Naturalness of three generations in free fermionic $Z_2^n \otimes Z_4$ string models

Alon E. Faraggi and D. V. Nanopoulos
Center For Theoretical Physics, Texas A&M University, College Station, Texas 77843-4242
and Astroparticle Physics Group, Houston Advanced Research Center (HARC), The Woodlands, Texas 77381
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We study the construction of free fermionic spin structure models with $Z_2^n \otimes Z_4$ boundary-condition vectors. We argue that requiring chiral space-time fermions in the massless spectrum and the existence of a well-defined hidden gauge group severely constrain the allowed boundary-condition vectors. We show that the minimal way to obtain these requirements is given by a unique set of Z_2^5 boundary-condition vectors. We classify the possible extensions to this basic set. We argue that a result of this fundamental set is that obtaining three generations in this construction is correlated with projecting out all the enhanced gauge symmetries which arise from nonzero vacuum expectation values of background fields. We propose that this correlation and the properties of the fundamental Z_2^5 subset suggest that three generations is natural in this construction.

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I. INTRODUCTION

It is by now well established that the number of light left-handed neutrinos is only three. The experimental results obtained at the SLAC Linear Collider (SLC) [1] and CERN e^+e^- collider LEP [2] clearly show that the width of the standard model Z boson can only accommodate three light left-handed neutrinos. This fundamental observation is one of the clues in the study of physics beyond the standard model. For over a decade it has been known that the idea of gauge unification [3] and the successful mass prediction for m_b/m_τ in this scheme can only be obtained in the presence of three generations of chiral fermions [4]. However, at present, we have no idea how nature has chosen to have only three generations. It could be argued more strongly that we do not even know that indeed nature has chosen to have only three generations as there may exist more generations where the left-handed component of the neutrino is for some reason very heavy, or at least heavier than $m_{\nu L} \sim 45$ GeV.

On the other hand, superstring theory [5] has been suggested as a possible candidate for a consistent formulation of all the known fundamental interactions, but so far lacks experimental support for its existence. Initially it was believed that to be consistent the superstring should be embedded in ten space-time dimensions which are then compactified on a Calabi-Yau manifold [6] or on an orbifold [7]. Further study revealed that one could construct a consistent string theory directly in four space-time dimensions and the extra degrees of freedom which are needed to cancel the conformal anomaly are interpreted as either [8] bosonic or fermionic [9] internal degrees of freedom. As a fundamental theory of nature, superstring theory should explain the origin of the number of generations. A general string model can, in principle, have an almost arbitrary number of generations. In the Calabi-Yau compactification of the heterotic string [10], the number of generations in the low-energy spectrum is dictated by a topological quantity, the Euler number χ of the compactified internal space. The Euler number of an arbitrary Calabi-Yau manifold can be very large and the number is reduced by moding out the compactified manifold by a freely acting symmetry group. Several models with three and four generations have been constructed in this way [11]. The most extensive studies are of models with (2,2) world-sheet supersymmetry and lead to $E_6 \times E_8$ gauge symmetry. The E_6 gauge symmetry corresponds to the observable gauge symmetry which is further broken by using the Hosotani flux-breaking mechanism [12]. Models with (2,0) world-sheet supersymmetry compactification in the Calabi-Yau framework have been proposed by Witten [13] as a way of obtaining $SO(10) \times E_8$ gauge symmetry and thus avoiding some of the phenomenological problems that are encountered with E_6 grand unification [13]. These models have the theoretical problem that (2,0) compactification may be unstable by the presence of world-sheet instantons [14]. Several studies of these models have been carried out in the literature [15]. In the orbifold formulation [7] the extra dimensions are compactified on a flat torus. In this formulation, the number of generations is again given by a topological quantity. Extensive studies have been carried out and semirealistic models have been obtained [16].

The most realistic string models [17] constructed so far have been constructed by using the free fermionic formulation [9]. In this formulation, the extra degrees of freedom which are needed to cancel the conformal anomaly are interpreted as internal fermionic degrees of freedom propagating on the string world sheet. Under parallel transport around a noncontractible loop the fermionic states pick up a phase and the specification of the phases for all the fermionic states around all the noncontractible loops constitutes the spin structure of the model. Requiring the partition function to be invariant under modular transformation leads to a set of constraints on the possible spin structures. A model is constructed by choosing a set of boundary conditions that satisfies those constraints.

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The massless physical spectrum is then obtained by applying the generalized Gliozzi-Scherk-Olive (GSO) projections. Several models have been constructed by applying this formalism. However, in this formulation, the geometrical insight that was gained in the Calabi-Yau and orbifold formulations is lost. Therefore, the current studies of these models lack the systematic approach which was achieved in geometrical formulations of the superstring. Moreover, in the fermionic formulation, construction of models with any simply laced gauge group with rank less than 22 is possible. Thus trying to perform a systematic analysis seems intractable. However, as we will show, the requirement that part of the gauge group is hidden, is a strong constraint on this class of models. We will argue that this requirement may be correlated in this construction with the fact that only three generations are observed in nature.

In this paper we study the construction of models in the free fermionic formulation with special attention to the number of generations in these models. We argue that in the class of models we are investigating there is a correlation between the number of degrees of freedom which are needed to cancel the conformal anomaly, the final gauge group and the number of chiral families. We present a first attempt to extract some general properties of this class of models and to study how the number of generations we observe in nature may be understood in this construction.

