Incorporation of QCD effects in basic corrections of the electroweak theory

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We study the incorporation of QCD effects in the basic electroweak corrections $\Delta\hat{r}$, $\Delta\hat{r}_W$, and Δr. They include perturbative $O(\alpha\alpha_s)$ contributions and $t\bar{t}$ threshold effects. The latter are studied in the resonance and Green-function approaches, in the framework of dispersion relations that automatically satisfy relevant Ward identities. Refinements in the treatment of the electroweak corrections, in both the modified minimal subtraction ($\overline{\rm MS}$) and the on-shell schemes of renormalization, are introduced, including the decoupling of the top quark in certain amplitudes, its effect on $\hat{e}^2(m_Z)$ and $\sin^2\hat{\theta}_W(m_Z)$, the incorporation of recent results on the leading irreducible $O(\alpha^2)$ corrections, and simple expressions for the residual, i.e., "nonelectromagnetic," parts of $\Delta\hat{r}$, $\Delta\hat{r}_W$, and Δr. The results are used to obtain accurate values for m_W and $\sin^2\hat{\theta}_W(m_Z)$, as functions of m_t and m_H. The higher-order effects induce shifts in these parameters comparable to the expected experimental accuracy, and they increase the prediction for m_t derived from current measurements. The $\overline{\rm MS}$ and the on-shell calculations of Δr, in a recently proposed formulation, are compared and found to be in excellent agreement over the wide ranges 60 GeV $\leq m_H \leq$ 1 TeV, $m_Z \leq m_t \leq$ 250 GeV.

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I. INTRODUCTION

It has been recently shown [1] that the modified minimal subtraction ($\overline{\rm MS}$) method of renormalization provides a very convenient framework to incorporate higher-order corrections to Δr [2] arising from resummation of one-loop effects. These include not only leading logarithms of $O([\alpha\ln(m_Z/m_f)]^n)$, where m_f is a generic fermion mass, and subleading logarithms of $O(\alpha^2\ln(m_Z/m_f))$ [3], but also terms of $O((\alpha\, m_t^2/m_W^2)^n)$ [1, 4, 5]. The reason can be traced to the fact that in this method one essentially subtracts the divergent parts of the amplitudes. In contrast with other approaches, this procedure circumvents the introduction of mass singularities and $O(\alpha\, m_t^2/m_W^2)$ terms via the finite parts of the counterterms. As a consequence, the renormalized perturbative expansion has a structure very similar to that of the bare theory, where resummation of one-loop effects is easy to implement [1, 4]. There are, of course, irreducible two-loop contributions of $O((\alpha\, m_t^2/m_W^2)^2)$. As discussed in Refs. [1, 4, 5], these can be gleaned from Ref. [6] and the more recent work of Ref. [7] on the $O(\alpha^2)$ corrections to the ρ parameter.

It has also been recently shown [8] that it is possible to derive a simple and accurate expression for Δr, within the on-shell method of renormalization [2], which contains the same leading and subleading contributions described above. On the other hand, the irreducible two-loop contributions of $O(\alpha^2 m_t^2/m_W^2)$ have not been computed, so that both the $\overline{\rm MS}$ and on-shell calculations of Δr become uncertain at this level of accuracy. In fact, one can see that the difference between the two calculations and their inherent theoretical uncertainty due to the neglect of higher-order electroweak corrections start with subleading terms of $O((\alpha/\pi s^2)(c^2/s^2)x_t)$, where x_t is the leading correction to the ρ parameter [cf. Eq. (17b)], or, equivalently, of $O((c^2/s^2)(\alpha_2/2\pi)(\kappa_t/2\pi))$, where $\alpha_2 \equiv g^2/4\pi$ and $\kappa_t \equiv (G_t)^2/4\pi$ are the SU(2) and the Yukawa couplings of the top quark, respectively. These are very small for low m_t and are expected to be $\approx 8\times10^{-4}$ for $m_t = 250$ GeV. (As we will see, over the large ranges $m_Z \leq m_t \leq 250$ GeV, 60 GeV $\leq m_H \leq$ 1 TeV, the actual numerical evaluation of the on-shell expression of Ref. [8] and the $\overline{\rm MS}$ calculations show a very small difference, reaching a maximum value of 2.5×10^{-4} at $m_t = 250$ GeV and $m_H = 1$ TeV, a very precise agreement which may be somewhat fortuitous.) This uncertainty is to be compared with an estimated error of $\pm9\times10^{-4}$ originating in the $O(\alpha)$ contributions of the first five quark flavors [9, 10].

