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Low-energy $\pi\pi$ scattering in the large-N_c limit of QCD

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An extension of Weinberg's approach is applied to the $\pi\pi$-scattering amplitude of the leading order in $1/N_c$. The resulting sum rules (SR's) restrict the masses and widths of $\pi\pi$ resonances providing duality-like properties of the $\pi\pi$ amplitude and enable us to express constants of chiral Lagrangians in terms of resonance parameters. The numerical verification of SR's displays agreement with present experimental information and consistency with the results of some models.

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I. INTRODUCTION

The method of effective Lagrangians (EL's) is widely used in modern particle physics due to the simplicity and convenience of its applications. The low-energy dynamics of hadrons is one of the areas where this method plays an essential role because the problem of a correct description of the hadronization regime from the QCD standpoint has not yet been resolved.

The standard way to obtain the EL for some "light" degrees of freedom of a given theory consists of performing the integration over all the rest ("heavy") degrees in a generating functional, the result giving one effective theory, which, as usual, contains nonlocal interaction terms. Expanding, further, nonlocal interactions in a series of local operators and neglecting (if possible) higher terms, one obtains the approximate EL for some selected ("light") degrees of freedom of the underlying theory. The structure of its vertices corresponds to the peculiarity of "low-energy" dynamics, while the information on "high-energy" physics is contained in coefficients of the expansion.

The EL technique is intensively applied now for the description of low-energy strong interactions (see, for example, [1–4]) as the long-distance QCD manifestation. The main difficulty arising in this way in comparison with the picture above is that, from the QCD point of view, hadrons, while being the collective excitations, cannot (in contrast with quarks and gluons) be treated as any elementary degrees of freedom. So, the problem of the direct derivation of EL's for light mesons from the first principles of QCD is, on the whole, that of the description of the long-distance QCD regime. The latter task, as mentioned above, is extremely hard and has not been solved until now.

Nevertheless, there exist methods [5–8] (based on some model assumptions) for the calculation of low-energy EL's for pions, the results of these methods being close enough to those obtained in experiments. For example, the EL for pions to the fourth order in momenta has been derived by the integration of the QCD nontopological chiral anomaly [5,6]. One of the important features of EL's provided by these methods is that their structure corresponds to that of QCD in the limit of a large color number N_c. Keeping in mind this circumstance we shall consider QCD in the leading order of $1/N_c$ expansion and obtain some relations between the parameters of the resonance spectrum and those of low-energy interactions of pions.

It is well known that in the limit of large color number N_c QCD is reduced to the theory of weakly interacting mesons [9,10] one of which, namely, the pion, being massless in the exact chiral $SU(2)\otimes SU(2)$ symmetry limit $m_u = m_d = 0$ (the so-called chiral limit). Thus, to obtain the EL for pions [effective chiral Lagrangian (ECL)] in the leading order of $1/N_c$ expansion, one has to integrate over all meson fields beyond that of the pion or, that is the same, to express the low-energy pion coupling constants in terms of heavy meson parameters. This latter task may be solved with the help of the expansion of the amplitude (calculated in the tree approximation, i.e., in the limit of $N_c \to \infty$) of an appropriate pion process in a power series of external momenta and subsequent identification of the corresponding coefficients with those

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extracted from the ECL.

Unfortunately, the coupling constants of the full meson theory are unknown. To overcome this difficulty one has to involve some additional information. It seems meaningful to use the phenomenological information on the asymptotic behavior of the scattering amplitudes at small t and large s. Indeed, if one supposes that the full amplitude for $\pi\pi$ scattering does not change drastically its asymptotic behavior in the limit $N_c\to\infty$, he must conclude that the sum of the graphs with an arbitrary fixed number of loops cannot violate the asymptotics of the physical amplitude. Hence, we arrive at the conclusion that it is natural to demand the observance of the asymptotic restrictions (AR's): the amplitude of any given process calculated in the limit of large color number N_c (or, the same, in the tree approximation) must not violate the phenomenologically known (i.e., Regge) asymptotic behavior.

