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Multifractal analysis of nucleus-nucleus interactions

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We have performed a multifractal (G-moment) analysis of 14.6–200 GeV/nucleon nucleus-nucleus and 200–800 GeV proton-nucleus interactions from KLM and Fermilab E-90 and E-508 emulsion data, including explicit corrections for the finite statistical sample. The corrected slopes of the G-moments for protons, ^{16}O, ^{28}Si, and ^{32}S nuclei show only slight evidence for departures from random behavior, while the normalized entropies appear to show a more consistent departure from randomness, particularly for protons. Given the size of the uncertainties, the results of the fractal analysis are not inconsistent either with results of intermittency analyses for nucleus-nucleus collisions or with the nonrandom behavior previously reported for leptonic and hadronic collisions. However, because of the effects of statistical noise, the fractal analysis is not as sensitive as the intermittency analysis for detecting nonrandom fluctuations.

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I. INTRODUCTION

High energy nucleus-nucleus collisions may provide the conditions necessary for producing a quark-gluon plasma. The possible existence of a deconfined phase of quarks and gluons at a central energy density in excess of 2–3 GeV/fm^3, with a subsequent phase transition to hadrons, has been suggested by a number of results [1]. For example, experiments have suggested the presence of anomalous high p_t events, the enhancement of strange particle production, and suppression of J/ψ in ultrarelativistic heavy ion interactions [2]. In no case, however, has the evidence been conclusive. One possible approach depends on observing the fluctuations in the particle density distributions: a phase transition may give rise to fluctuations in individual events which manifest themselves as peaks or spikes in narrow phase space domains [3].

In the field of hydrodynamic turbulence, the related phenomenon of intermittency is studied via the scaling properties of the moments of the relevant distributions as a function of the bin size in phase space [4]. Typically, a set of generalized exponents or dimensions may be introduced in order to quantify the extent of the dynamical fluctuations in a chaotic system. Bialas and Peschanski [5] adapted the method of generalized scaling exponents to study intermittency in multihadron production. They suggested that the behavior of the factorial moments of the multiplicity distributions as a function of successively smaller and smaller phase space bins, down to the experimental resolution, might reveal fundamental properties of the interaction process. In particular, they suggested the importance of a power-law (or intermittent) dependence of the scaled factorial moments on the bin size $\delta\eta$, and a connection between the power-law exponents and the scaling properties of the collision process. Evidence for intermittency has now been reported for e^+e^- [6], μp [7], hadron-hadron [8], and nucleus-nucleus [9,10] collisions where, for sufficiently small $\delta\eta$, the moments approach a power law $\delta\eta^{-\phi}$. At least for hadrons and nuclei, this intermittency behavior is not reproduced by standard models of particle production [11]. We have previously demonstrated [9] the presence of an intermittent power-law growth of the moments with decreasing bin size for 200 and 800 GeV protons, 60 and 200 GeV/nucleon ^{16}O, and 200 GeV/nucleon ^{32}S interactions with the Ag-Br in emulsion, and found that the intermittency patterns were weaker for central nucleus-nucleus collisions than for proton-nucleus events. Furthermore, the effect was weakest for the heaviest projectile nucleus (^{32}S).

It is not at all clear whether the reported intermittency

*Deceased.

effects reflect anything more than conventional short-range hadronic correlations. However, if one looks for a link between a phase transition and intermittency, then one is naturally led to a thermodynamic formulation of fractal dynamics (of which intermittency is a special case) [12]. A fractal or self-similar object has the property that it satisfies a power-law scaling which reflects the underlying dynamics [13]. The most complete description of the scaling properties of a fractal is given in terms of the power-law exponent or dimension—or, for a multifractal, in terms of an infinite set of generalized dimensions. In this work, we use the method of multifractal moments (or G moments) to study the scaling properties of the KLM nucleus-nucleus and Fermilab E-90 and E-508 hadron-nucleus interaction data. We describe the multifractal moment analysis in Sec. II and present our results in Secs. III and IV. We discuss the interpretation of the results in Sec. V, and in particular discuss the connection between the fractal analysis and intermittency.

II. MULTIFRACTAL MOMENTS AND MASS EXPONENTS

Assume a pseudorapidity range $\Delta\eta$ is divided into M_0 bins of width $\delta\eta = \Delta\eta/M_0$, where $\eta = -\ln\tan\theta/2$. Let k_j be the number of particles in the jth bin. Since there may be bins that have no particles, we define M to be the number of nonempty bins. A multifractal multiplicity moment of order q is defined by [7,14]

$$G_q = \sum_{j=1}^{M} p_j^q\,\theta(k_j - q) \tag{2.1}$$

where $p_j = k_j/N$ with $N = \sum k_j$ and $\sum p_j = 1$, and q is an integer. The step function $\theta(k_j - q)$ here is 1 for $k_j \geq q$ and 0 for $k_j < q$. (The standard definition of G_q does not include the θ function. Here we use the modified formulation of Derado *et al.* [7] and Hwa and Pan [14], who introduce the θ function to minimize the effects of finite multiplicity.) We note that for q positive, G_q is sensitive to peaks in the pseudorapidity distribution; and q negative provides sensitivity to dips in the η distribution.