II. THE NAHE[1] SET

In the free fermionic formulation in four dimensions we need 18 left and 44 right real internal fermionic degrees of freedom. The gauge bosons of the low-energy interactions come from the right moving part. The existence of chiral fermions in the massless spectrum leads to the projection of the left-moving gauge bosons [18]. This would seem to give a rank 22 gauge group at the string level. However, we certainly do not live in a rank 22 world. The standard model has a rank 4 group and the idea of unification, which is supported by calculation of $\sin^2\theta_W$ and m_b/m_τ, may increase the rank to rank 5 [SO(10)] or rank 6 (E_6). Thus we need to find ways to reduce the rank of the observable gauge group. In the free fermionic formulation there are two ways to reduce the rank. One is to construct the model in such a way that part of the gauge group will be hidden. The matter representations of the low-energy gauge group will not transform under the hidden gauge group. The second way is to couple left with right fermionic states to form Ising model parity operators [19]. In this way the rank can be reduced at the most by six because of the eighteen left-moving fermionic states, six are combined to give the supersymmetry generators. We are left with 12 left movers which can be combined with 12 right movers leading to a reduction in the rank by six. The second constraint on the gauge group is that the low-energies gauge group, after the Gliozzi-Scherk Olive (GSO) projections, has to be a gauge group that does not require adjoint representations to break the gauge symmetry to the standard model, as level 1 Kac-Moody algebras do not allow adjoint or higher level scalar representations in their spectrum [20,21] (while higher level Kac-Moody algebras do have adjoint representations, their phenomenological promise is very questionable [21]). The gauge group is broken at the string level by using the generalized GSO projections. A set of boundary conditions that satisfies the modular invariance constraints and world-sheet supersymmetry is constructed. The set defines a set of projections on the physical massless spectrum. In this way some of the gauge bosons will be projected out and the dimension of the gauge group will be reduced. For each such sector there is a corresponding massless spectrum that similarly gets truncated by the other GSO projections. For the sake of finiteness we confine ourselves to a restricted class of models. We consider models of the form $Z_2^n\otimes Z_4$. In this case the chiral generations come from the Z_2 vectors. This is the simplest choice which we can consider. The Z_4 is used to break the symmetry from an O(n) to a U(n) gauge symmetry. The supercurrent gauge group G [22] is then $SU(2)^6$, which is the only case that allows $N=1$ space-time supersymmetry [22]. It should be stressed that even in this restricted class of models, *a priori* there is a huge range of models that can be contemplated [23,24]. Moreover, in the free fermionic formulation all the degrees of freedom which are needed to cancel the conformal anomaly have the same interpretation. The advantage is that the formulation is directly in four space-time dimensions. The disadvantage is that the immediate geometrical information is lost and a systematic analysis is more difficult.

We start to build a set of basis vectors. The first is the vector **1**, with only periodic boundary conditions. This vector must be in the basis to keep the modular invariance rules. The second is the vector[2] $\mathbf{S}=(\psi^\mu,\chi_{12},\chi_{34},\chi_{56}|0_R)$, the space-time supersymmetry generator, which generates $N=4$ space-time supersymmetry. $N=4$ space-time supersymmetry has to be broken to $N=1$ space-time supersymmetry. This is done by adding more vectors to the basis. For any vector b_j with only periodic and antiperiodic boundary conditions $b_j\cdot S\equiv 0\,\mathrm{mod}2$. The requirement of $N=1$ space-time supersymmetry gives that if $b_j\cdot S=0$ or 4, then b_j does not contribute to the massless spectrum unless $(b_j)_L\equiv 0$, in which case the vector gives additional gauge bosons. Therefore, only vectors with $b_j\cdot S=2$ will contribute to the massless matter spectrum. For a vector of this form to give massless states, $(b_L\cdot b_L)\equiv 4$. Therefore, in these vectors $\{\psi^\mu,\chi_{12}\}$, $\{\psi^\mu,\chi_{34}\}$, or $\{\psi^\mu,\chi_{56}\}$ plus some $\{y,w\}_L$ will receive periodic boundary conditions. In this way the number of the supersymmetries is reduced to $N=2$ and 1 space-time supersymmetry. The vectors that

[1]Nahe means pretty in Hebrew.

[2]We follow closely here the notation of Ref. [19]; the fermions in the brackets have periodic boundary conditions and the rest are antiperiodic. The vertical line separates real from complex fermions.

lead to the chiral generations must have $(b\cdot b)_L\equiv 4$. In this case from the modular invariance rules, only $(b\cdot b)_R=0,4,8$ will give massless states. For the case $(b\cdot b)_R=0$, this vector will give rise to more gravitinos and will increase the number of space-time supersymmetries. The case $(b\cdot b)_R=4$ gives vector representations and therefore only the case $(b\cdot b)_R=8$ will give rise to the generations of chiral fermions. The chiral generations of the low-energy spectrum cannot carry quantum numbers under the hidden gauge group which has to be, as we will argue below, in this case, rank 8 [25]. They must transform under the low-energy gauge group which, because of the, level 1 Kac-Moody algebra, no-adjoint theorem, can include at the most a rank 4 simple group. Therefore, five of the periodic fermions in the vector $b_{\rm gen}$ must transform under the low-energy gauge group and form the weight vector of SO(10). Five is a minimal number in this case because three have to give rise to $SU(3)_C$ and two to $SU(2)_L$. Of the remaining six periodic real fermions in $b_{\rm gen}$, two have to be complexified to keep the two components of the Weyl spinor in the spectrum, and must take the value $\frac{1}{2}$ in the Z_4 twist. We are left with two complex periodic fermions, or four real periodic fermions in the right sector of $b_{\rm gen}$ and four real periodic fermions in the left sector.

The next vector in the basis, $\mathbf{b}_1$, gives rise to a one generation vector. Without loss of generality it is taken to be

$$b_1=(\psi^\mu,\chi^{12},y^{3,\ldots,6},\bar{y}^{3,\ldots,6}|\bar{\psi}^{1,\ldots,5},\bar{\eta}^1)\ . \tag{1}$$

This vector breaks the number of supersymmetries from $N=4$ to 2. All the left movers are real and of the right movers $\bar{\psi}_{1,\ldots,5}$, $\bar{\eta}^1$ are complex and $\bar{y}^{3,\ldots,6}$ are real. The next vector b_2 must have either $\{\psi^\mu,\chi_{34}\}$ or $\{\psi^\mu,\chi_{56}\}$ periodic to reduce $N=2$ to 1 space-time supersymmetry. Take, for example, the first choice. The super current condition then dictates that the remaining left periodic fermions must come from the $\{12\}_L\{56\}_L$ triplets. Now, to obtain from a given vector a full spinorial **16** representation of SO(10) with the same chirality we need in the basis a second vector for which $\{\psi^\mu,\bar{\psi}_{1,\ldots,5}\}$ are periodic in both vectors and the intersection between the remaining boundary conditions is empty. If this condition is not satisfied then, as long as the SO(10) symmetry is not broken, a given vector $b_{\rm gen}$ will give an equal number of **16** and $\overline{\mathbf{16}}$ and thus will not contribute to the net number of generations. We refer to this condition as the "chirality condition" and it will be of importance to the following discussion. Therefore, without loss of generality we can take

$$b_2=(\psi^\mu,\chi^{34},y^1\bar{y}^1,y^2\bar{y}^2,\omega^5\bar{\omega}^5,\omega^6\bar{\omega}^6|\bar{\psi}^{1,\ldots,5},\bar{\eta}^2)\ . \tag{2}$$