Weinberg [11] (see also [12]) first recognized the significance of AR's. In his own approach these restrictions (for the case $t=0$ and pion mass $\mu=0$) had been exploited for the investigation of some purely algebraic aspects of broken chiral symmetry $SU(2)\otimes SU(2)$ (see, also [13–19] where Weinberg's results were generalized for more complicated groups, and [20–24] where a similar approach was used to study Reggeon couplings). In this paper we shall take advantage of another, slightly different, approach [25] which is more convenient for our purposes. The only assumptions on which our approach is based are the following: (a) the equivalence of QCD in the large-N_c limit to the theory of weakly interacting mesons (tree approximation); (b) Weinberg's suggestion [11] that at $t=0$ the tree-level amplitude does not violate Regge asymptotics is valid also in some small region of $t\neq 0$.

The result of the application of this approach to the pion elastic-scattering process may be written in the form of the sum rules (SR's) for the parameters of resonance spectrum and coupling constants. This (in the case of $t=0$) was shown in [25–28]; this paper is devoted to the extension of corresponding results concerning the elastic $\pi\pi$ scattering to the case of nonzero values of t. We show that the SR obtained in this way can be divided in two large groups. SR's from the first group express the low-energy pion-pion coupling constants in terms of masses and widths of $\pi\pi$ resonances. The second group consists of SR's expressing the restrictions of the resonance spectrum themselves. The latter group of SR's originates from a certain dualness property (i.e., some infinite sum of poles in t and u turns out to be equal to another, also infinite, sum of poles in s and u) of the $\pi\pi$ amplitude which, as we show, follows from the condition of the compatibility of the AR's with requirements of exact symmetries (Bose and crossing).

The results of our calculation of ECL parameters of the p^4 order (first group of SR's) are compared with the experimental values as well as with the values obtained in low-energy QCD models. Then it is shown that some SR's of the second group (those which can be verified) are confirmed by existing experimental data. This provides extra confirmation of the reasonability of the approach.

Perhaps the most interesting result of this paper is connecting the values of some low-energy $\pi\pi$ parameters with those obtained with the help of Veneziano-type models. This result may be treated as a hint on the close relation between dual models and the large-N_c limit of QCD at low energy.

II. THE STRUCTURE OF $\pi\pi$-SCATTERING AMPLITUDE IN THE LARGE-N_c LIMIT

Let us consider the elastic $\pi\pi$-scattering process

$$\pi_a(k_1)+\pi_b(k_2)\to\pi_c(k_3)+\pi_d(k_4)\ .$$

($a,b,c,d=1,2,3$ are the isotopic indices and $k_1,\ldots,k_4$ are the pion momenta.) Its amplitude M^{abcd} can be written in the form

$$M^{abcd}=\delta^{ab}\delta^{cd}A+\delta^{ac}\delta^{bd}B+\delta^{ad}\delta^{bc}C\ , \tag{1}$$

where A,B,C are the scalar functions of Mandelstam variables s,t,u

$$s=(k_1+k_2)^2,\quad t=(k_1-k_3)^2,\quad u=(k_1-k_4)^2\ ,$$

obeying the Bose symmetry requirements

$$\begin{aligned} &A(s,t,u)=A(s,u,t)\ ,\\ &B(s,t,u)=A(t,s,u)\ ,\\ &C(s,t,u)=A(u,t,s)\ . \end{aligned} \tag{2}$$

The amplitudes M^I

$$\begin{aligned} &M^0=3B+A+C\ ,\\ &M^1=A-C\ ,\\ &M^2=A+C\ , \end{aligned} \tag{3}$$

with a definite t-channel isospin value I in accordance with Regge theory prescriptions must possess (at fixed value of t) the following asymptotics at $s\to\infty$:

$$M^I\sim\beta_I(t)s^{\alpha_I(t)}\ , \tag{4}$$

where

$$\alpha_I(t)=\alpha_I(0)+t\alpha_I' \tag{5}$$

and

$$\alpha_0(0)=1,\quad \alpha_1(0)\simeq 0.5,\quad \alpha_2(0)<0\ . \tag{6}$$

Thus, according to assumptions (a) and (b) (Sec. I) and formulas (3)–(6) we demand the following behavior of the amplitudes A,B,C at small t (the exact meaning of the term "small" will be explained lower) and large s:

$$\begin{aligned} &A(s,t)\underset{\substack{s\to\infty\\ t\sim 0}}{\longrightarrow} 0\ ,\\ &B(s,t)\underset{\substack{s\to\infty\\ t\sim 0}}{\longrightarrow} B_0(t)\ ,\\ &C(s,t)\underset{\substack{s\to\infty\\ t\sim 0}}{\longrightarrow} 0\ . \end{aligned} \tag{7}$$

It should be stressed here that we do not suppose that functions A,B,C possess the correct Regge asymptotics in the leading order of $1/N_c$ expansion; we suggest only that they do not violate it. This suggestion is quite enough for the derivation of all of our subsequent results.