A self-similar particle production process may be characterized by a power law

$$G_q \propto \delta\eta^{\tau_q} \tag{2.2}$$

such that the "mass exponent" τ_q is invariant under a scale change $\delta\eta \to a\delta\eta$. In physical systems, the scaling relation (2.2) holds only down to a characteristic cutoff in pseudorapidity (determined, for example, by an external parameter such as Reynolds number in turbulence or in our case by the experimental resolution $\delta\eta_{\text{expt}} \sim 0.1$).

If a set is self-similar, it can be usefully characterized by an infinite spectrum of generalized dimensions,

$$D_q = \frac{\tau_q}{q-1} \tag{2.3}$$

given by [15]

$$D_q = \begin{cases} \dfrac{1}{q-1}\lim\limits_{\delta\eta\to 0}\dfrac{\ln\sum\limits_{j=1}^{M} p_j^q\,\theta(k_j-q)}{\ln\delta\eta} & \text{for } q\neq 1\,, \\[2ex] \lim\limits_{\delta\eta\to 0}\dfrac{\sum\limits_{j=1}^{M} p_j\ln p_j\,\theta(k_j-q)}{\ln\delta\eta} & \text{for } q=1. \end{cases} \tag{2.4}$$

D_0 is the fractal dimension, which in most cases coincides with the topological dimension of the set. D_1 is the information dimension, which describes how the information entropy

$$S = -\sum p_j\ln p_j\,\theta(k_j-1) \tag{2.5}$$

varies with decreasing widths of the pseudorapidity bins. D_2 is the correlation dimension, determined by the correlation function of the set.

The generalized dimensions D_q are a direct manifestation of any underlying fractal nature of the emission mechanism. They are related to the mass exponents by Eq. (2.3) and (in the case of high multiplicity $N/M >> q$) to the intermittency exponents by [14,16]

$$\phi_q = (q-1)(1-D_q). \tag{2.6}$$

It should be noted that the high multiplicity condition $N/M >> q$ is generally not satisfied over the full range of values of M and q even for nucleus-nucleus data samples, and so the intermittency slopes can be expected to give only an indirect measure of the nature of the collisions. One might hope that the fractal analysis might be a more direct and more sensitive probe of the underlying nature of the interaction process. Bialas and Hwa [12] have pointed out that the fractal dimensions are sensitive to the presence of a phase transition: If a quark-gluon plasma is produced in central high-energy heavy ion collisions, with a subsequent first- or second-order phase transition to hadrons, then the fractal dimensions D_q will approach a limiting value with increasing order. In particular, if the phase transition is first order with a large latent heat and a short correlation length, then the $D_q = 0$.

The discussion so far has been appropriate for single events of fixed multiplicity N. In practice, however, it is generally necessary to average over an ensemble of events from the experimental database. A (horizontal) average is defined as

$$\langle G_q\rangle_h = \frac{1}{N_{\text{ev}}}\sum_{i=1}^{N_{\text{ev}}}\sum_{j=1}^{M} p_j^q\,\theta(k_j-q) \tag{2.7}$$

where N_{ev} is the number of events in the sample. (We note that the horizontal average $\langle G_q\rangle_h$ gives different results from the vertical average over bins [13].)

Finally, we must account for the effects of limited statistics. In the case of a Gaussian parent distribution, we can compute the effect $\langle G_q^{\text{stat}}\rangle$ of statistical noise due to the finite multiplicity of the events in the experimental sample: for every event with N particles in $\Delta\eta$ we distribute N particles through the $\Delta\eta$ window according to a random Gaussian with standard deviation σ and then average the calculated values of G_q over all the events in the sample. In the absence of correlations, Eq. (2.1) averaged over a Gaussian distribution will yield

$$\langle G_q^{\rm stat} \rangle = \tilde{G} M^{1-q} \ , \tag{2.8}$$

$$\tilde{G} = \frac{1}{\sqrt{q}} \left(\frac{2\chi}{\sqrt{\pi}} \right)^{q-1} (\mathrm{erf}\chi)^{-q} \mathrm{erf}(\sqrt{q}\chi) \tag{2.9}$$

where $\chi = \Delta\eta/(2\sqrt{2}\sigma)$, although for low multiplicity data where $N/M \leq 1$ (i.e., e^+e^-, μp, μd, and hadron-hadron data) the requirement of fixed total multiplicity N may lead to bin-to-bin correlations and consequent departures of $\langle G_q^{\rm stat} \rangle$ from the expected value. We therefore correct for the finite multiplicity effect by scaling $\langle G_q \rangle$ by $\tilde{G} M^{1-q} / \langle G_q^{\rm stat} \rangle$:

$$\langle G_q^{\rm dyn} \rangle = \frac{\langle G_q \rangle}{\langle G_q^{\rm stat} \rangle} \tilde{G} M^{1-q}. \tag{2.10}$$

Real dynamical effects then manifest themselves as departures of the resulting dynamical mass exponent

$$\tau_q^{\rm dyn} = \tau_q - \tau_q^{\rm stat} + q - 1 \tag{2.11}$$

from $q-1$. [Chiu *et al.* [17] have considered the case of a flat parent distribution, where $\tilde{G} = 1$. Equation (2.11) remains the same.]