In order to set the stage for our discussion, it is convenient to recall at this point some of the basic relations of the on-shell and $\overline{\rm MS}$ frameworks [1–3, 11, 12],

$$s^2 = \frac{A^2}{m_W^2(1-\Delta r)}, \tag{1}$$

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$$\hat{s}^2 = \frac{A^2}{m_W^2(1-\Delta\hat{r}_W)}, \tag{2}$$

$$\hat{s}^2\hat{c}^2 = \frac{A^2}{m_Z^2(1-\Delta\hat{r})}, \tag{3}$$

where m_W and m_Z are the physical masses, $A = [\pi\alpha/(\sqrt{2}G_\mu)]^{1/2} = (37.2802\pm0.0003)$ GeV, s^2, $\hat{s}^2$, and $\hat{c}^2$ are abbreviations for $\sin^2\theta_W \equiv 1 - m_W^2/m_Z^2$ and the $\overline{\rm MS}$ parameters $\sin^2\hat{\theta}_W(m_Z)$ and $\cos^2\hat{\theta}_W(m_Z)$, respectively, and Δr, $\Delta\hat{r}_W$, and $\Delta\hat{r}$ are radiative corrections. The 't Hooft mass μ has been set equal to m_Z in Eqs. (2) and (3).

It follows from the analysis of Ref. [13] that m_Z and m_W in Eqs. (1) and (2) can be identified, phenomenologically, with the masses measured in current experiments and, theoretically, with the definition $m_1^2 = m_2^2 + \Gamma_2^2$, where $\bar{s} = m_2^2 - i\,m_2\Gamma_2$ is the relevant complex pole position [14]. The latter is given by $\bar{s} = m_0^2 + A(\bar{s})$, where m_0 is the bare mass and $A(s)$ the conventional self-energy, which includes tadpoles, tadpole counterterms, and, in the Z^0 case, γZ mixing effects that start in $O(\alpha^2)$. On general grounds, it is expected that m_2 and Γ_2, and therefore m_1, are gauge invariant to all orders [13, 15]. Over a large class of gauges, including those in which explicit calculations have been carried out, m_1 differs from the "field-theoretic" or "on-shell" definition, $m^2 = m_0^2 + {\rm Re}A(m^2)$, by gauge-dependent terms of $O(\alpha^3)$ [13]. Because contributions of this order are well beyond the accuracy that may be achieved in the foreseeable future, the replacement of m by the more rigorous definition m_1 does not require a modification of the radiative corrections Δr, $\Delta\hat{r}_W$, and $\Delta\hat{r}$. Using the expression

$$s^2 = \frac{1}{2}\left\{1 - \left[1 - \frac{4A^2}{m_Z^2(1-\Delta r)}\right]^{1/2}\right\}, \tag{4}$$

equivalent to Eq. (1), the analogous one with $s^2 \to \hat{s}^2$ and $\Delta r \to \Delta\hat{r}$, equivalent to Eq. (3), and the accurately known value $m_Z = (91.187\pm0.007)$ GeV [16], the corrections Δr and $\Delta\hat{r}$ lead to precise evaluations of s^2 and $\hat{s}^2$, as functions of m_t and m_H. These, in turn, can be compared with other determinations of s^2 and $\hat{s}^2$ to constrain the value of m_t and, in the future, that of m_H. They are also important input parameters in the prediction of the Z^0 partial widths and on-resonance asymmetries, as some of these observables depend very sensitively on the weak mixing angle (see, for example, Ref. [17]). More generally, the basic corrections Δr, $\Delta\hat{r}_W$, and $\Delta\hat{r}$ are frequently employed to verify the standard model (SM) at the level of its quantum corrections and in searches for signals of new physics [18–21]. It was also explained in Ref. [1] how $\Delta\hat{r}$ and $\Delta\hat{r}_W$, relevant corrections in the $\overline{\rm MS}$ framework, can be employed to evaluate the on-shell quantity Δr.

The aim of the present paper is to incorporate the leading QCD corrections in the calculations of the radiative corrections Δr, $\Delta\hat{r}$, and $\Delta\hat{r}_W$. We also introduce some refinements in our previous treatment of the electroweak corrections [1].