We emphasize that up to this stage the analysis is completely general. For the phases

$$c\begin{bmatrix} b_i \\ b_j \end{bmatrix}\ ,$$

we take

$$c\begin{bmatrix} b_i \\ b_j \end{bmatrix}=-1$$

for all i and j. For

$$c\begin{bmatrix} S \\ b_j \end{bmatrix}$$

we take

$$c\begin{bmatrix} S \\ b_j \end{bmatrix}=1$$

for all j to insure $N=1$ space-time supersymmetry. The rest of the phases are fixed by the modular invariance rules. In the choice of the next vector in the basis we have several possibilities. However, we argue that there exists only one realistic possibility. The gauge group after the b_1 and b_2 projections is

$$SO(10)\times SO(6)^2\times SO(22)\ .$$

SO(10) will give rise to the low-energy standard model gauge symmetry and the $SO(6)^2$ corresponds to horizontal symmetries of the observable sector. The SO(22) gauge symmetry has to be broken to a hidden, rank 8 group [25], and a rank 3 part. In Ref. [25] it is shown that the hidden gauge group lattice has to be self-dual in order for the whole string lattice to be so. This confines the allowed hidden group lattices to be $E_8=D_8^+,E_8\times E_8$, and D_{16}^+. Furthermore, it is shown that a grand unifiable standard model with an arbitrary number of generations, with or without the usual choice of the Higgs bosons, will be nonchiral if the hidden gauge group is of dimension 16. Therefore, the hidden gauge group must be rank 8. We use their result as a guideline to argue that in the construction we are investigating there is a unique way to achieve this result for the hidden gauge group. The second requirement is that we need to have at least three generations of chiral fermions. Each of the vectors b_1 and b_2 gives 16 generations, a total of 32, which is not divisible by 3. There exists the possibility that one of the vectors b_1 or b_2 will give two generations, while the other will give one generation. We view this possibility as unnatural. There is a simple and unique way to obtain both of these points by constructing the vector

$$b_3=(\psi^\mu,\chi^{56},\omega^1\bar{\omega}^1,\omega^2\bar{\omega}^2,\omega^3\bar{\omega}^3,\omega^4\bar{\omega}^4|\bar{\psi}^{1,\ldots,5},\bar{\eta}^3)\ . \tag{3}$$

The set $\{1,S,b_1,b_2,b_3\}$ will give $SO(6)^6\times SO(10)\times E_8$, with $N=1$ space-time supersymmetry, where E_8 is the hidden gauge group. What is meant here by hidden is that the representations which are identified with the low-energy matter representations, namely, the **16** representations of SO(10), do not transform under the hidden E_8 gauge group. Practically from the point of view of the spin structure the sectors which lead to the **16** representations of SO(10) and which do not break the SO(10) symmetry, cannot have periodic fermions under the E_8 hidden gauge group ($\bar{\phi}^{1,\ldots,8}\equiv 0$ in these sectors). At a later stage in the spin structure construction, in particular when the SO(10) symmetry is broken, the hidden E_8 symmetry is broken to a subgroup of E_8 which will depend on the specific model. In the case we are investigating the SO(10) symmetry is broken by the Z_4 twist. This breaking has many phenomenological consequences. One example is the presence of fractionally charged states in the massless sector of the string model [26]. With respect to the hidden sector, it will lead to states which may mix the observable sector with the hidden sector. However, the states which are identified with the chiral generations do not transform under the hidden gauge group and, in this respect, it is hidden.

Overall, there are 48 spinorial **16** representation of SO(10) at this step which do not transform under the hidden gauge group. (Here, instead of identifying how the states from a given vector b_{gen} transform under the horizontal symmetries, as was done in previous discussions, we simply count the total number of **16** representation of SO(10) from a given vector b_{gen}.)

Let us examine how the 44 right-moving internal fermions and 18 left-moving internal fermions are divided. Eight complex right-moving fermions $\bar{\phi}^{1,\ldots,8}$ give rise to the hidden gauge group. Five complex right-moving fermions $\bar{\psi}^{1,\ldots,5}$ give rise to the low-energy observable gauge group. This number is minimal and maximal because of the no-adjoint theorem and because we must have three complex fermions to give rise to $SU(3)_C$ and two complex fermions to give rise to $SU(2)_L$, a total of five. Three complex fermions $\bar{\eta}^1$, $\bar{\eta}^2$, and $\bar{\eta}^3$ will give rise to three horizontal U(1) symmetries. They are needed in the spectrum to keep the two components of the Weyl spinor of the chiral generations in the spectrum and must receive the value $\frac{1}{2}$ for their boundary condition in the Z_4 twist. Thus we are left with 12 real right-moving fermionic states. Of the 18 left-moving fermions, six give rise to the supersymmetry charges $\chi^{1,\ldots,6}$ and we are left with 12 real left-moving states. These 12 right-moving and 12 left-moving fermionic states completely determine the number of spinorial **16** representation of SO(10) and thus completely determine the number of generations in the model. It is interesting to note that from the point of view of the number of generations, the same number of degrees of freedom that determines this number in bosonic formulations, namely, the 6 bosonic compactified dimensions, is the same in this free fermionic construction. This point is further clarified by adding the vector

$$X=\{0,\ldots,0|\psi^{1,\ldots,5},\eta^{1,2,3}\} \tag{4}$$

to $\{1,S,b_1,b_2,b_3\}$, which extends the gauge symmetry to

$$E_6\times U(1)^2\times SO(4)^3 .$$

The sectors $(b_1;b_1+X)$, $(b_2;b_2+X)$, and $(b_3;b_3+X)$ each give eight **27** of E_6. The $(NS;NS+X)$ sector gives in addition to the vector bosons and spin two states, three copies of scalar representations in $\mathbf{27}+\overline{\mathbf{27}}$ of E_6.