To proceed further we need to display the detailed structure of the scattering amplitude in the tree approximation (or, the same, in the leading order in $1/N_c$). The isoscalar amplitudes A,B,C can be written in the form (see, for example, [29])

$$A(s,t)=\sum_{\text{odd}} V_1(J=2l+1)\frac{P_J[z_1(t)]}{u-M_1^2}-\sum_{\text{even}} V_0(J=2l)\frac{P_J[z_0(t)]}{s-M_0^2}+\cdots \ , \tag{8}$$

$$B(s,t)=-\sum_{\text{odd}} V_1(J=2l+1)P_J[z_1(t)]\left[\frac{1}{s-M_1^2}+\frac{1}{u-M_1^2}\right]+B_0(t)+\cdots \ , \tag{9}$$

$$C(s,t)=\sum_{\text{odd}} V_1(J=2l+1)\frac{P_J[z_1(t)]}{s-M_1^2}-\sum_{\text{even}} V_0(J=2l)\frac{P_J[z_0(t)]}{u-M_0^2}+\cdots \ , \tag{10}$$

where

$$z_i(x)=1+2x/h_i,\quad h_i=M_i^2-4\mu^2\quad (i=0,1)\ ,$$

μ is the pion mass and constants $V_0(J)$ and $V_1(J)$ can be related to the decay widths of the corresponding resonances:

$$V_1(J)=8\pi(2J+1)\frac{M_1^2}{k_\pi}\Gamma(R\to\pi\pi)\ ,$$

$$V_0(J)=\tfrac{2}{3}8\pi(2J+1)\frac{M_0^2}{k_\pi}\Gamma(R\to\pi\pi)\ ,$$

k_π being the pion c.m. system (c.m.s.) momentum.

In Eqs. (8)–(10) the summation is implied to be performed over all resonances of the same (odd or even) spin J ($J=2l+1$ or $J=2l$) and different masses M_I ($I=0,1$ corresponds to isospin value) as well as over all values of J ($l=0,1,\ldots$). Later on we shall imply such summation, the corresponding symbols being sometimes omitted.

The ellipses in (8)–(10) mean terms explicitly violating the restrictions (7). Such terms may originate from the pointlike 4π vertices as well as from off-shell $\pi\pi R$ interactions. The graphs with t-channel poles are an important source of such terms. The unknown function $B_0(t)$ in (9) gives an example of a term exactly of this nature.

In order to satisfy asymptotic restrictions (7), we demand the sum of terms omitted in (8)–(10) to be equal to zero; this implies that there must exist a fine mutual cancellation among these terms. So, expressions (8)–(10) (with ellipses being dropped out) provide the complete form of the $\pi\pi$ amplitude in the leading order of $1/N_c$ expansion with the AR being taken into account.

III. BOSE SYMMETRY REQUIREMENTS AND SELF-CONSISTENCY SUM RULES

Even a short glance at formulas (8)–(10) is enough to notice that they do not possess the Bose symmetry [Eqs. (2)] required. The origin of the puzzle is obvious: starting from *symmetric* expressions constructed of resonance exchanges and pointlike 4π vertices, we then take advantage of *asymmetric* conditions (4) of the proper asymptotic behavior at small t and large s which explicitly makes the choice between s and t variables. Of course, we cannot reconcile ourselves to the loss of such a fundamental property of an amplitude as Bose symmetry; hence, we are forced to consider the way to preserve it. It is not difficult to understand that the only possibility consists of imposing certain limitations on the resonance parameters $V_I(J)$ and $M_I^2(J)$ appearing in expressions (8)–(10).