The relevant QCD contributions occur in the vacuum-polarization functions associated with the $W^\pm$ and Z^0 bosons and have been extensively discussed in the literature [22–31]. In particular, the QCD corrections involving the (t,b) isodoublet are known to be significant for large m_t. There are actually two types of effects that may be classified as perturbative $O(\alpha\alpha_s)$ and threshold contributions. In the literature, the latter are frequently referred to as "nonperturbative." The perturbative $O(\alpha\alpha_s)$ components have been studied with two different methods: (*i*) direct evaluation of the two-loop Feynman diagrams in dimensional regularization, an approach that goes back to the pioneering work of Djouadi and Verzegnassi [22]; (*ii*) calculation of the imaginary parts and computation of the full amplitude by means of suitably defined dispersion relations (DR's) [23–26]. It has been shown [30, 31] that the two approaches are equivalent in the evaluation of the perturbative contributions to Δr and $\Delta\rho$, a welcome fact. On the other hand, the $\overline{\rm MS}$ scheme is implemented in the framework of dimensional regularization and, for a full determination of the subtraction constants, one must appeal to method (*i*).

Threshold effects on the absorptive parts of the self-energies have, in turn, been discussed in two different approaches: (*a*) in Ref. [24] the contributions of densely spaced, narrow quarkonium resonances were taken into account on the basis of a specific quark-antiquark potential; (*b*) in Refs. [27, 28] one considers the imaginary part of the Green function for the nonrelativistic Schrödinger equation that characterizes the $t\bar{t}$ system near threshold. The latter formulation effectively resums the contributions of soft-gluon exchanges in the ladder approximation (see also Ref. [29]).

For sufficiently low m_t there should be, near threshold, a rich spectrum of distinct nonrelativistic states bound by strong long-range forces and the approach (*a*) is very natural. For increasing m_t, however, the weak decay of a single top quark inside the bound states becomes important and, for $m_t \gtrsim 130$ GeV, the partial width of $t \to W^+ b$ is so large that the revolution period of a $t\bar{t}$ bound state would exceed its lifetime. As a consequence, the individual resonances lose their distinctiveness and are smeared out to a coherent structure [27–29]. In that regime, the Green-function method is more appropriate. In summary, one expects approaches (*a*) and (*b*) to be preferable for lower ($\lesssim 130$ GeV) and higher ($\gtrsim 130$ GeV) values of m_t, respectively. Both formulations deal directly with the absorptive parts of the amplitudes. To obtain the real parts it is then necessary to employ DR's. This was done in detail in Ref. [31] using DR's for the vacuum-polarization functions directly constructed from relevant Ward identities [30]. In conjunction with very plausible assumptions concerning the asymptotic behavior of the threshold effects for large q^2, this procedure leads to specific results for the real parts.

In Sec. VII, we compare four different calculations of m_W and $\sin^2\hat{\theta}_W(m_Z)$, as functions of m_t and m_H: (*i*) only electroweak corrections; (*ii*) electroweak plus perturbative $O(\alpha\alpha_s)$ corrections; (*iii*) the above, plus threshold effects in the resonance approach; (*iv*) same as (*iii*) with threshold effects in the Green-function framework. This

allows us to demonstrate the magnitude of the QCD corrections and, at the same time, to separate the threshold effects from the more established perturbative $O(\alpha\alpha_s)$ contributions. Although the two approaches to treat the threshold effects are certainly not identical, we find the encouraging and fortunate result that their overall effects on m_W and $\sin^2\hat{\theta}_W(m_Z)$ are quite close over the entire range $m_Z \leq m_t \leq 250$ GeV.

As mentioned before, aside from incorporating the QCD effects in the relevant self-energies, we introduce some refinements in our treatment of the $O(\alpha)$ electroweak corrections. Conceptually, the most interesting modification is a slight change in the definition of the fundamental couplings $\hat{e}^2(m_Z)$ and $\sin^2\hat{\theta}_W(m_Z)$ of the $\overline{\text{MS}}$ scheme, which is introduced in order to make them essentially independent of heavy particles such as the top quark or unknown massive excitations. In the case of $\sin^2\hat{\theta}_W(m_Z)$, we follow a convention recently proposed at the one-loop level by Marciano and Rosner [32, 33], and explain how to extend it when $O(\alpha\alpha_s)$ corrections are included. We emphasize that these modifications in the definitions of the fundamental $\overline{\text{MS}}$ parameters do not affect, to the order of the calculations, the relations between physical observables because they are compensated by corresponding changes in the appropriate radiative corrections. A second change is that we use an updated calculation by Jegerlehner [10] for the contribution of the first five quark flavors to the photon vacuum-polarization function. A third modification is that we incorporate the very recent results of Ref. [7] concerning the irreducible two-loop corrections of $O((\alpha\, m_t^2/m_W^2)^2)$. In the rest of the calculations, as we did in Ref. [1], we treat the u, d, and s quarks as massless but we now include terms of $O(\alpha\, m_f^2/m_W^2)$, where $f = c, b, \tau, \ldots$. Although they are very small—they contribute to Δr only at the $\lesssim 1 \times 10^{-4}$ level—their incorporation may facilitate detailed comparisons with calculations by other authors.