In this model the only internal fermionic states which count the multiplets of E_6 are the real internal fermions $\{y,w|\bar{y},\bar{w}\}$. This is observed by writing the degenerate vacuum of the sectors b_j in a combinatorial notation. The vacuum of the sectors b_j contains twelve periodic fermions. Each periodic fermion gives rise to a two-dimensional degenerate vacuum $|+\rangle$ and $|-\rangle$ with fermion numbers 0 and -1, respectively. The GSO operator is a generalized parity operator, which selects states with definite parity. After applying the GSO projections, we can write the degenerate vacuum of the sector b_1 in combinatorial form:

$$\left[\binom{4}{0}+\binom{4}{2}+\binom{4}{4}\right]\times\left\{\binom{2}{0}\left[\binom{5}{0}+\binom{5}{2}+\binom{5}{4}\right]\binom{1}{0}+\binom{2}{2}\left[\binom{5}{1}+\binom{5}{3}+\binom{5}{5}\right]\binom{1}{1}\right\}, \tag{5}$$

where

$$4=\{y^3y^4,y^5y^6,\bar{y}^3\bar{y}^4,\bar{y}^5\bar{y}^6\}\ ,\quad 2=\{\psi^\mu,\chi^{12}\}\ ,$$

$5=\{\bar{\psi}^{1,\ldots,5}\}$ and $1=\{\bar{\eta}^1\}$. The combinatorial factor counts the number of $|-\rangle$ in a given state. The two terms in the curly brackets correspond to the two components of a Weyl spinor. The $\mathbf{10}+\mathbf{1}$ in the **27** of E_6 are obtained from the sector b_j+X. From Eq. (8) it is observed that the states which count the multiplicities of E_6 are the internal fermionic states

$$\{y^{3,\ldots,6}|\bar{y}^{3,\ldots,6}\}\ .$$

A similar result is obtained from the sectors b_2 and b_3 with

$$\{y^{1,2},\omega^{5,6}|\bar{y}^{1,2},\omega^{5,6}\}$$

and

$$\{\omega^{1,\ldots,4}|\bar{\omega}^{1,\ldots,4}\}\ ,$$

respectively, which suggests that these 12 states correspond to a six-dimensional compactified orbifold with Euler characteristic equal to 48. The number of generations in the fermionic language, from each sector, corresponds to the number of zero modes of the periodic fermions with definite chirality, being positive in Eq. (5). The remaining 16 right-moving complex fermions correspond to the 16-dimensional torus of the heterotic string. In the language of toroidal compactification, the three SO(4) horizontal symmetries arise from nonzero vacuum expectation values (VEV's) of the background fields at the critical point [27].

A vector that contributes to the generations of chiral fermions will not transform under the hidden gauge group. It must have

$$b_{\text{gen}}(\psi^\mu,\chi_j|\bar{\psi}^{1,\ldots,5},\bar{\eta}_j)=1$$

[where j denotes the following sets $\{\chi_j|\bar{\eta}_j\}=\{(\chi_{12}|\bar{\eta}_1);(\chi_{34}|\bar{\eta}_2);(\chi_{56}|\bar{\eta}_3)\}$]. Thus, to satisfy the massless condition each vector b_{gen} must have four right-moving real fermionic states and four real left-moving states. We see that the number of such vectors is naturally 3. To summarize, the vectors b_1, b_2, and b_3 perform three functions.

(1) They give rise to the generations of chiral fermions and the number of exactly 3 such vectors hints to the fact that in this construction three generations can naturally be obtained.

(2) They separate the hidden gauge group from the observable gauge group through a "comb mechanism." By adding $b_1+b_2+b_3$ we get a vector for which $b\{\bar{\phi}_{1,\ldots,8}\}=0$ and the rest of the fermionic states receive periodic boundary conditions.

(3) They provide the necessary "chirality projection" for each other.

This set was first constructed by Antoniadis, Ellis, Hagelin, and Nanopoulos [17] in the construction of the flipped SU(5)×U(1). We will refer to it as the NAHE (Nanopoulos, Antoniadis, Hagelin, and Ellis) set. At this stage, we postulate it to be a unique set and deviations from it will be studied in future work.

III. BEYOND THE NAHE SET

At this stage we ask the following questions.

(1) Is there a maximum to the number of vectors that can lead to full spinorial representations of SO(10), which do not transform under the hidden gauge group and obey the "chirality condition," so as to maximize the number of generations?

(2) Is the number of 3 such vectors favored?

We work in the original flipped SU(5)×U(1) construction [19] in which the 12 real left movers are coupled with the 12 real right movers to form 12 real Ising model fermionic states.

For a vector to contribute full spinorial representation **16** of SO(10) it must satisfy the "chirality condition," it must have

$$b\{\psi^\mu,\chi_j|\bar{\psi}^{1,\ldots,5},\eta_j\}=1\ ,$$

and it cannot carry any quantum numbers under the hidden gauge group. Regarding the number of generations, the problem is reduced to examining the 12 Ising model states. The other constraints which we impose are the following.

(1) We want to break the SO(10) symmetry to a group which does not use adjoint representations. This is done by using the Z_4 twist. [The alternative is to break the SO(10) gauge symmetry to SO(6)×SO(4) by using only periodic and antiperiodic boundary conditions. The results will be similar to the results presented here.]