III. ANALYSIS OF THE FRACTAL MOMENTS

The present analysis focuses on central collisions, i.e., interactions in which essentially all projectile nucleons participate. We have used high energy proton-nucleus emulsion data at 200 GeV [18] and 800 GeV [19] from Fermilab and KLM nucleus-nucleus data [9,20,21] for the analysis. (See Table I for a list of the various beams used.) Interactions were analyzed by studying the tracks produced by relativistic secondary pions and protons (n_s) and the tracks of slow heavily ionizing particles (N_h) produced in target fragmentation. Emission angles of all the tracks were measured with respect to the primary track with an accuracy in pseudorapidity of 0.1 units. To ensure a sample of central (small impact parameter) collisions in which a target Ag or Br nucleus is completely broken up, we select events with no charge $Z \geq 2$ fragments and with $N_h \geq 15$. These central event selection criteria have been applied to interactions of 14.6 GeV/nucleon ^{16}O and ^{28}Si, 60 GeV/nucleon ^{16}O, 200 GeV/nucleon ^{16}O and ^{32}S, 200 GeV protons, and 800 GeV protons. In addition to these samples of central events, we have also included minimum bias samples of 200 GeV/nucleon ^{16}O and ^{32}S, 200 GeV protons, and 800 GeV protons. The number of events $N_{\rm ev}$, mean pseudorapidity $\langle \eta \rangle$, and average multiplicity $\langle n_s \rangle$ within a window of width $\Delta\eta = 2$ centered on $\langle \eta \rangle$ are given in Table I. We note that $N_{\rm ev}$ given in Table I is the actual number of events satisfying the N_h cut and with at least one track in the central $\Delta\eta = 2$ window. The experimental details regarding scanning, measurement, and the method of analyzing proton-nucleus collisions in emulsion can be found elsewhere [9,19–21].

We have used Eq. (2.1) to calculate G_q in different $\delta\eta$ bins for the data as well as for the corresponding randomized events. Horizontal averaging was then performed using Eq. (2.7). In Fig. 1 the dependence of $\ln\langle G_q \rangle_h$ on the widths of the pseudorapidity windows $\delta\eta$ is shown for samples of the KLM nucleus-nucleus and proton-nucleus

TABLE I. Nucleus-nucleus and hadron-nucleus data sets.

Projectile	Energy (GeV/nucleon)	Sample	$N_{\rm ev}$	$\langle \eta \rangle$	$\langle n_s \rangle$
^{32}S	200	Central	461	3.1	134.0
^{32}S	200	Min. bias	845	3.1	42.3
^{28}Si	14.6	Central	154	1.8	58.5
^{16}O	200	Central	158	2.9	75.4
^{16}O	200	Min. bias	812	3.1	30.6
^{16}O	60	Central	244	2.2	56.6
^{16}O	14.6	Central	246	1.7	30.4
P	800	Central	284	2.7	13.6
P	800	Min. bias	1665	3.3	8.2
P	200	Central	448	2.1	11.5
P	200	Min. bias	2431	2.6	6.9

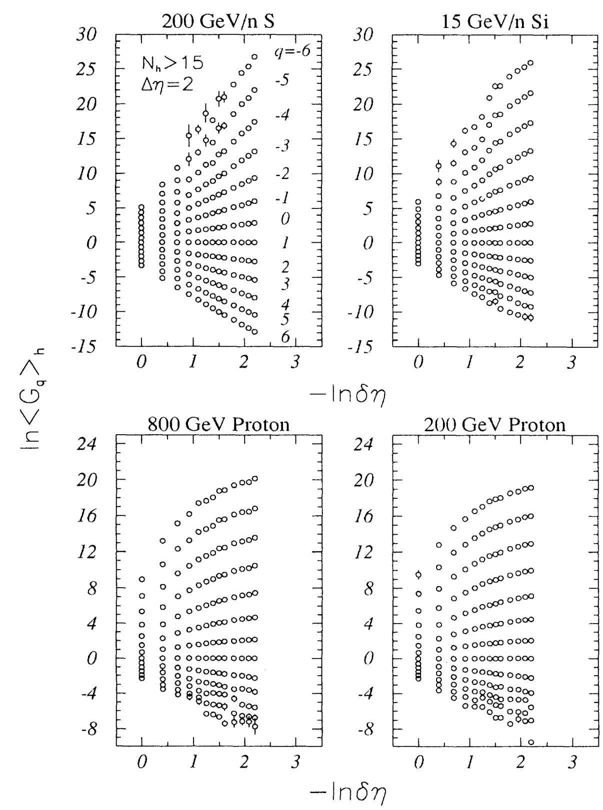

FIG. 1. Multifractal moments $\ln\langle G_q \rangle_h$ vs $-\ln\delta\eta$ for central nucleus-nucleus and proton-nucleus interactions with $N_h \geq 15$ and $\Delta\eta = 2$. Representative error bars are shown.

