# Current algebras and the heavy quark limit

Fayyazuddin
*Physics Department, King Abdulaziz University, P.O. Box 9028 Jeddah 21413, Saudi Arabia*
(Received 14 December 1992)

Current algebra is used to obtain the matrix elements for $B,D\to\pi$ in semileptonic $B$ and $D$ decays. The relation $g_{D^*D\pi}=(m_D^* m_D)^{1/2}/2f_\pi$ is also obtained, using SU(4) current algebra, PCAC, and the heavy quark limit.

PACS number(s): 14.40.Jz, 11.40.Ha, 13.20.Fc, 13.20.Jf

Recently [1,2], the chiral and heavy quark symmetries [3–5] have been combined. An effective Lagrangian [1,2] has been constructed to describe interactions of low-momentum pions with heavy quarks.

In this paper current algebra and partial conservation of axial-vector current (PCAC) are used to derive results similar to those obtained in Refs. [1,2]. We also obtain $g_{D^*D_\pi}=(m_D^* m_D)^{1/2}/2f_\pi$ [6] using SU(4) current algebra, PCAC, and the heavy quark limit (HQL).

We define the matrix elements

$$\langle \pi^-,q|i\bar{d}\gamma_\mu c|D,p\rangle=[F_+(p+q)_\mu+F_-(p-q)_\mu] , \quad (1)$$

$$\langle \pi^-,q|i\bar{d}\gamma_\mu\gamma_5 c|D^*,p\rangle =i[G_1\epsilon_\mu+G_2 q\cdot\epsilon(p+q)_\mu+G_3 q\cdot\epsilon(p-q)_\mu] , \quad (2)$$

$$\langle O|i\bar{u}\gamma_\mu\gamma_5 c|D\rangle=i\sqrt{2}f_D p_\mu , \quad (3)$$

$$\langle O|i\bar{u}\gamma_\mu c|D^*\rangle=\sqrt{2}f_{D^*}\epsilon_\mu , \quad (4)$$

$$\langle D,p'|J_\pi^-|D^{*+},p\rangle=\sqrt{2}g_{D^*D\pi}(p-2p')\cdot\epsilon , \quad (5)$$

where $\epsilon_\mu$ is the polarization vector of $D^*$. We have suppressed the kinematic factors. In our normalization, $f_\pi=93$ MeV. The form factors $F$ and $G$ are functions of $t$: $t=-(p-q)^2=m_D^2+2m_D v\cdot q$, where $p=m_D v$ and we have put $m_\pi^2=0$. In the rest frame of the heavy quark, $t=m_D^2-2m_D E_\pi$.

Writing a once-subtracted dispersion relation for $(F_++F_-)$ and an unsubtracted dispersion relation for $(F_+-F_-)$, we obtain [1]

$$F_++F_-=+\frac{f_D}{f_\pi}\left[1-2\frac{f_\pi g_{D^*D\pi}}{m_D}\frac{v\cdot q}{\Delta-v\cdot q}\right] , \quad (6)$$

$$F_+-F_-=-\frac{2f_D g_{D^*D\pi}}{\Delta-v\cdot q} , \quad (7)$$

where $\Delta=m_{D^*}-m_D$. In deriving the above relations, only the contribution of the $D^*$ pole has been retained in the dispersion integral and the current algebra result [7]

$$F_+(m_D^2)+F_-(m_D^2)=+f_D/f_\pi \quad (8)$$

has been used. Since only the $D^*$ pole has been retained in the dispersion integral, it is therefore obvious that Eqs. (6) and (7) hold in the neighborhood of $t=m_D^2$. The range of validity of these results has been discussed in Refs. [2,8,9]. A similar result for the form factor $G_1$ can be derived. In the chiral limit, the current algebra result [10] is

$$G_1(m_{D^*}^2)=-\frac{f_{D^*}}{f_\pi} . \quad (9)$$

Heavy quark spin symmetry [3] gives

$$f_{D^*}=m_D f_D , \quad (10)$$

$$-\langle \pi^-|i\bar{d}\gamma_4 c|D\rangle=\langle \pi^-|i\bar{d}\gamma_3\gamma_5 c|D^*\rangle . \quad (11)$$