(2) We want to break the horizontal SO(6) symmetries, at the most to factors of U(1)'s, as there is no evidence for those horizontal symmetries in nature. $\bar{\eta}_j$ will be separated due to the Z_4 twist and will give rise to three U(1) horizontal symmetries. To break the symmetries of the real fermions we need a minimum of three vectors; otherwise, we will be left with either a gauge symmetry or a residual Z_2 symmetry. In the minimal case of only three vectors, the breaking of all the horizontal symmetries which arise from the real fermions forces the number of **16** representations of SO(10) from b_1, b_2, and b_3 to be exactly 3. In the minimal case we cannot add more vectors that will obey the "chirality condition" and with all the horizontal symmetries broken. Thus, the minimal case must be $Z_2\otimes Z_2\otimes Z_4$ and the number of **16** representations of SO(10) from each vector is reduced by

$$\frac{\text{No. of gen}(b_j)}{Z_2\times Z_2\times Z_4}=\frac{16}{2\times 2\times 4}=1\ . \qquad (6)$$

Thus in the minimal case the existence of only three generations is correlated with the breaking of all the horizontal symmetries that arise from the part of the real fermions. In the toroidal compactification language it corresponds to projecting out all the gauge bosons which arise from the nonzero VEV's of the background fields at the critical point.

Let us now examine how the basis may be enlarged to include more vectors that lead to additional full spinorial **16** representation of SO(10). These vectors must obey the "chirality condition" and must be of the form described above.

In Tables I and II we have classified all the vectors that have empty real intersection with one of the sectors b_1, b_2, or b_3 and thus may lead to additional full generations. Additional vectors may have empty real intersection with only one of the vectors b_1, b_2, or b_3. In the notation of Tables I and II, the upper index indicates with which vec-

TABLE I. In the notation used here χ^i_j j denotes the sets $\{(\chi_{12}|\bar{\eta}^1),(\chi_{34}|\bar{\eta}^2),(\chi_{56}|\bar{\eta}^2)\}$, the upper index i indicates that the given sector has empty real intersection with the sector b_i. In all this sectors ψ_μ and $\bar{\psi}^{1,\ldots,5}$ are periodic. To obtain the sectors $\bar{\chi}^j_i$ the sector $\{\bar{\psi}^{1,\ldots,5},\bar{\eta}^1,\bar{\eta}^2,\bar{\eta}^3\}$ is added to χ^j_i.

	$y_3\bar{y}_3$	$y_4\bar{y}_4$	$y_5\bar{y}_5$	$y_6\bar{y}_6$	$y_1\bar{y}_1$	$y_2\bar{y}_2$	$\omega_5\bar{\omega}_5$	$\omega_6\bar{\omega}_6$	$\omega_1\bar{\omega}_1$	$\omega_2\bar{\omega}_2$	$\omega_3\bar{\omega}_3$	$\omega_4\bar{\omega}_4$
χ^3_{12}	1	1	0	0	0	0	1	1	0	0	0	0
χ^3_{34}	0	0	1	1	1	1	0	0	0	0	0	0
χ^3_{56}	1	1	0	0	1	1	0	0	0	0	0	0
χ^2_{12}	1	1	0	0	0	0	0	0	0	0	1	1
χ^2_{34}	0	0	1	1	0	0	0	0	1	1	0	0
χ^2_{56}	1	1	0	0	0	0	0	0	1	1	0	0
χ^1_{12}	0	0	0	0	0	0	1	1	0	0	1	1
χ^1_{34}	0	0	0	0	1	1	0	0	0	0	1	1
χ^1_{56}	0	0	0	0	1	1	0	0	1	1	0	0

TABLE II. Additional vectors with empty real intersection. The notation of Table I is used. In these vectors an odd number of internal fermions are periodic.

	$y_3\bar{y}_3$	$y_4\bar{y}_4$	$y_5\bar{y}_5$	$y_6\bar{y}_6$	$y_1\bar{y}_1$	$y_2\bar{y}_2$	$\omega_5\bar{\omega}_5$	$\omega_6\bar{\omega}_6$	$\omega_1\bar{\omega}_1$	$\omega_2\bar{\omega}_2$	$\omega_3\bar{\omega}_3$	$\omega_4\bar{\omega}_4$
χ_{12}^3	1	1	1	0	0	0	0	1	0	0	0	0
χ_{34}^3	0	0	0	1	1	1	1	0	0	0	0	0
χ_{12}^2	1	1	1	0	0	0	0	0	0	0	0	1
χ_{56}^2	1	0	0	0	0	0	0	0	1	1	1	0
χ_{34}^1	0	0	0	0	0	1	1	1	1	0	0	0
χ_{56}^1	0	0	0	0	1	0	0	0	0	1	1	1

tors of the NAHE set the given vector has an empty real intersection. The lower index indicates which of the three χ pairs is periodic. To obtain the vector $\bar{\chi}_i^j$, the vector

$$\{\bar{\psi}^{1,\ldots,5},\bar{\eta}^1,\bar{\eta}^2,\bar{\eta}^3\}$$

is added to the vector χ_i^j. The number of generations can be enlarged by adding one or a combination of such vectors that satisfies the modular invariance rules and world-sheet supersymmetry. We investigate in detail the addition of vectors from Table I. Adding vectors from Table II will yield similar results. From Table I we see that only one vector from each group 1, 2, or 3 could be added. Otherwise we have one of the two cases: (1) $\chi_{12}^3\oplus\chi_{34}^3$; (2) $\chi_{12}^3\oplus\chi_{56}^3$ and $\chi_{34}^3\oplus\chi_{56}^3$.

Case (1) will not satisfy the canonical condition $\sum_i m_i b_i=0$ if $m_i\equiv 0$, as can be seen by taking $\chi_{12}^3+\chi_{34}^3+b_1+b_2=0$.