To obtain an explicit form of these limitations, let us analyze at the beginning the first of Eqs. (2), which dictates the function $A(s,t,u)$ to be symmetric under the interchange of t and u. With the help of Eq. (8) one obtains

$$\sum_{\text{odd}} V_1(J=2l+1)\left[\frac{P_J[z_1(y)]}{x-M_1^2}-\frac{P_J[z_1(x)]}{y-M_1^2}\right] =\sum_{\text{even}} V_0(J=2l)\frac{P_J[z_0(y)]-P_J[z_0(x)]}{(x+y)-(M_0^2-4\mu^2)}\ . \tag{11}$$

Similarly, the second equation in Eqs. (2), connecting the amplitude $B(s,t,u)$ with the function $A(s,t,u)$, may be written [with the help of (8) and (9)] in the form

$$\sum_{\text{odd}} V_1(2l+1)\left\{P_J[z_1(y)]\left[\frac{1}{x-M_1^2}-\frac{1}{x+y+(M_1^2-4\mu^2)}\right]-\frac{P_J[z_1(x)]}{x+y+(M_1^2-4\mu^2)}\right\}-\sum_{\text{even}} V_0(2l)\frac{P_J[z_0(x)]}{y-M_0^2}=B_0(y)\ . \tag{12}$$

It can be shown that the substitution of Eqs. (8) and (10) into the last of Eqs. (2) results in the condition completely equivalent to (11). Moreover, it is possible to prove that Eq. (11) is, in its turn, a direct consequence of (12).

Thus, Eq. (12) expresses the general form of Bose symmetry restrictions on the resonance parameters $V_I(J)$ and M_I^2

appearing in the representation (8)–(10) of the $\pi\pi$-scattering amplitude (1) in the leading order of $1/N_c$ expansion. This equation may be regarded as a compact record of an infinite series of self-consistency sum rules for the parameters mentioned above. Indeed, expanding both sides of (12) in a power series in x [this procedure is permissible because Eqs. [(8)–(10) are correct at small values of x,y] and comparing the corresponding coefficients one obtains

$$B_0(t)=-\sum V_1(2l+1)P_J[z_1(t)]\left\{\frac{1}{M_1^2}+\frac{1}{h_1+t}\right\}-\sum\frac{V_1(2l+1)}{h_1+t}+\sum\frac{V_0(2l)}{M_0^2-t} \qquad (13)$$

and

$$-b_m\delta_{n,0}+\sum V_0(2l)\pi_J^{(k)}M_0^{-2(m+1)}h_0^{-k}$$
$$=\sum V_1(2l+1)\left\{\pi_J^{(m)}M_1^{-2(k+1)}h_1^{-m}+h_1^{-(m+n+1)}\left[\sum_{l=k}^{m+k}(-1)^l\pi_J^{(m+n-l)}C_l^k+\sum_{l=m}^{m+k}(-1)^l\pi_J^{(m+n-l)}C_m^{l-m}\right]\right\}. \qquad (14)$$

Here, $m,k=0,1,\ldots$; C_l^k are binomial coefficients; b_m are defined as

$$B_0(y)=\sum_{m=0}^{\infty}b_m y^m$$

and

$$\pi_J^{(k)}\equiv\frac{2^k}{k!}\frac{\partial^k}{\partial z^k}P_J(z)\Bigg|_{z=1}=\frac{1}{(k!)^2}\frac{(J+k)!}{(J-k)!}.$$

Equation (13) expresses the previously unknown function $B_0(t)$ in terms of resonance parameters only. So, the functional dependence of the $\pi\pi$-scattering amplitude (1) of its arguments s,t,u is now completely determined. All unknown coupling constants of pointlike 4π vertices proved to be excluded.

The representation (14) of the infinite set of self-consistency SR's is too complicated. Thus, for the sake of visuality, it would be useful to write down some first SR's explicitly. In the chiral limit $\mu=0$ (which simplifies considerably the form) these SR's (of order M^{-4} and M^{-6}) are

$$\sum\frac{V_1(2l+1)}{M_1^4}[\pi_{2l+1}^{(1)}-1]=\sum\frac{V_0(2l)}{M_0^4}\pi_{2l}^{(1)}, \qquad (15)$$

$$\sum\frac{V_1(2l+1)}{M_1^6}[\pi_{2l+1}^{(1)}-4]=-\sum\frac{V_0(2l)}{M_0^6}\pi_{2l}^{(1)},$$
$$\sum\frac{V_1(2l+1)}{M_1^6}[\pi_{2l+1}^{(2)}-\pi_{2l+1}^{(1)}+3]=\sum\frac{V_0(2l)}{M_0^6}\pi_{2l}^{(2)}. \qquad (16)$$

The relatively fast convergence of SR's (15) and (16) permits one to check their validity numerically (with the help of experimental data [30]). It should be noted that, due to the presence of multipliers $\pi_{2l}^{(1)}$, the right-hand side (RHS) of these SR's contains no contributions of scalar mesons ($J=0$), this circumstance being of importance since the corresponding data are extremely unreliable. The results of the numerical verification of Eqs. (15) and (16) together with the detailed analysis of the influence of possible corrections will be presented in the next section.