From Eq. (11) we obtain

$$-[F_+(p+q)_4+F_-(p-q)_4] =i[G_1\epsilon_3+G_2 q\cdot\epsilon(p+q)_3+G_3 q\cdot\epsilon(p-q)_3] . \quad (12)$$

Evaluating Eq. (12) in the rest frame of the heavy quark, we get

$$-[(F_++F_-)m_D+(F_+-F_-)E_\pi]=G_1+G_2E_\pi^2-G_3E_\pi^2 . \quad (13)$$

From Eqs. (13), (6), and (7), we get

$$G_1+G_2E_\pi^2-G_3E_\pi^2=-\frac{f_D}{f_\pi}m_D . \quad (14)$$

In the chiral limit ($E_\pi\to 0$), we get back the current algebra result when Eq. (10) is used. The form factor $G_3$ is dominated by the $D$ pole, viz.,

$$G_3=\frac{2f_D}{m_D^*}\frac{g_{D^*D\pi}}{\Delta+v\cdot q} . \quad (15)$$

The relation (10) in the form $f_{D^*}=(m_D^* m_D)^{1/2}f_D$ was known before the advent of heavy quark spin symmetry (see, for example, Ref. [6]). It can be derived in a nonrelatvistic bound state picture for $D$ and $D^*$. Relation (10) follows from it by taking $m_D^*=m_D$.

The only unknown in the form factors $F_+$, $F_-$, $G_1$, and $G_3$ is $g_{D^*D\pi}$. This can be fixed as follows. We use the SU(4) algebra generated by axial charges [11,12], viz.,

 

$$[S_i^{(+)},S_j^{(-)}]=2\delta_{ij}I_3 \ , \qquad (16)$$

where

$$S_i^+=\int A_i^{(+)}(x)d^3x \ , \qquad (17)$$

$$A_\mu^{(+)}=i\bar{u}\gamma_\mu\gamma_5 d \ , \quad A_i^{(+)}=-u^\dagger\sigma_i d \ , \quad \sigma_i=-i\gamma_4\gamma_i\gamma_5 \ ,$$

and

$$I_3=-i\int V_{43}(x)d^3x \ , \quad V_{4i}=\tfrac{1}{2}q^\dagger\tau_3 q \ . \qquad (18)$$

Taking the matrix elements of Eq. (16), between $|D^+\rangle$, we get

$$\langle D^+|[S_i^{(+)},S_j^{(-)}]|D^+\rangle=2\delta_{ij}\langle D^+|I_3|D^+\rangle=\delta_{ij} \ . \qquad (19)$$

We define

$$\langle D^+,p'|A_\mu^+|D^{*0},p\rangle = \frac{1}{\sqrt{2p_0 2p'_0}}i[A_1\epsilon_\mu+A_2 p'\cdot\epsilon(p+p')_\mu + A_3 p'\cdot\epsilon(p-p')_\mu] \ . \qquad (20)$$

In order to evaluate the left-hand side of Eq. (19), we introduce the complete set of allowed intermediate states. We retain only $D^{*0}$ state; i.e., we assume that the algebra is saturated by isospin multiplet $(D^{*+},D^{*0})$. In the rest frame of the $D$ meson, we obtain

$$A_1^2=4m_D m_D^* \ . \qquad (21)$$

PCAC gives

$$A_1+A_2(m_{D^*}^2-m_D^2)=-4f_\pi g_{D^*D\pi} \ . \qquad (22)$$

In the HQL, $m_{D^*}^2-m_D^2=\text{const}$. But the second term on the left-hand side of Eq. (22) can be neglected in the HQL because $A_2$ is suppressed with respect to $A_1$ by an inverse power of the heavy mass. Hence we obtain

$$g_{D^*D\pi}=\frac{(m_D m_D^*)^{1/2}}{2f_\pi} \ . \qquad (23)$$

It may be noted that Eq. (23) follows from Eq. (21) plus PCAC and the HQL.