The plan of the paper is the following: in Sec. II, we discuss the definitions of the basic $\overline{\text{MS}}$ parameters $\hat{e}^2(m_Z)$ and $\sin^2\hat{\theta}_W(m_Z)$ in the presence of the $O(\alpha\alpha_s)$ corrections, with emphasis on the decoupling of heavy particles. In Sec. III, we incorporate the perturbative $O(\alpha\alpha_s)$ contributions in $\Delta\hat{r}_W$ and $\Delta\hat{r}$. We emphasize the important fact that the magnitude of the $O(\alpha\alpha_s)$ effects depends sensitively on the precise definition of m_t. Our detailed calculations, as well as the other papers in the literature, employ the "on-shell" definition of m_t. In the discussion we give a brief argument to indicate why this choice is useful and appropriate. In Sec. IV, we present a simple method to separate the residual, i.e., "nonelectromagnetic," parts of $\Delta\hat{r}$, $\Delta\hat{r}_W$, and Δr. In Sec. V, we include the perturbative $O(\alpha\alpha_s)$ corrections in the calculation of Δr in the on-shell scheme, using the formulation of Ref. [8]. In Sec. VI, we discuss the incorporation of threshold effects in $\Delta\hat{r}$, $\Delta\hat{r}_W$, and Δr. We include estimates, based on a simple "Bohr-atom" model, of the contribution of the $1S$ toponium resonance to the imaginary part of the self-energies. We find that this simple model gives values roughly similar to the calculations carried out with more realistic quark-antiquark potentials. In Sec. VII, we use the theoretical results to carry out precise calculations of m_W and $\sin^2\hat{\theta}_W(m_Z)$, in the manner explained before. We find that the QCD and other higher-order corrections induce shifts in m_W and $\sin^2\hat{\theta}_W(m_Z)$ comparable with the expected experimental precision. Interestingly, all of them increase the value of m_t derived from current measurements. We also compare the $\overline{\text{MS}}$ and on-shell calculation of Δr, in the formulation of Ref. [8]. We find that, as was the case in the absence of perturbative $O(\alpha\alpha_s)$ corrections, the $\overline{\text{MS}}$ and on-shell calculations of Δr are very close over a large range of m_t and m_H values. The Appendixes discuss basic expressions for the perturbative $O(\alpha\alpha_s)$ corrections, the effect of top-quark decoupling in $\Delta\hat{r}_W$ and $\Delta\hat{r}$, and the very small contribution from finite fermion masses.

II. THE PARAMETERS $\hat{e}^2(m_Z)$ AND $\sin^2\hat{\theta}_W(m_Z)$

In our previous treatment [1] we defined these parameters, at the one-loop level, by simply subtracting from the radiatively corrected cofactors the terms involving

$$\delta = \frac{1}{n-4} + \frac{1}{2}[\gamma - \ln(4\pi)], \tag{5a}$$

and setting the 't Hooft mass scale, μ, equal to m_Z. Because at one loop δ always occurs in combination with $-\ln\mu$, this is equivalent to subtracting only the pole terms, $(n-4)^{-1}$, rescaling μ according to

$$\mu = \frac{\mu'\, e^{\gamma/2}}{(4\pi)^{1/2}}, \tag{5b}$$

and then setting $\mu' = m_Z$. The second formulation can be conveniently generalized to higher-order corrections and one can define the $\overline{\text{MS}}$ renormalization procedure as the subtraction of pole terms of the form $(n-4)^{-m}$, where m is an integer ≥ 1, and the identification of the rescaled 't Hooft parameter μ' with the relevant mass scale, in this case m_Z. As is well known, the factor $e^{\gamma/2}(4\pi)^{-1/2}$ is appended in Eq. (5b) to cancel relatively large numerical constants that are an artifact of dimensional regularization [34].