In case (2), by taking $b_1+\chi_{34}^3+\chi_{56}^3$, we will get a vector

$$X=(0_L|\bar{\psi}^{1,\ldots,5},\bar{\eta}_1,\bar{\eta}_2,\bar{\eta}_3)\ .$$

This vector will enlarge the observable gauge group to $E_6\times U(1)^2\times SO(4)^3$. The vector X gives $\mathbf{16}+\overline{\mathbf{16}}$ gauge bosons, which are needed to enlarge the gauge symmetry from SO(10) to E_6. The $\mathbf{5},\bar{\mathbf{5}}$ and the singlet which are needed to enlarge the $\mathbf{16}$ representation of SO(10) from a vector b_j to $\mathbf{27}$ representation of E_6 come from the vectors b_j+X. In principle, we could enlarge the gauge symmetry to E_6 by simply adding the vector X to the set $\{1,S,b_1,b_2,b_3\}$. However, it is interesting to note again how the vectors of the form $b_{\rm gen}$ give rise to the generation structure and at the same time determine the gauge symmetry. (The extension of the NAHE set to include an E_6 gauge symmetry is an interesting exercise in its own right as it may be easier in this case to understand the correspondence with the geometrical formulations.) For similar reasons, at the most two vectors can be added from Table I to the NAHE set. We have classified all the possible combinations. Overall, from Table I there are 33 combinations. We generated all possible combinations by using a simple computer program and checked the equivalence of the additive groups of all those combinations. There are four different models with the addition of two vectors to the NAHE set and three different models with the addition of one vector to the NAHE set. After using the cyclic symmetry between $b_1\to b_2\to b_3\to b_1$ the number is reduced to 2 possible models with the addition of 2 vectors and 1 possible model with the addition of 1 vector: (1) NAHE$\oplus\chi_{12}^3$; (2) NAHE$\oplus\chi_{12}^3\oplus\chi_{56}^2$; (3) NAHE$\oplus\chi_{56}^3\oplus\chi_{34}^2$.

Overall, case (1) will give

$$\frac{\chi_{12}^{23}}{Z_2}+\frac{\chi_{34}^{13}}{Z_2}+\chi_{56}^{12}+\frac{\chi_{12}^{3}}{Z_2}+\frac{\chi_{34}^{3}}{Z_2}$$

for a total of 48 generations. Case (2) will give

$$\frac{\chi_{12}^{23}}{Z_2}+\frac{\chi_{34}^{13}}{Z_2}+\frac{\chi_{56}^{12}}{Z_2}+\frac{\chi_{12}^{3}}{Z_2}+\frac{\chi_{34}^{3}}{Z_2}+\frac{\chi_{12}^{2}}{Z_2}+\frac{\chi_{34}^{2}}{Z_2}$$

for a total of 56 generations. Case (3) will give

$$\frac{\chi_{12}^{23}}{Z_2}+\frac{\chi_{34}^{13}}{Z_2}+\frac{\chi_{56}^{12}}{Z_2}+\frac{\chi_{56}^{3}}{Z_2}+\frac{\chi_{34}^{2}}{Z_2}+\frac{\chi_{12}^{1}}{Z_2}$$

for a total of 48 generations.

In addition to the vectors of the for $b_{\rm gen}$, there are in the spectrum vectors of the form $b=\bar{\chi}_j^k$. For example,

$$\bar{\chi}_{56}^1=(\psi^\mu,\chi^{56},y^1\bar{y}^1,y^2\bar{y}^2,\omega^3\bar{\omega}^3,\omega^4\bar{\omega}^4,|\bar{\eta}^1,\bar{\eta}^2)\ .$$

These vectors give rise to the representations $\mathbf{5},\bar{\mathbf{5}}$ and singlets of SO(10).

(1) $\bar{\chi}_{56}^1$.

(2) $\bar{\chi}_{56}^1,\bar{\chi}_{34}^1,\bar{\chi}_{34}^2\chi_{56}^3$.

(3) $\bar{\chi}_{34}^1,\bar{\chi}_{56}^1,\bar{\chi}_{12}^2,\bar{\chi}_{56}^2,\bar{\chi}_{12}^3,\bar{\chi}_{34}^3$.

Vectors of this form mix between generations from different vectors. Therefore, they may be used in the construction of realistic mass matrices by using the $\mathbf{5}$ and $\bar{\mathbf{5}}$ as Higgs fields.

At this stage, each horizontal symmetry is broken to factors of $SO(2)^k\times SO(4)^j$, which we break further. We can now add vectors that will obey the "chirality condition" with b_4 or b_5. We must, however, keep in mind that in the Z_4 twist, to keep the full $\mathbf{16}$ representation of SO(10), there must remain a Z_2 symmetry from the part of the real fermionic states, which must be broken when the Z_4 twist is applied. Otherwise, only part of the $\mathbf{16}$ will remain in the spectrum. [This will also be true in the case where the symmetry is broken to $SO(6)\times SO(4)$. The conclusion is that in a vector that breaks the SO(10) symmetry there must exist a Z_2 symmetry from the real part to insure that the full spinorial $\mathbf{16}$ of SO(10) will remain in the spectrum.] Therefore, we can add at the most one more vector of the form $b_{\rm gen}$.

Thus the conclusion is that in this construction, a maximum of six vectors of the form b_j, to give an additive group of $Z_2^8 \otimes Z_4$, can be constructed. In cases (2) and (3) we cannot add more vectors that obey the "chirality condition." Therefore, in these models, after the breaking of the horizontal symmetries, we will have one generation from each of the vectors. Therefore, in these two cases we will have 7 and 6 full spinorial **16** representation of SO(10), respectively, after all the horizontal symmetries are broken. In the case of model 1, we may add the two vectors

$$b_5 = (\psi^\mu, \chi^{34}, y^1\bar{y}^1, \omega^2\bar{\omega}^2, y^5\bar{y}^5, y^6\bar{y}^6 | \bar{\psi}^{1,\ldots,5}, \bar{\eta}^2) , \tag{7}$$

$$b_6 = (\psi^\mu, \chi^{56}, \omega^1\bar{\omega}^1, y^2\bar{y}^2, \omega^3\bar{\omega}^3, y^4\bar{y}^4 | \bar{\psi}^{1,\ldots,5}, \bar{\eta}^3) . \tag{8}$$

The model spanned by this basis will have 11 vectors of the form $b_{\rm gen}$ that obey the "chirality condition" and therefore after the application of the Z_4 twist and the breaking of all the horizontal symmetries, this model will lead to 11 generations of chiral fermions. This is the maximal possible number of generations in this construction with all the horizontal symmetries broken.

