles, Phys. Rev. D **37**, 3406 (1988).
[10] W. Chen and Y. S. Wu, Phys. Rev. D **41**, 695 (1990).
[11] J. C. Carvalho, J. A. S. Lima, and I. Waga, Phys. Rev. D **46**, 2404 (1992).
[12] M. S. Berman, M. M. Som, and F. M. Gomide, Gen. Relativ. Gravit. **21**, 287 (1989).
[13] M. S. Berman, Int. J. Theor. Phys. **29**, 567 (1990).
[14] O. Bertolami, Nuovo Cimento B **93**, 36 (1986).
[15] O. Bertolami, Fortschr. Phys. **34**, 829 (1986).
[16] M. S. Berman and M. M. Som, Int. J. Theor. Phys. **29**, 1411 (1990).
[17] M. S. Berman, Int. J. Theor. Phys. **29**, 1419 (1990).
[18] Y. K. Lau, Aust. J. Phys. **29**, 339 (1985).
[19] A.-M. M. Abdel-Rahman, Gen. Relativ. Gravit. **22**, 655 (1990).
[20] R. F. Sistero, Gen. Relativ. Gravit. **23**, 1265 (1991).
[21] M. S. Berman, Gen. Relativ. Gravit. **23**, 465 (1991).
[22] D. Kalligas, P. Wesson, and C. W. F. Everitt, Gen. Relativ. Gravit. **24**, 351 (1992).
[23] M. S. Berman, Phys. Rev. D **43**, 1075 (1991).
[24] M. S. Berman, Nuovo Cimento B **74**, 182 (1983).
[25] O. Gron, Astrophys. Space Sci. **173**, 191 (1990).
[26] C. W. Misner, Nature (London) **214**, 40 (1967).
[27] C. W. Misner, Phys. Rev. Lett. **19**, 533 (1967).
[28] C. W. Misner, Astrophys. J. **151**, 431 (1968).
[29] J. M. Stewart, Astrophys. Lett. **2**, 133 (1968).
[30] J. M. Stewart, Mon. Not. R. Astron. Soc. **145**, 347 (1969).
[31] A. G. Doroshkevich, Ya. B. Zeldovich, and I. D. Novikov, Zh. Eksp. Teor. Fiz. **53**, 644 (1967) [Sov. Phys. JETP **26**, 408 (1968)].
[32] A. G. Doroshkevich, Ya. B. Zeldovich, and I. D. Novikov, Astrofizika **5**, 539 (1969).
[33] C. B. Collins and J. M. Stewart, Mon. Not. R. Astron. Soc. **153**, 419 (1971).
[34] G. L. Murphy, Phys. Rev. D **8**, 4231 (1973).
[35] V. A. Belinskii and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **69**, 401 (1975) [Sov. Phys. JETP **42**, 205 (1976)].
[36] V. A. Belinskii and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **72**, 3 (1977) [Sov. Phys. JETP **45**, 1 (1977)].
[37] J. D. Nightingale, Astrophys. J. **185**, 105 (1973).
[38] N. Caderni and R. Fabri, Nuovo Cimento **44**, 288 (1978).
[39] I. Waga, R. C. Falcao, and R. Chandra, Phys. Rev. D **33**, 1839 (1986).
[40] T. Pacher, J. A. Stein-Schabes, and M. S. Turner, Phys. Rev. D **36**, 1603 (1987).
[41] V. A. Belinkskii and I. M. Khalatnikov, Pis'ma Zh. Eksp. Teor. Fiz. **21**, 223 (1975) [JETP Lett. **21**, 99 (1975)].
[42] Z. Golda, M. Heller, and M. Szydlowski, Astrophys. Space Sci. **90**, 313 (1983).
[43] J. D. Barrow, Nucl. Phys. **B310**, 743 (1988).
[44] M. Heller and M. Szydlowski, Astrophys. Space Sci. **90**, 327 (1983).
[45] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[46] J. D. Barrow, Phys. Lett. B **180**, 335 (1986).
[47] M. Novello and R. A. Araujo, Phys. Rev. D **22**, 260 (1980).
[48] M. Novello and J. B. S. d'Olival, Acta Phys. Pol. B **11**, 3 (1980).
[49] M. Novello and J. M. Salim, in *Galaxies and Cosmology*, edited by V. M. Canuto and B. G. Elmegreen (Gordon and Breach, New York, 1988).
[50] C. Romero, Rev. Bras. Fis. **18**, 75 (1988).
[51] H. F. M. Goenner and F. Kowalewski, Gen. Relativ. Gravit. **21**, 467 (1989).
[52] J. E. Jones, Proc. R. Soc. London **A340**, 263 (1974).