central collision data for integer orders $-6 \le q \le 6$. We plot the data here for the case $\Delta\eta = 2$, where we consider particles within ± 1 pseudorapidity unit of the central peak in the η distribution. The experimental resolution $\delta\eta_{\rm expt} \sim 0.1$ (corresponding to $-\ln\delta\eta = 2.3$) defines the cutoff in $-\ln\delta\eta$. The moments with positive values of q typically show approximate linearity over the full range of $-\ln\delta\eta$, but those with negative q values tend to saturate at larger values of $-\ln\delta\eta$. The saturation of $\langle G_q\rangle_h$ at negative values of q is a typical feature of the collisions with small multiplicities per bin. For example, in the ^{28}Si data set at 14.6 GeV/nucleon, and the 200 and 800 GeV proton data, the saturation effects are more pronounced than in the ^{32}S data at 200 GeV/nucleon. We note that at $-\ln\delta\eta = 1.1$ (or $M_0 = 6$), near the onset of the saturation, the average multiplicity per bin $\langle N\rangle/M_0$ is 22.3 for central 200 GeV/nucleon ^{32}S. On the other hand, for central 14.6 GeV/nucleon ^{28}Si, the corresponding value of $\langle N\rangle/M_0$ is 9.8; and for the central 200 and 800 GeV proton samples, $\langle N\rangle/M_0$ =1.9 and 2.3. Near the edges of the $\Delta\eta$ window especially, there are a significant number of empty bins for the low multiplicity data sets. The results are similar for the other data sets.

In Fig. 2 we show an expanded plot of $\ln\langle G_q\rangle_h$ versus $-\ln\delta\eta$ for 200 GeV/nucleon central ^{32}S, 800 GeV central protons, 200 GeV/nucleon central ^{16}O, and 14.6 GeV/nucleon central ^{16}O. The open circles denote the measured data and the crosses show the results of a simulation of random events (assuming a Gaussian background distribution with the same N and σ as the average measured distribution, and averaging the simulated results over $50N_{\rm ev}$ events in order to reduce event-to-event fluctuations). The solid lines are weighted straight-line fits to the data; the dashed lines show the results of fitting the simulated points. The 200 GeV/nucleon central ^{32}S and ^{16}O show relatively little scatter and very similar results for the data and the random samples. The fractal moments are clearly determined almost completely by random statistics. For the lower multiplicity data sets (800 GeV protons and 14.6 GeV/nucleon ^{16}O), we see more scatter in the data but still a reasonable agreement between data and simulations. We obtain similar results if we distribute the background according to a binomial parent distribution instead of a Gaussian.

The generalized dimensions calculated from the difference of the slopes of the fitted lines according to Eqs. (2.3) and (2.11) are shown in Fig. 3 as a function of the order q for positive q. (For negative q, the results would just reflect the saturation due to the finite multiplicity.) We have corrected for the finite multiplicity and scaled the mass exponents by $q-1$; i.e., we have plotted $D_q^{\rm dyn} = \tau_q^{\rm dyn}/(q-1)$. (We consider only $q \ge 2$ here, and treat the special case $q = 1$ in the next section.) The error bars reflect the statistical dispersion $[(\langle\tau_q^2\rangle - \langle\tau_q\rangle^2)/N_{\rm ev}]^{1/2}$ only. The results are tabulated (both for central and minimum bias events) in Table II.

If there is a phase transition, we expect D_q to approach a constant [i.e., $\tau_q^{\rm dyn} \to (q-1)D_q$]. In the particular case where there is no self-similarity, we expect $D_q = 1$. For low multiplicity e^+e^-, μp, μd, hadron-hadron, and hadron-nucleus data, departures of D_q from 1 have been reported [7,22]. Departures of D_q from 1 have also been