Considering Eq. (14) (in the chiral limit—for the sake of simplicity) one obtains, after some algebraic transformations, the following set of SR's ($n,k=1,2,\ldots$):

$$\sum_{\text{odd}}\frac{V_1(2l+1)\pi_J^{(k)}}{M_1^{2(n+1)}}+\sum_{\text{even}}\frac{V_0(2l)\pi_J^{(k)}}{M_0^{2(n+1)}}=\sum_{\text{odd}}\frac{V_1(2l+1)\pi_J^{(n-k)}}{M_1^{2(n+1)}}+\sum_{\text{even}}\frac{V_0(2l)\pi_J^{(n-k)}}{M_0^{2(n+1)}}. \qquad (17)$$

The remarkable feature of (17) is that all terms are positive definite. So, since $\pi_J^{(k)}\sim J^k$ and $M_J^2\sim J$ at large J, we conclude that the values of constants $V_0(J)$ and $V_1(J)$ must decrease faster than J^{-n} (for arbitrary n) to provide the convergence in the usual sense of this latter SR.

Our next note concerns some characteristic features of self-consistency conditions (11) and (12). From the formal point of view they demonstrate the properties peculiar for dual models: the summation of poles in one set of variables results in a sum of poles in another one (or, poles in some combination of both variables). In other words, to satisfy the conditions of the large-N_c limit of QCD, AR, and Bose symmetry simultaneously, the $\pi\pi$-scattering amplitude must possess certain dual properties; this means that (at infinitely large N_c) the spectrum of $\pi\pi$ resonances must be infinite and strongly restricted by Eqs. (14).

It is our point here to explain the correct meaning of the above-used term "small values of t." The best example of an explanation is given by the second of Eqs. (7) where, in accordance with (4), we admit the presence of $B_0(t)$. This term does not vanish at $s\to\infty$, and, hence, its presence coincides with the Regge theory prescriptions only for

$$|t|\le 1/\alpha_0'.$$

Similar limitations can be obtained from the consideration of two other equations (7). Then, in the course of our calculations we use the power-series expansions in t. These expansions are valid only if

$$|t|<M^2,$$

M being the mass of the lightest $\pi\pi$ resonance ($M\ge 0.5$ GeV). Remembering that $\alpha_I'\sim(1\text{ GeV})^{-2}$ ($I\ne 0$) and $\alpha_0'\sim(0.5\text{ GeV})^{-2}$, one concludes that the term "small value of t" stands for

$$|t|<(0.5\ \text{GeV})^2\ . \tag{18}$$

Of course, (18) does not mean that Eqs. (8)–(10) become invalid for larger values of $|t|$. It means only that to obtain the correct representation of $B_0(t)$ for $|t|\geq(0.5\ \text{GeV})^2$ one must use a suitable power-series expansion in (12), the SR (14) remaining unchanged.

IV. PARAMETERS OF THE EFFECTIVE CHIRAL LAGRANGIAN AND $\pi\pi$ RESONANCES

Formulas (8)–(10) express the $\pi\pi$ amplitude in terms of parameters of the resonance spectrum only. They are valid for any values of s. Near threshold, however, it is more convenient to deal with the low-energy expansion of the form

$$A(s,t)=\sum_{i,j} a_{ij}s^i t^j\ . \tag{19}$$

The coefficients a_{ij} appearing in (19) can be related to the parameters of the ECL. On the other hand, these coefficients can be expressed in terms of resonance parameters; to do this one has to expand (8) in a power series of s and t. So, it becomes possible to calculate numerically at least some of the ECL parameters with the help of experimental data [30] on $\pi\pi$ resonances.