In Ref. [1] we applied this procedure uniformly, independently of whether the top quark is more or less massive than m_Z. On the other hand, it is desirable to treat heavy particles, as much as possible, as decoupled. For example, when $m_t > m_Z$ it is convenient to subtract from the amplitude terms involving $\ln(m_t/m_Z)$ and to absorb them in the coupling constants. In Ref. [1] we did not follow this route for two reasons: (*i*) $\sin^2\hat{\theta}_W(m_Z)$ appears as a cofactor in several important radiatively corrected amplitudes and it is not possible to absorb completely the $\ln(m_t/m_Z)$ terms occurring in all of them; (*ii*) some important relations, such as Eq. (3), contain terms proportional to m_t^2, which certainly do not decouple. Recently, however, Marciano and Rosner [32, 33] proposed to implement the decoupling idea, at the one-loop level, by absorbing in $\sin^2\hat{\theta}_W(m_Z)$ all $\ln(m/m_Z)$ terms with $m > m_Z$ occurring in a specific amplitude, namely the γZ self-energy evaluated at $q^2 = m_Z^2$, $\text{Re}A_{\gamma Z}(m_Z^2)$. Here

m is the mass of the top quark or any unknown heavy particle with $m > m_Z$. With this convention, a heavy top or a heavy unknown particle decouples in the limit $m/m_Z \gg 1$ from the cofactor $\hat{\kappa}(q^2)$ multiplying $\hat{s}^2$ in most neutral-current processes; as a consequence, this parameter can effectively be determined from the on-resonance asymmetries without hindrance from unknown "heavy physics."

We now explain how we implement the decoupling idea in the presence of the $O(\alpha\alpha_s)$ corrections. It is convenient to start with $\hat{e}^2(m_Z)$, which we frequently abbreviate as $\hat{e}^2$. We recall the relation between the bare charge e_0 and the conventional renormalized charge e,

$$e^2 = e_0^2 \left\{ 1 + e_0^2 \Pi_{\gamma\gamma}^{(f)}(0) + \frac{7e_0^2}{8\pi^2}\left[\frac{1}{n-4} + \ln\frac{m_W}{\mu'} - \frac{1}{21}\right]\right\}^{-1}, \tag{6}$$

where $\Pi_{\gamma\gamma}^{(f)}(0)$ is the usual fermionic vacuum-polarization function evaluated at $q^2 = 0$ and the last term represents $O(e_0^2)$ bosonic contributions to charge renormalization that must be included in the SM. The latter can be gleaned, for example, from Ref. [2]. As explained in Refs. [1, 2], because of the existence of mass singularities associated with the light quarks, it is not possible to calculate perturbatively the contribution of the first five quark flavors to $\Pi_{\gamma\gamma}^{(f)}(0)$. Calling this contribution $\Pi_{\gamma\gamma}^{(5)}(0)$, the problem is circumvented by writing $\Pi_{\gamma\gamma}^{(5)}(0) = \mathrm{Re}[\Pi_{\gamma\gamma}^{(5)}(0) - \Pi_{\gamma\gamma}^{(5)}(m_Z^2)] + \mathrm{Re}\Pi_{\gamma\gamma}^{(5)}(m_Z^2)$. The first term represents the five-flavor contribution to the renormalized photon vacuum-polarization function at $q^2 = m_Z^2$ and can be evaluated using dispersion relations, experimental data on $e^+e^- \to$ hadrons and QCD corrections [2, 9, 10]. The presence of a large invariant momentum, $q^2 = m_Z^2$, in the third term prevents the occurrence of fermionic mass singularities and, as a consequence, $\mathrm{Re}\Pi_{\gamma\gamma}^{(5)}(m_Z^2)$ can be analyzed perturbatively. The same is, of course, true for the leptonic and bosonic contributions and, because of its large mass, for the top contribution $\Pi_{\gamma\gamma}^{(t)}(0)$. Including irreducible two-loop contributions of $O(\hat{\alpha}\hat{\alpha}_s, \hat{\alpha}^2)$ to $\hat{e}^2\Pi_{\gamma\gamma}^{(f)}(0)$ arising from virtual gluon and photon interchanges, we obtain (see Appendix A)