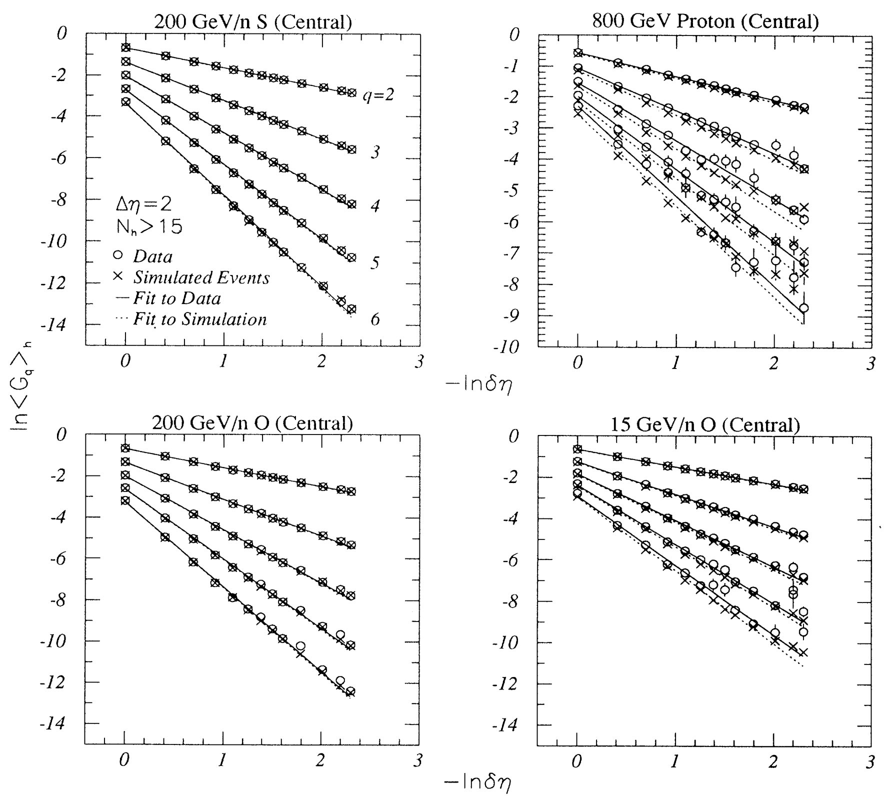

FIG. 2. $\ln\langle G_q\rangle_h$ as a function of $-\ln\delta\eta$, shown for $2 \le q \le 6$ for nucleus-nucleus and proton-nucleus interactions. The solid lines are the weighted fits to the data, while the dashed lines are those for the Monte Carlo simulations.

reported for nucleus-nucleus collisions [23]. However, in none of the previous nucleus-nucleus analyses has the separation of the dynamical effects from the statistical background been performed; i.e., previous analyses have been for τ_q, not $\tau_q^{\rm dyn}$. As is shown in Fig. 2, τ_q is dominated by the statistical background effects so that the separation of $\tau_q^{\rm dyn}$ is essential to a meaningful analysis. Hołynski *et al.* [9] have reported intermittency slopes $\phi_q/(q-1)$ of $0.005 - 0.07$ for central proton-, ^{16}O-, and ^{32}S-emulsion interactions. According to Eq. (2.6), then, although the high multiplicity condition $N/M >> q$ is not well satisfied by the present data set, we might expect at least roughly comparable values of $1 - D_q$.

The exact numerical values in Fig. 3 depend somewhat on the detailed weighting and the particular fitting prescription used, on whether we assume a Gaussian or a binomial background distribution, and on the precise value of the minimum N_h. But in all cases we see qualitatively similar behavior: There are some suggestions of a suppression of D_q below unity by a few percent for some of the data, but for the 200 GeV protons (at least for $q = 6$) we see $D_q > 1$. The D_q values for 200 GeV/nucleon ^{32}S are consistently less than 1 (0.98–0.99), as are 15 and 60 GeV/nucleon ^{16}O. The largest deviations for D_6, in the 15 and 60 GeV/nucleon ^{16}O data, give values of 0.96, corresponding to 2.8σ and 4.2σ, respectively. However, the ^{28}Si and 200 GeV/nucleon ^{16}O results fall at ~ 0.995, totally consistent, within the uncertainties, with random behavior (i.e., $D_q = 1$). For 800 GeV protons, D_4 is 3.1σ below unity, but D_6 is consistent with unity. The 200 GeV proton points are all consistent with or above $D_q = 1$. Thus, there are no clear trends in the data, either with increasing energy or multiplicity, and definite claims about departures of D_q from randomness are not possible. We see similar results for the minimum bias sample in Table II.

IV. NORMALIZED ENTROPIES

Since the intermittency slopes appear to increase with increasing q, one is initially tempted to look for a de-

TABLE II. Compilation of q, τ_q, $\tau_q^{\rm stat}$, and $D_q^{\rm dyn}$ for central and minimum bias interactions.