We carry out this program in the chiral limit. The main advantage of this limit is that, while being a correct enough approximation to the real world, it simplifies considerably the resulting formulas. We use the notation F_0, L_1, L_2, L_3 of [2,4] for the low-energy constants of the ECL; their relations with the coefficients a_{ij} of (19) in the chiral limit are

$$a_{00}=0,\quad a_{10}=\frac{1}{F_0^2},\quad a_{01}=0\ ,$$

$$a_{20}=\frac{4(2L_2+L_3)}{F_0^4},\quad a_{11}=a_{02}=\frac{8L_2}{F_0^4}\ .$$

Thus, expanding (8) in power series (19) and comparing the corresponding coefficients with those obtained with the help of an ECL [2], one finds

$$\begin{aligned}
0&=-\sum\frac{V_1(2l+1)}{M_1^2}+\sum\frac{V_0(2l)}{M_0^2}\ ,\\
1/F_0^2&=\sum\frac{V_1(2l+1)}{M_1^4}+\sum\frac{V_0(2l)}{M_0^4}\ ,\\
0&=-\sum\frac{V_1(2l+1)}{M_1^4}(\pi_J^{(1)}-1)+\sum\frac{V_0(2l)}{M_0^4}\pi_J^{(1)}\ ,\\
\frac{4(2L_2+L_3)}{F_0^4}&=-\sum\frac{V_1(2l+1)}{M_1^6}+\sum\frac{V_0(2l)}{M_0^6}\ ,\\
8L_2/F_0^4&=-\sum\frac{V_1(2l+1)}{M_1^6}(\pi_J^{(1)}-2)+\sum\frac{V_0(2l)}{M_0^6}\pi_J^{(1)}\ ,\\
8L_2/F_0^4&=-\sum\frac{V_1(2l+1)}{M_1^6}(\pi_J^{(2)}-\pi_J^{(1)}+1)\\
&\quad+\sum\frac{V_0(2l)}{M_0^6}\pi_J^{(2)}\ .
\end{aligned}\tag{20}$$

Equations (20) have the form of the SR for the parameters of the ECL. In contrast with SR (14) expressing the consequences of our assumptions (a) and (b) (see Sec. I), they are caused by an external (for our approach) reason—the chiral symmetry restrictions. These equalities (in addition to their being interesting by themselves) provide the possibility to estimate numerically the values of the above-mentioned ECL parameters and compare results with those obtained from experiment (see, for example, [2]). Hence, both groups [(20) and (14)] of SR's can be used for the verification of the correctness of our approach.

Before advancing the numerical calculations let us discuss possible sources of uncertainties. These are (a) the uncertainties of experimental data, (b) the incompleteness of information on heavy ($M\geq 2$ GeV) mesons, (c) corrections caused by the next orders of $1/N_c$, (d) insufficiently fast convergence of the SR's under consideration, (e) corrections due to nonvanishing light quark masses. To avoid troubles caused by points (a), (b), and (d), one has to deal with rapidly converging SR's containing contributions of well-established mesons only. Then, the strongest $1/N_c$ corrections (strong violation of the Zweig rule) are expected in the scalar channel which is the most doubtful from the experimental point of view also.

Therefore one expects the results of numerical estimations to be most reliable for those (rapidly converging) SR's which do not obtain the contribution of the scalar channel. With respect to the corrections of point (e) we would like to note that they are commonly believed to be small ($\sim$5–10 %).

Keeping in mind all the reasons expressed above, we select for the numerical verification the following SR obtained from (14) and (20):

$$\sum\frac{V_1(2l+1)}{M_1^4}\pi_J^{(1)}=\sum\frac{V_0(2l)}{M_0^4}\pi_J^{(1)}+\sum\frac{V_1(2l+1)}{M_1^4}\ , \tag{21}$$

$$\sum\frac{V_1(2l+1)}{M_1^6}\pi_J^{(1)}+\sum\frac{V_0(2l)}{M_0^6}\pi_J^{(1)}=4\sum\frac{V_1(2l+1)}{M_1^6}\ , \tag{22}$$

$$1/F_0^2=\sum\frac{V_1(2l+1)}{M_1^4}+\sum\frac{V_0(2l)}{M_0^4}\ , \tag{23}$$

$$4L_2/F_0^4=\sum\frac{V_1(2l+1)}{M_1^6}\ , \tag{24}$$

$$4(2L_2+L_3)/F_0^4=-\sum\frac{V_1(2l+1)}{M_1^6}+\sum\frac{V_0(2l)}{M_0^6}\ . \tag{25}$$