It is clear that phenomenologically the extension of the NAHE set along the lines suggested above is not realistic, as perturbative unification will be impossible in these cases. However, can we find a reason why the string will not choose to go in this direction? The consistency requirement implemented by requiring modular invariance and vanishing of the conformal anomaly does not forbid those extensions. It is interesting to note that even though models with more vectors which obey the "chirality condition" are possible, the NAHE set which is postulated here to be a unique set, forbids the construction of models with only four such vectors and therefore a model with only four $b_{\rm gen}$ is impossible. If a vector from Table I or Table II, which obeys the "chirality condition," is added to the NAHE set, then there will always be another vector which is a combination of this vector and a combination of the vectors b_1, b_2, and b_3 which obeys the "chirality condition" as well. Therefore, the extensions will always be of at least five vectors which obey the "chirality condition" and a model with the NAHE set and only four generation vectors is not possible. The only way to obtain a four generation model is by not using the full NAHE set and leaving some of the horizontal symmetries unbroken, as is done in some of the examples in Ref. [24], in which case there is not a well-defined hidden sector. The second reason why the string does not go in this direction may be the following. The NAHE set establishes a cyclic symmetry between b_1, b_2, and b_3, extending the basis to include more vectors which obey the "chirality condition" and therefore constructing more vectors that may lead to additional generations destroys this cyclic symmetry. To preserve this symmetry we either cannot add more vectors which obey the "chirality condition" or we have to add at least two such vectors and therefore we will always have either only the NAHE vectors which obey the "chirality condition" or we will have five or more such vectors.

So far we have avoided the question of what is the observable gauge group which emerges from the string after the application of the GSO projections. In the case of the minimal basis the net chirality of three generations is true irrespective of the choice of the observable gauge group after the GSO projections. The no-adjoint theorem restricts the observable gauge group to gauge groups which do not use adjoint representations, at the field theory level, to break the gauge symmetry to the standard model. Only three possibilities are permitted: $SU(5)\times U(1)$, $SO(6)\times SO(4)$, and $SU(3)\times SU(2)\times U(1)^2$. In the first two cases there are four **16** and one $\overline{\mathbf{16}}$ representations of SO(10), a net chirality of three generations. The additional **16** and $\overline{\mathbf{16}}$ are coming from vectors which do not obey the "chirality condition." In the third case there are exactly three **16** representations of SO(10) and no $\overline{\mathbf{16}}$. Therefore, a net chirality of three generations is a unique characteristic of the NAHE set and the breaking of all the real horizontal symmetries in the minimal basis. In the language of toroidal compactification these symmetries arise from the nonzero VEV's of background fields. Thus, in the minimal basis a net chirality of three generations is correlated with the projection of all the enhanced gauge symmetries, due to the nonzero VEV's of background fields at the critical point. We would like to remark that even though there exist many models with a net chirality of three generations in the Calabi-Yau and orbifold formulations, most of these models have generations and mirror generations that cancel out. It is at present not clear what the role of these generations and mirror generations is and whether they can survive the phenomenological scrutiny.

Before turning to the conclusion, let us briefly comment on the question of the number of generations in string theories in general. In the geometrical formulations of the superstring the number of generations is related to the Euler characteristic of the compactified space. Once this is realized a systematic classification, with respect to the number of generations is easily achieved. The free fermionic formulation has the advantage of being formulated directly in four space-time dimensions and has led to the most realistic string models to date. Its disadvantage is that the geometrical information, with respect to the number of generations is lost. Therefore, the ability to perform a systematic classification is weakened. As progress is made in the understanding of string theory, the different formulations are expected to unify. Then, a complete understanding of the geometrical origin behind the free fermionic construction, and the number of generations, will be gained. At the moment, why free fermionic constructions naturally lead to three generations, and most realistic models, is still a puzzle. One possible reason behind it may be the fact that the free fermionic formulation is formulated in the most symmetric point in the moduli space. Where the gauge symmetries are enhanced from U(1) to SU(2).

IV. CONCLUSION

In this paper we have investigated in detail the construction of free fermionic spin structure models with $Z_2^n \otimes Z_4$ boundary conditions, with special attention to the number of generations in these models. The number of generations of chiral fermions has been observed to be only 3. We have shown that in the restricted class of models that we examined the number of 3 generations is a natural consequence of the following.

(1) The initial number of degrees of freedom which are needed to cancel the conformal anomaly.

(2) As a consequence of 1 and low-energy observations the low-energy gauge group has to be split to a hidden rank 8 gauge group and an observable part. This, as we have shown, naturally introduces three vectors of the form b_{gen} into the model construction.

(3) Requiring minimality of the basis and the breaking of all the horizontal symmetries that arise from the part of the real fermions.

From the point of view of gauge theories and point field theories, the number of light left-handed neutrinos has been observed to be only 3. Dynamical calculations of m_b/m_τ are in agreement with experimental observations only in the presence of three generations of chiral fermions. However, in unified gauge theory there is no understanding of how nature has chosen to have only three generations. Moreover, in unified gauge theory, because of the nonobservation of proton decay and the big desert picture, the origin of the number of generations must have an explanation at the scale which is above the grand unified theory scale, or at the Planck scale. Indeed, in the geometrical formulation of the string the number of generations is determined by the Euler characteristic which is an intrinsic characteristic of the compactification scale, or of the Planck scale. The question is how does the string choose three generations. In this paper we have investigated this question in detail within the framework of the free fermionic formulation with $Z_2^n \otimes Z_4$ boundary conditions. As the number of chiral generations depend only on the choice of boundary conditions, it is a characteristic of the chosen string spin structure. Therefore, the number of generations of chiral fermions could provide circumstantial evidence for the string. At this stage it is safe to claim that in the free fermionic formulation of the superstring with $Z_2^n \otimes Z_4$ boundary conditions, 3 generations is the most natural number. Models with 3 generations are simpler, more elegant, and more economical than models with a different number. Further investigation is required to see if 3 generations, in this formulation, is practically unique. In this respect, our work here may be instrumental in identifying the real string vacua. One way to enhance our understanding is by constructing the NAHE set (or its E_6 extension) in the geometrical formulations of the string as those formulations will give a geometrical understanding of the correct string vacua.

Further and elaborate investigation is required to affirm the claims made in this paper. We hope that we have been able to convince the reader of the value of such investigation. What we may gain is twofold. On the one hand, we may develop an understanding of the origin of the three generations. On the other hand, the experimental observation of only three generations, because of the reasons discussed above, is the first experimental observation that we may be able to connect to the string. In this respect the somewhat disappointing result at LEP and SLC may eventually turn out to be one of the most fundamental observations ever made.