For $K^*$ and $\rho$, the SU(6) relations $g_{K^*\bar{K}\pi}=m_{K^*}/2f_\pi$ and $g_{\rho\pi\pi}=m_\rho/f_\pi$ have been known since 1966 [11,13]. These relations differ from Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSRF) values [14] $g_{K^*\bar{K}\pi}=m_{K^*}/2\sqrt{2}f_\pi$ and $g_{\rho\pi\pi}=m_\rho/\sqrt{2}f_\pi$ by a factor of $\sqrt{2}$. Experimentally, the SU(6) relation for $K^*$ or $\rho$ is not tenable. But for heavy quarks relation (23) is expected to hold for the following reasons. (i) For SU(4) algebra the natural frame is the $\mathbf{p}=0$ frame, a good frame for heavy mesons. For heavy quarks the static limit is a good approximation. The spin of a $D$ or $B$ meson is flipped with the emission of a pion, with negligible recoil. (ii) The PCAC relation (22) reduces to $A_1=-4f_\pi g_{D^*D\pi}$ in the HQL. This relation is exact in this limit and is crucial in deriving relation (23). This is not the case for $\rho$ and $K^*$ mesons. If we take the matrix element of Eq. (16) between $|D^{*+}\rangle$ and assume that the left-hand side is saturated by the $D$ state, we get Eq. (21).

Let us consider the possible contribution of $p$-wave states $D_J$ $(J=0,1,2)$ to the sum rule (19). The $D$ $(J=1^+)$ states do not contribute to the sum rule (19) due to the following argument. From Lorentz and parity invariance, the matrix element $\langle D^+,p'|A_\mu^+|D(J=1^+),p\rangle$ is proportional to $\epsilon_{\mu\nu\alpha\beta}\eta_\nu p'_\alpha p_\beta$, where $\eta_\nu$ is the polarization vector for $D(J=1^+)$. Thus

$$\langle D^+\cdot p'|A_i^+|D(J=1^+)\rangle\sim\epsilon_{ijk}\eta_j p_k$$

and hence it vanishes in the static limit.

Equation (23) was derived in Ref. [6], using PCAC and the nonrelativistic bound state picture for mesons with a heavy quark. The derivation of Eqs. (10) and (23) shows a close relation between the bound state picture and algebraic approach. A relation similar to Eq. (23) was obtained in Ref. [15], using a Adler-Weisberger-type sum rule with only a pole contribution in the sum rule. Note that for this type of algebra the natural frame is the $\mathbf{p}\to\infty$ frame.

Using Eq. (23) and neglecting terms of order $v\cdot q/m_D$, we obtain

$$F_+=-\frac{f_D(m_D m_D^*)^{1/2}}{2f_\pi}\frac{1}{\Delta-v\cdot q}=-F_- \ , \qquad (24a)$$

$$G_1=-(m_D m_D^*)^{1/2}f_D/f_\pi \ , \qquad (24b)$$

$$G_3=\frac{f_D(m_D m_D^*)^{1/2}}{m_D^* f_\pi}\frac{1}{\Delta+v\cdot q} \ . \qquad (24c)$$

Needless to say, the above relations are equally applicable to $B$ mesons. Since for $B$ mesons $\Delta=m_{B^*}-m_B=50$ MeV as compared to 140 MeV for $m_{D^*}-m_D$ and $m_B$ is much larger than $m_D$, the above results are expected to be more accurate for $B$ mesons. However, the decay $B^*\to B\pi$ is not allowed energetically; the relation $g_{B^*B\pi}=(m_B m_B^*)^{1/2}/2f_\pi$ cannot be tested directly.