Projective			Central				Min. bias	
(GeV/nucleon)	q	τ_q	$\tau_q^{\rm stat}$	$D_q^{\rm dyn}$	q	τ_q	$\tau_q^{\rm stat}$	$D_q^{\rm dyn}$
	2	0.759	0.777	0.982±0.009	2	0.690	0.700	0.989±0.006
	3	1.368	1.427	0.971±0.014	3	1.209	1.258	0.976±0.008
p 800	4	1.861	1.992	0.956±0.014	4	1.722	1.755	0.989±0.010
	5	2.328	2.443	0.971±0.014	5	2.171	2.216	0.989±0.011
	6	2.874	2.912	0.992±0.016	6	2.425	2.678	0.949±0.012
	2	0.751	0.757	0.994±0.008	2	0.661	0.674	0.987±0.005
	3	1.356	1.356	1.000±0.009	3	1.161	1.180	0.991±0.007
p 200	4	1.849	1.870	0.993±0.011	4	1.597	1.631	0.989±0.009
	5	2.284	2.293	0.998±0.013	5	2.080	2.055	1.006±0.010
	6	2.846	2.698	1.030±0.012	6	2.487	2.439	1.010±0.012
	2	0.951	0.953	0.998±0.001	2	0.805	0.813	0.992±0.006
	3	1.859	1.867	0.996±0.001	3	1.560	1.588	0.986±0.008
^{32}S 200	4	2.733	2.748	0.995±0.001	4	2.279	2.379	0.967±0.012
	5	3.576	3.602	0.994±0.002	5	3.101	3.126	0.994±0.010
	6	4.396	4.444	0.990±0.002	6	3.757	3.874	0.977±0.011
	2	0.903	0.915	0.988±0.003	2	0.806	0.802	1.005±0.006
	3	1.767	1.780	0.993±0.003	3	1.534	1.542	0.996±0.009
^{16}O 200	4	2.580	2.603	0.992±0.004	4	2.260	2.286	0.991±0.011
	5	3.358	3.385	0.993±0.005	5	3.079	3.010	1.017±0.009
	6	4.096	4.140	0.991±0.006	6	3.821	3.740	1.016±0.010
	2	0.878	0.893	0.985±0.003				
	3	1.683	1.727	0.978±0.004				
^{16}O 60	4	2.444	2.493	0.984±0.006				
	5	3.133	3.248	0.971±0.007				
	6	3.780	3.971	0.962±0.009				
	2	0.835	0.843	0.992±0.005				
	3	1.568	1.603	0.983±0.007				
^{16}O 15	4	2.248	2.308	0.980±0.008				
	5	2.864	2.941	0.981±0.011				
	6	3.358	3.540	0.964±0.013				
	2	0.890	0.895	0.995±0.003				
	3	1.709	1.722	0.993±0.005				
^{28}Si 15	4	2.481	2.505	0.992±0.007				
	5	3.203	3.241	0.991±0.009				
	6	3.945	3.955	0.998±0.010				

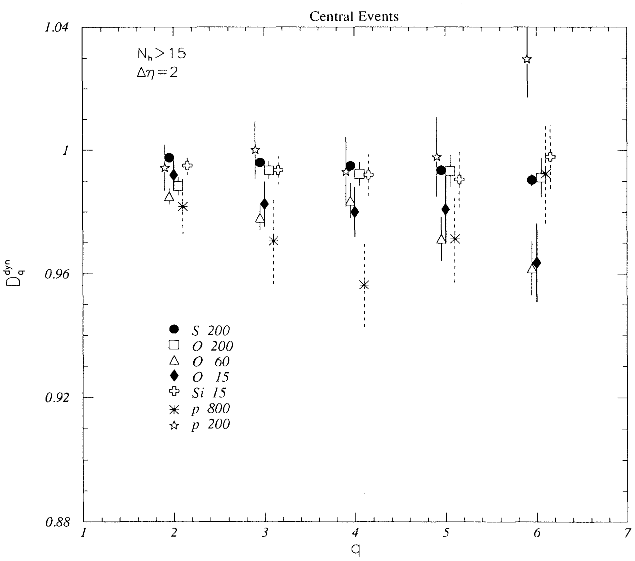

FIG. 3. D_q^{dyn} as a function of q for central events. In order to show the error bars more clearly, the plotted points have been displaced slightly to the left or right of their actual positions.

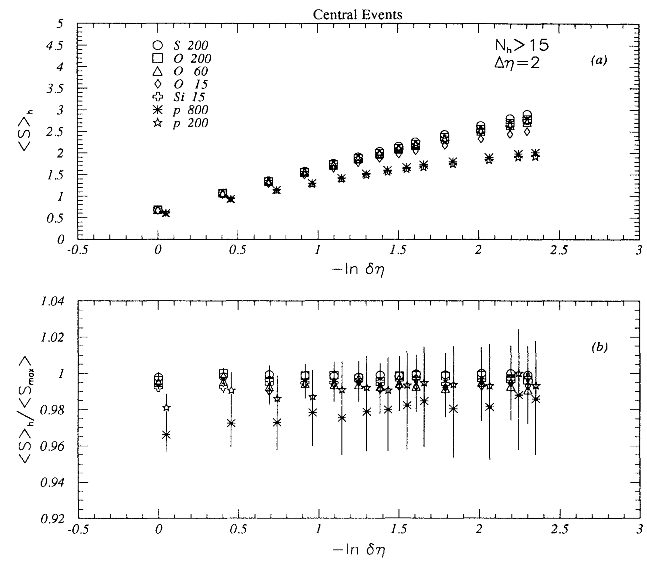

FIG. 4. (a) $\langle S\rangle_h$ as a function of $-\ln\delta\eta$ for central events. (b) $\langle S\rangle_h/\langle S_{\max}\rangle$ vs $-\ln\delta\eta$ for central events. For clarity, we have displaced the proton data by 0.05 units to the right in Figs. 4(a) and 4(b).