The corresponding numerical results (with contributions of all the mesons placed in [30] being taken into account) are

$$(51\pm10)\ \text{GeV}^{-2}=(38\pm10)\ \text{GeV}^{-2}\ , \tag{21$'$}$$

$$(390\pm20)\ \text{GeV}^{-4}=(370\pm20)\ \text{GeV}^{-4}\ , \tag{22$'$}$$

$$F_0=(90\pm5)\ \text{MeV}\ \underline{(94.3)}\ , \tag{23$'$}$$

$$L_2/F_0^4=(23\pm1)\ \mathrm{GeV}^{-4}\ \ \underline{(23\pm8)}\ , \tag{24'}$$

$$(2L_2+L_3)/F_0^4=(20\pm20)\ \mathrm{GeV}^{-4}\ \ \underline{(12\pm10)}\ . \tag{25'}$$

The underlined numbers on the RHS of (23')–(25') correspond to the values derived independently (see [4]) from the analysis of existing experimental information (scattering lengths, slope parameters, etc.); we would like to recall that in the course of our calculations we are using the data on masses and decay widths only.

Thus, the confirmation of the results (21)–(25) of our approach by experimental data on meson spectrum parameters demonstrated by Eqs. (21')–(25') looks to be quite convincing. Therefore, it would be interesting to draw this approach for the calculation of low-energy coefficients of amplitudes for various processes (such as $\gamma\pi\to\pi\pi$, πK elastic scattering, etc.). The corresponding results will be published elsewhere; here instead we would like to point another way to use the capabilities of the approach.

To calculate ECL coupling constants one can use models [5,6,8] based on consideration of the QCD chiral anomaly. These models, however, tell us nothing about the parameters of the resonance spectrum. Hence, it seems natural to combine the capabilities of two approaches and, using our SR along with the values of ECL parameters from the above models, to try to obtain some limitations on low-lying meson masses. For example, using the results [5,6,8]

$$2L_1=L_2\ , \tag{26}$$

$$2L_2+L_3=0\ , \tag{27}$$

$$L_2=\frac{1}{12}\frac{N_c}{16\pi^2} \tag{28}$$

[Eq. (26) being the consequence of the large-N_c limit] and our SR (23)–(25), one can obtain the inequality

$$M_\sigma^2\le\frac{F_0^2[1-g_{\rho\pi\pi}^2F_0^2/M_\rho^2]}{4(3L_2+L_3)}\simeq(800\ \mathrm{MeV})^2\ , \tag{29}$$

where M_σ is the mass of the lightest scalar (and isocalar) meson. Thus, we arrive at the conclusion that the existence of a light scalar meson is the necessary condition for the correspondence of results of approaches in question. It is interesting to note that, if the approximate equality in (29) is reached, this scalar meson would be wide enough ($g_\sigma\sim g_\rho$). Further, if one drops out all contributions but that of ρ and σ mesons in our SR [see the first of SR's (20) and (25)] and uses Eq. (27), he obtains the famous Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSFR) relation [31]

$$M_\rho^2=2F_\pi^2g_{\rho\pi\pi}^2\ . \tag{30}$$

This example shows that corrections to (30) have their origin in contributions of heavy resonances and those of the next order in the $1/N_c$ expansion.

V. PARAMETERS OF ECL, DUAL MODELS, AND THE QCD CHIRAL ANOMALY

As shown above, the spectrum of $\pi\pi$ resonances in the large-N_c limit must possess certain features to ensure the dualness property of the $\pi\pi$-scattering amplitude. The features expressed by SR (14) provide the possibility to calculate the ECL parameters F_0, L_1, L_2, L_3 in an unambiguous manner. It would be interesting, therefore, to compare our results for these parameters with those given by Veneziano-type dual models (see, for example, [29,32]) and some models [5,6,8] of low-energy regime in QCD.