From Eq. (23), we obtain $\Gamma(D^{*+}\to D^0\pi^+)=184$ keV, $\Gamma(D^{*+}\to D^+\pi^0)=84$ keV, and $\Gamma(D^{*0}\to D^0\pi^0)=121$ keV. Since a direct experimental measurement of these decay widths is difficult, to make contact with experiment, we use the branching ratios $B_{\pi^0}^0, B_\gamma^0, B_{\pi^+}^+, B_{\pi^0}^+, B_\gamma^+$. Recent CLEO data [16] give the branching ratios (%) $B_{\pi^0}^0=63.6\pm2.3\pm3.3$, $B_\gamma^0=36.4\pm2.3\pm3.3$, $B_{\pi^+}^+=68.1\pm1.0\pm1.3$, $B_{\pi^0}^+=30.8\pm0.4\pm0.8$, and $B_\gamma^+=1.1\pm1.4\pm1.6$. Taking $30\%\le B_\gamma^0\le40\%$, we find $52\le\Gamma(D^{*0}\to D^0\gamma)\le81$ keV. Most models [17] for the $M1$ transition for $D^*\to D\gamma$ give $B_\gamma^+<1.4\%$. Thus we find $\Gamma(D^{*+}\to D^+\gamma)<4$ keV. Equation (23) combined with the present experimental branching ratios give $M1$ transition decay widths for $D^{*+,0}$ to be higher than those given by most models [17] for these decays.

I would like to thank Professor Riazuddin for many useful discussions.

[1] M. B. Wise, Phys. Rev. D **45**, 2188 (1992).
[2] G. Burdman and J. F. Donoghue, Phys. Lett. B **280**, 287 (1992).
[3] N. Isgur and M. B. Wise, Phys. Lett. B **232**, 113 (1989); **237**, 527 (1990).
[4] H. Georgi, Phys. Lett. B **240**, 447 (1990).
[5] F. Hussain, J. G. Korner, K. Schilcher, G. Thomson, and Y. L. Wu, Phys. Lett. B **249**, 295 (1990); J. G. Korner and G. Thomson, *ibid.* **264**, 185 (1991).
[6] M. Suzuki, Phys. Rev. D **37**, 239 (1988).
[7] C. G. Callan and S. B. Treiman, Phys. Rev. Lett. **16**, 153 (1966); V. S. Mathur, S. Okubo, and L. K. Pandit, Phys. Rev. Lett. **16**, 371 (1966).
[8] G. Burdman and J. F. Donoghue, Phys. Rev. Lett. **68**, 2887 (1992).
[9] L. Wolfenstein, Phys. Lett. B **291**, 177 (1992).
[10] Riazuddin and Fayyazuddin, Phys. Rev. **147**, 1071 (1966).
[11] Fayyazuddin, Riazuddin, and M. S. K. Razmi, Phys. Rev. **141**, 1509 (1966).
[12] B. W. Lee, Phys. Rev. Lett. **14**, 676 (1965).
[13] See also J. J. Sakurai, Phys. Rev. Lett. **17**, 552 (1966).
[14] K. Kawarabayashi and M. Suzuki, Phys. Rev. Lett. **16**, 255 (1966), and Ref. [7].
[15] C. A. Domingues and N. Paver, Z. Phys. C **41**, 270 (1988).
[16] CLEO Collaboration, F. Butler *et al.*, Phys. Rev. Lett. **69**, 2041 (1992).
[17] A. N. Kamal and Q. P. Xu, Phys. Lett. B **284**, 421 (1992) (references to previous literature can be found in this reference); L. Angelas and G. P. Lepage, Phys. Rev. D **45**, 3021 (1992); Fayyazuddin and O. H. Mobarek, Phys. Rev. D **48**, 1220 (1993).