$$\begin{aligned}\hat{e}^2\Pi_{\gamma\gamma}^{(f)}(0) = {} & \frac{2\hat{\alpha}}{3\pi}\sum_l\left[\ln\frac{\mu'}{m_l}\left(1+\frac{3\hat{\alpha}}{4\pi}\right)+\frac{45\hat{\alpha}}{32\pi}\right]+\frac{8\hat{\alpha}}{9\pi}\left[\ln\frac{\mu'}{m_t}\left(1+\frac{\hat{\alpha}_s}{\pi}+\frac{\hat{\alpha}}{3\pi}\right)+\frac{15}{8}\left(\frac{\hat{\alpha}_s}{\pi}+\frac{\hat{\alpha}}{3\pi}\right)\right] \\ & +\frac{\hat{\alpha}}{\pi}\sum_{q\neq t}Q_q^2\left[2\ln\frac{\mu'}{m_Z}\left(1+\frac{\hat{\alpha}_s}{\pi}+\frac{3\hat{\alpha}}{4\pi}Q_q^2\right)+f_1(r_q)+\left(\frac{\hat{\alpha}_s}{\pi}+\frac{3\hat{\alpha}}{4\pi}Q_q^2\right)f_2(r_q)\right] \\ & +\hat{e}^2\mathrm{Re}[\Pi_{\gamma\gamma}^{(5)}(0)-\Pi_{\gamma\gamma}^{(5)}(m_Z^2)]-\frac{\hat{\alpha}}{\pi}\frac{I}{(n-4)},\end{aligned} \tag{7a}$$

where the l and q sums run over leptons and quarks, respectively, the color factor 3 is henceforth explicitly included, $r_q = m_Z^2/(4m_q^2)$,

$$\begin{aligned} f_1(r) &= \ln(4r) - \left(2+\frac{1}{r}\right)\left(1-\frac{1}{r}\right)^{1/2}\cosh^{-1}\sqrt{r}+\frac{5}{3}+\frac{1}{r} \\ &= \frac{5}{3}+\frac{3}{2r}+O\left(\frac{\ln r}{r^2}\right), \end{aligned} \tag{7b}$$

$$\begin{aligned} f_2(r) &= \ln(4r)+\frac{\mathrm{Re}V_1(r)}{r}-4\zeta(3)+\frac{55}{12} \\ &= -4\zeta(3)+\frac{55}{12}-\frac{3}{r}\ln(4r)+O\left(\frac{\ln^2 r}{r^2}\right), \end{aligned} \tag{7c}$$

$\zeta(3) = 1.20206\ldots$, $V_1(r)$ is a complicated function defined in Ref. [25], and $I = 16/3 + 5\hat{\alpha}_s/(3\pi) + 11\hat{\alpha}/(9\pi)$. The first and second terms in Eq. (7a) are the finite parts of the leptonic and top contributions, while the sum over q is the finite part of the perturbative evaluation of $\hat{e}^2\mathrm{Re}\Pi_{\gamma\gamma}^{(5)}(m_Z^2)$. In the latter the terms proportional to r_q^{-1} are extremely small and we can replace the functions $f_1(r)$ and $f_2(r)$ by their asymptotic values $f_1(\infty) = 5/3$ and $f_2(\infty) = 55/12 - 4\zeta(3) \approx -0.22491$. The last term in Eq. (7a) represents the divergent part up to terms of $O(\hat{\alpha}^2, \hat{\alpha}\hat{\alpha}_s)$. As is well known, up to two loops $\Pi_{\gamma\gamma}(q^2)$ is linear in $(n-4)^{-1}$ and $\ln\mu'$. We note that the cofactors of $(n-4)^{-1}$ are equal to those of $\ln(1/\mu')$ at one loop and to one-half those of $\ln(1/\mu')$ at two loops.

In order to obtain the relation between $\hat{e}^2$ and e^2, one writes $e_0^2 = \hat{e}^2/\hat{Z}_e$ in Eq. (6), uses the counterterms present in $\hat{Z}_e$ to cancel the $(n-4)^{-1}$ terms in Eqs. (6) and (7a), and sets $\mu' = m_Z$ in those equations. The mass scale employed in $\hat{\alpha}_s$ is discussed in greater detail at the end of this section. With the exception of the top quark, all the particles contributing to Eqs. (6) and (7a) are less massive than m_Z and their contribution is retained. To implement decoupling in Eq. (6), we also subtract the finite top contribution when $m_t > m_Z$, so that $\hat{Z}_e$ in that case contains an additional finite counterterm and reads