The Veneziano-type form of the $\pi\pi$-scattering amplitude

$$A(s,t)=V(u,s)+V(s,t)-V(t,u)\ , \tag{31}$$

$$V(x,y)=\lambda\frac{\Gamma(1-\alpha_\rho(x))\Gamma(1-\alpha_\rho(y))}{\Gamma(1-\alpha_\rho(x)-\alpha_\rho(y))}\ , \tag{32}$$

where

$$\alpha_\rho(x)=\frac{1}{2}+\frac{x}{2M_\rho^2}\ , \tag{33}$$

ensures the low-energy theorem

$$\lim_{s,t\to0}A(s,t)=\frac{s}{F_\pi^2}+O(p^4)\ , \tag{34}$$

provided the constant λ is related to the pion decay constant $F_\pi=93$ (MeV):

$$\lambda=-\frac{M_\rho^2}{\pi F_\pi^2}\ .$$

Expanding (31) in a power series of pion momenta (up to terms of order p^4), one obtains the following expressions for the ECL parameters L_1 to L_3:

$$L_2=2L_1\ , \tag{35}$$

$$L_3=-2L_2\ , \tag{36}$$

$$L_2=\frac{1}{8}\frac{F_\pi^2}{M_\rho^2}\ln(2)\simeq1.25\times10^{-3}\ . \tag{37}$$

Equation (35) is identical to Eq. (26); it is equivalent to the consequence of the large-N_c limit. It can be shown that this equation does not depend on the absence or presence of satellites. Equation (37) expresses the consequences of the particular form of the amplitude (32). It is interesting mainly because of the numerical value of its RHS. This value is close enough to that obtained in the models based on a nontopological chiral anomaly:

$$2L_1=L_2=\frac{1}{12}\frac{N_c}{16\pi^2}\simeq1.51\times10^{-3}\ .$$

The most remarkable is Eq. (36) which can [similar to Eq. (35)] be shown to be independent of the satellites added. The nullification of the combination $(2L_2+L_3)$ occurs due to the general property of the Veneziano-type amplitudes—the existence of straight lines in the (s,t)

plane where the amplitude $V(s,t)$ has its zeros.

It is interesting to note that Eq. (36) follows from the models of the low-energy QCD regime also. Careful analysis (see [33,34]) shows that this equation is affected both by perturbative gluon corrections and finite cutoff. It can be shown, however, that there exists the regularization scheme (inspired by the instanton liquid model [8]), preserving the form of Eq. (27) even with the finite cutoff. As for the influence of perturbative gluon corrections, we would like to refer to [35], where it was shown that they do not change the form of the equation in question.

VI. CONCLUSION

The above results demonstrate that the extension of Weinberg's approach [11,12] to the case of nonzero values of t gives one the possibility to connect unambiguously the parameters of low-energy $\pi\pi$ scattering with those of $\pi\pi$-resonance spectrum. The latter parameters must strongly correlate among themselves. The structure of correlations of spectrum parameters is shown to be similar to that appearing in dual models. So, we conclude that duality is the necessary condition to provide the $\pi\pi$ amplitude with the desired properties in the large-N_c limit of QCD.

Then, because of the well-known dynamical chiral symmetry of low-energy $\pi\pi$ interactions, the threshold parameters of the $\pi\pi$ amplitude obey some additional restrictions. The above-mentioned correspondence of low-energy coupling constants with masses and widths of $\pi\pi$ resonances allows one to conclude that these latter quantities must inherit certain features of chiral symmetry. Possible ways of realizing this idea has been discussed in Weinberg's original paper [11]. In that paper, however, the author considered the case of $t=0$ only; this limitation did not allow him to take into account the properties of duality. It would be extremely interesting, therefore, to combine the advantages of two approaches for the elucidation of connections between duality and chiral symmetry. It should be noted that some authors (see, for example, [36,37]) used the spectrum of the Veneziano model for the saturation of Weinberg's commutation relations and obtained quite reasonable results.

The last point we would like to stress here is the fruitfulness of any kind of asymptotic conditions (unitarity, Regge, Froissart, etc.) for the studying of low-energy dynamics and properties of resonance spectrum. In addition to Weinberg's original paper [11], this fruitfulness was strikingly manifested by many authors, among which we would like to single out papers by Ecker *et al.* [38] (the equivalence of two different kinds of spin-1 field formalism), Tiktopoulos [39] (Veneziano amplitude as a consequence of meromorphy and Regge behavior), Levin and Tiktopoulos [40] (the Salam-Weinberg model as a consequence of "unitarylike" behavior of trees), Truong [41], and some others.

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