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[1] Mark II Collaboration, G. S. Abrams *et al.*, Phys. Rev. Lett. **63**, 724 (1989); **63**, 2780 (1989).

[2] ALEPH Collaboration, D. Decamp *et al.*, Phys. Lett. B **231**, 519 (1989); OPAL Collaboration, M. Akrawy *et al.*, *ibid.* **231**, 530 (1989); L3 Collaboration, B. Adeva *et al.*, *ibid.* **231**, 509 (1989); DELPHI Collaboration, P. Aarnio *et al.*, *ibid.* **231**, 539 (1989).

[3] C. Kounnas, A. Masiero, D. V. Nanopoulos, and K. A. Olive, *Grand Unification With and Without Supersymmetry and Cosmological Implications* (World Scientific, Singapore, 1984).

[4] D. V. Nanopoulos and D. A. Ross, Nucl. Phys. **B157**, 273 (1979); Phys. Lett. **108B**, 351 (1982); **118B**, 99 (1982).

[5] M. Green, J. Schwarz, and E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, England, 1987), Vols. 1 and 2.

[6] P. Candelas, G. T. Horowitz, A. Strominger, and E. Witten, Nucl. Phys. **B258**, 46 (1985).

[7] L. Dixon, J. A. Harvey, C. Vafa, and E. Witten, Nucl. Phys. **B261**, 678 (1985); **B274**, 285 (1986).

[8] K. S. Narain, Phys. Lett. **169B**, 41 (1986); W. Lerche, D. Lüst, and A. N. Schellekens, Nucl. Phys. **B287**, 477 (1987).

[9] I. Antoniadis, C. Bachas, C. Kounnas, and P. Windey, Phys. Lett. B **171**, 51 (1986); I. Antoniadis, C. Bachas, and C. Kounnas, Nucl. Phys. **B289**, 87 (1987); I. Antoniadis and C. Bachas, *ibid.* **B298**, 586 (1988); H. Kawai, D. C. Lewellen, and S. H. H. Tye, Phys. Rev. Lett. **57**, 1832 (1986); Phys. Rev. D **34**, 3794 (1986); Nucl. Phys. **B288**, 1 (1987); R. Bluhm, L. Dolan, and P. Goddard, *ibid.* **B309**, 330 (1988).

[10] D. J. Gross, J. A. Harvey, E. Martinec, and R. Rohm, Phys. Rev. Lett. **54**, 502 (1985); Nucl. Phys. **B256**, 253 (1985).

[11] B. Greene, K. H. Kirklin, P. J. Miron, and G. G. Ross, Phys. Lett. B **180**, 69 (1986); Nucl. Phys. **B278**, 667 (1986); **B292**, 606 (1987); S. Kalara and R. N. Mohapatra, Phys. Rev. D **36**, 3473 (1987); R. Arnowitt and P. Nath, *ibid.* **39**, 2006 (1989); **42**, 2498 (1990); Phys. Rev. Lett. **62**, 222 (1989).

[12] Y. Hosotani, Phys. Lett. **126B**, 309 (1983); **129B**, 193 (1983); E. Witten, *ibid.* **126B**, 351 (1984).

[13] E. Witten, Nucl. Phys. **B268**, 79 (1986).

[14] J. Ellis, C. Gomez, D. V. Nanopoulos, and M. Quiros, Phys. Lett. B **173**, 59 (1986); **174**, 465(E) (1986); M. Dine, N. Seiberg, X. G. Wen, and E. Witten, Nucl. Phys. **B287**, 769 (1986); **B289**, 319 (1987).

[15] H. Okada and K. Sasaki, Phys. Rev. D **40**, 3743 (1989), and references therein.

[16] L. E. Ibañez, J. E. Kim, H. P. Nilles, and F. Quevedo, Phys. Lett. B **191**, 282 (1987); L. E. Ibañez, J. Mas, H. P. Nilles, and F. Quevedo, Nucl. Phys. **B301**, 157 (1988).

[17] I. Antoniadis, J. Ellis, J. S. Hagelin, and D. V. Nanopoulos, Phys. Lett. B **231**, 65 (1989); A. E. Faraggi, D. V. Nanopoulos, and K. Yuan, Nucl. Phys. **B335**, 347 (1990); I. Antoniadis, G. K. Leontaris, and J. Rizos, Phys. Lett. B **245**, 161 (1990).

[18] L. Dixon, V. S. Kaplunovsky, and C. Vafa, Nucl. Phys. **B294**, 43 (1987).

[19] I. Antoniadis, J. Ellis, J. S. Hagelin, and D. V. Nanopoulos, Phys. Lett. B **205**, 459 (1989); J. L. Lopez, S. Kalara, and D. V. Nanopoulos, *ibid.* **245**, 421 (1990); Nucl. Phys. **B353**, 650 (1991).

[20] H. Dreiner, J. L. Lopez, D. V. Nanopoulos, and D. Reiss, Phys. Lett. B **216**, 283 (1989); D. C. Lewellen, Nucl. Phys. **B337**, 61 (1990).

[21] J. Ellis, J. L. Lopez, and D. V. Nanopoulos, Phys. Lett. B **245**, 375 (1990).

[22] H. Dreiner, Jorge L. Lopez, D. V. Nanopoulos, and D. Reiss, Nucl. Phys. **B320**, 401 (1989).

[23] D. Sénéchel, Phys. Rev. D **39**, 3717 (1989).

[24] J. A. Schwartz, Phys. Rev. D **42**, 1777 (1990).

[25] T. Gannon and C. S. Lam, Phys. Rev. D **41**, 492 (1990).

[26] A. N. Schellekens, Phys. Lett. B **237**, 363 (1990); J. Ellis, J. L. Lopez, and D. V. Nanopoulos, *ibid.* **247**, 257 (1990); A. E. Faraggi, Phys. Rev. D **46**, 3204 (1992).

[27] K. S. Narain, M. H. Sarmadi, and E. Witten, Nucl. Phys. **B279**, 369 (1987).