parture of D_q from 1 at large q. However, the error bars in Fig. 3 also grow with q. If $\phi_q/(q-1)$ is related to $1-D_q$, then one might also search for a departure of the generalized dimension D_q from unity where the statistics are best, i.e., for $q=1$. We therefore look in more detail at the particular case $q=1$, where D_1 is the derivative of the entropy S [Eq. (2.5)] with respect to $\ln\delta\eta$. In this case, Šimák *et al.* [24] have analyzed pd, αp, and $\alpha\alpha$ collisions at Fermilab and the CERN Intersecting Storage Rings (ISR), and demonstrated that the entropy derived from the negative particle multiplicity distributions obeys a simple scaling law. Furthermore, they found that $S/\ln(\sqrt{s}/m_\pi)$, where $\sqrt{s}$ is the center-of-mass energy, also obeys a scaling law when computed in various rapidity intervals, and suggested that the normalized entropy $S/S_{\max}$ should also be independent of energy.

We plot the horizontally averaged entropy $\langle S\rangle_h$ vs $-\ln\delta\eta$ for the KLM data in Fig. 4(a), and find that as projectile mass and energy increase, the points approach a straight line with characteristic slope D_1. It is interesting to ask whether this tendency to approach a straight line with increasing multiplicity and energy is a real dynamical effect, or just a result of saturation (bendover) at large M due to finite multiplicity. We therefore calculate $\langle S^{\text{stat}}\rangle$ in a manner similar to the calculation of $\langle G^{\text{stat}}\rangle$: for every event with N particles in $\Delta\eta$, we distribute each particle randomly (assuming a Gaussian background), and then calculate $\langle S^{\text{stat}}\rangle$ using Eq. (2.5). $\langle S^{\text{stat}}\rangle$ of the randomized events is then interpreted as the averaged maximum entropy $\langle S_{\max}\rangle$ attainable for every event. It should be pointed out that our procedure for calculating $\langle S_{\max}\rangle$ differs from that of Šimák *et al.* For hadron-hadron interactions at $\sqrt{s}\geq 20$ GeV, they defined $S_{\max}$ as the entropy in the maximum rapidity interval $y_{\max}=\ln(\sqrt{s}/m_\pi)$ of the produced particles:

$$S_{max}=\ln\overline{N}+1-\left(\frac{a}{\ln\overline{N}}+\frac{b}{(\ln\overline{N})^2}+\frac{c}{(\ln\overline{N})^3}\right)+O\left(\frac{1}{(\ln\overline{N})^4}\right) \tag{4.1}$$

where a, b, and c are constants and $\overline{N}$ is the average charged multiplicity. Evidently, $S_{\max}$ calculated from Eq. (4.1) is a fixed number, while in this work $\langle S_{max}\rangle$ has a roughly linear dependence on $-\ln\delta\eta$. We should also note that our definition of S [Eq. (2.5)] includes the θ function from Eq. (2.1); this is another difference from the treatment of Šimák *et al.*

We plot $\langle S\rangle_h/\langle S_{\max}\rangle$ in Fig. 4(b) for central events with $N_h\geq 15$ and $\Delta\eta=2$ (i.e., central events with tracks with pseudorapidities near the peak of the η distribution). Two interesting observations can be made from Figs. 4(a) and 4(b).

(1) Although the entropy $\langle S\rangle_h$ increases with increasing multiplicity, the normalized entropy $\langle S\rangle_h/\rangle S_{\max}\rangle$ is essentially constant and equal to the maximum value of unity.

(2) Although $\langle S\rangle_h$ for protons is significantly less than for the nuclei, the normalized entropy for protons is somewhat closer to the value for nuclei. $\langle S\rangle_h/\langle S_{\max}\rangle$ for the protons only falls significantly below the nuclear value (given the large size of the proton error bars) for the smallest values of $-\ln\delta\eta$.

In Fig. 5, however, we show the normalized entropy for minimum bias samples and see a clear difference between proton-nucleus and nucleus-nucleus collisions. We note in addition that the values of $\langle S\rangle_h/\langle S_{\max}\rangle$ for nuclei are now

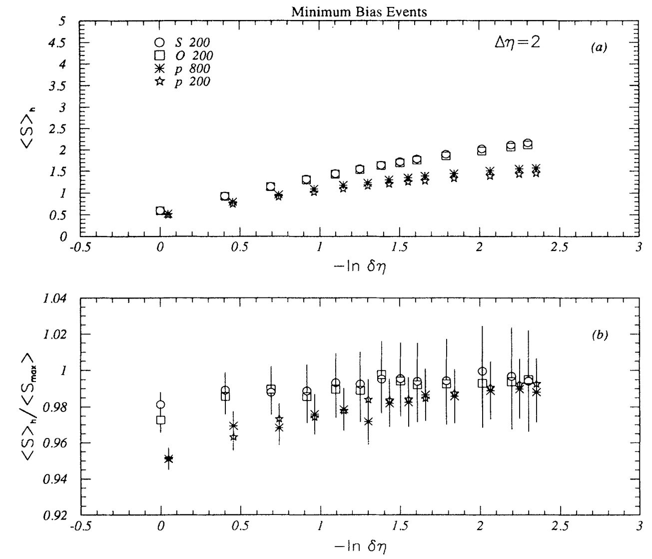

FIG. 5. Same plots as in Fig. 4 but for minimum bias events.

less than 1, suggesting that the minimum bias sample (in particular the proton sample) is less chaotic than the sample of central events.

V. CONCLUSIONS

We have determined the slopes of the horizontally averaged multifractal moments, explicitly correcting for finite statistics, to obtain $\tau_q^{\rm dyn}$ for projectiles from protons to ^{32}S at energies from 14.6 to as high as 800 GeV/nucleon. Only for central collisions of ^{32}S at 200 GeV/nucleon do we observe a possible departure from $\tau_q^{\rm dyn} = q - 1$ (or, equivalently, $D_q = 1$). For all of the other systems investigated, D_q is consistent with unity, within the uncertainties, or exhibits a large scatter in values as a function of q. A constant value of D_q implies either a second-order phase transition [12] or (in the particular case $D_q = 1$) random, nonscaling behavior in the particle production process. We find no systematic trends in D_q with projectile energy or projectile mass.

It should be emphasized that for central nucleus-nucleus collisions, the fluctuations are small and are dominated by statistical background. It is therefore a crucial and rather difficult task, often dependent on the detailed analysis method, to disentangle dynamical and statistical fluctuations.

The case of ^{32}S is the most interesting from the point of view of having the smallest statistical uncertainty in the derived values of $\tau_q^{\rm dyn}$. The central collisions of ^{32}S with Ag/Br targets exhibit a decreasing value of D_q with increasing q: the $q = 6$ point is 5 standard deviations below unity. For other central collision data sets, the deviations are typically at the 1–2 σ level. In the proton-nucleus interactions, both minimum bias and central, as well as the minimum bias nucleus-nucleus collisions, no trends in the data are evident. The ^{32}S represents the heaviest system that has been investigated and shows the smallest intermittency slopes [9]. Yet, it is only this system that shows any possible evidence of nonrandom behavior. It should be noted, however, that the error bars are purely statistical, and that the systematic effects of correlations among the data points (since the data points all represent the same experimental data with just different binning) have been neglected. Especially given the lack of evidence for $D_q < 1$ in the rest of the data, it is not at all clear that the low values of D_q for ^{32}S are significant.

Departures of D_q from 1 have been reported by e^+e^-, μp, μd, and hadron-hadron experiments. Previous reports of nonrandom behavior in nucleus-nucleus or hadron-nucleus experiments have failed to account for the statistical fluctuations in the data. The fact that we do not see such effects in our corrected proton-nucleus and minimum bias nucleus-nucleus results might indicate the onset of a transition between the lepton-hadron regime and the higher multiplicity nucleus-nucleus and proton-nucleus regime studied here. Such behavior is consistent with that suggested by Bialas and Hwa [12], who remark that if a quark-gluon plasma is produced in central nucleus-nucleus collisions, then one might expect that D_q depends on q for *low* values of transverse energy and multiplicity, and D_q becomes independent of q for large transverse energy and multiplicity. (They point out, however, that cascading can also produce D_q = constant, so that, again, this condition is not a guarantee of a phase transition.) Here, again, the ^{32}S results stand as a possible contradiction.

We have also looked in more detail at the particular case $q = 1$, and studied the normalized entropy $\langle S\rangle_h/\langle S_{\rm max}\rangle$ as a function of pseudorapidity. For minimum bias events especially, we see normalized entropies somewhat less than unity and a clear difference between proton-nucleus and nucleus-nucleus collisions. These results are consistent with intermittency analyses and suggest nonrandom behavior in proton-hadron and low multiplicity collisions, but not in higher multiplicity central heavy ion collisions. In this case, the central ^{32}S results show normalized entropies near unity.

Although our central nucleus-nucleus events do not reveal the same departures of D_q from unity reported for lower multiplicity data, we note that there is no inconsistency with intermittency analyses. Intermittency slopes have been found to be smaller for more complex (central nucleus-nucleus) interactions than for simpler (e^+e^-, meson-proton) interactions [9]. The intermittency slopes $\phi_q/(q-1)$ ~0.5–7% measured by Hołyński *et al.* [9] correspond [from Eq.(2.6)] to values of $1 - D_q$ smaller than or comparable to the error bars in the present fractal analysis (Fig. 3 and Table II).

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