$$\hat{Z}_e = 1 + \frac{\hat{\alpha}}{\pi}\left(I - \frac{7}{2}\right)\frac{1}{n-4} - \hat{e}^2\Pi^{(t)}_{\gamma\gamma}(0)\Big|_{\overline{\rm MS}}\theta(m_t - m_Z), \tag{7d}$$

where $\overline{\rm MS}$ denotes the "finite part" after the $\overline{\rm MS}$ renormalization has been carried out, i.e., the remainder after the $(n-4)$ poles have been subtracted and μ' has been set equal to m_Z, and the superscript (t) refers to the top-quark contribution. Specifically,

$$\hat{e}^2\Pi^{(t)}_{\gamma\gamma}(0)\Big|_{\overline{\rm MS}} = \frac{8\hat{\alpha}}{9\pi}\left[\ln\frac{m_Z}{m_t}\left(1 + \frac{\hat{\alpha}_s}{\pi} + \frac{\hat{\alpha}}{3\pi}\right) + \frac{15}{8}\left(\frac{\hat{\alpha}_s}{\pi} + \frac{\hat{\alpha}}{3\pi}\right)\right]. \tag{7e}$$

The term proportional to $(n-4)^{-1}$ in Eq. (7d) cancels the divergent parts in Eqs. (6) and (7a). The other term subtracts the finite top contribution, i.e., the second term in Eq. (7a), when $m_t > m_Z$. Because this contribution does not exactly vanish at $m_t = m_Z$, the above prescription leads to a small discontinuity at $m_t = m_Z$. An alternative that would ensure continuity would be to do the matching at the point where the finite top correction vanishes, which is $m_t = 1.073\,m_Z = 97.8$ GeV. However, because the decoupling at $m_t > m_Z$ is easy to implement and is analogous to what is done in some QCD calculations [35], we will adopt it as our convention.

As the mass range $m_t < 91$ GeV has been excluded at the 95% confidence level [36], we will henceforth assume that $m_t > m_Z$, in which case Eqs. (6), (7a), and (7d) lead to

$$e^2 = \frac{\hat{e}^2}{1 + (\hat{\alpha}/\pi)\Delta_\gamma}, \tag{8a}$$

where

$$\Delta_\gamma = \frac{7}{4}\ln c^2 - \frac{1}{6} + \frac{2}{3}\sum_l\left[\ln\frac{m_Z}{m_l}\left(1 + \frac{3\hat{\alpha}(m_Z)}{4\pi}\right) + \frac{45\hat{\alpha}(m_Z)}{32\pi}\right] + \frac{55}{27} + \left(\frac{11\hat{\alpha}_s(m_Z)}{9\pi} + \frac{35\hat{\alpha}(m_Z)}{108\pi}\right)\left(\frac{55}{12} - 4\zeta(3)\right)$$
$$+4\pi^2{\rm Re}[\Pi^{(5)}_{\gamma\gamma}(0) - \Pi^{(5)}_{\gamma\gamma}(m_Z^2)], \tag{8b}$$

and c^2 is an abbreviation for $\cos^2\theta_W \equiv m_W^2/m_Z^2$. Solving for $\hat{e}^2$, we have

$$\hat{e}^2 = \frac{e^2}{1 - (\alpha/\pi)\Delta_\gamma}. \tag{8c}$$

In $(\alpha/\pi)\Delta_\gamma$ we have retained very small terms of $O(\alpha^2)$ arising from virtual-photon interchange. We have done this because of the analogy with gluon contributions and the fact that they contain interesting leptonic mass singularities. It should be understood, however, that this does not represent a complete $O(\alpha^2)$ calculation as there are other irreducible two-loop contributions of this order arising from the bosonic sector and from $W^\pm$, Z^0, and H interchanges in the fermionic sector.

Equation (8c) allows us to compute $\hat{e}^2$ in terms of α, independently of m_t or unknown particles heavier than m_Z. Using $m_Z = 91.187$ GeV [16], $m_W = 80.22$ GeV [37, 38], $\hat{\alpha}_s = \hat{\alpha}_s(m_Z) = 0.118$ [39], and $e^2{\rm Re}[\Pi^{(5)}_{\gamma\gamma}(0) - \Pi^{(5)}_{\gamma\gamma}(m_Z^2)] = 0.0282 \pm 0.0009$ [10], we find $(\alpha/\pi)\Delta_\gamma = 0.0668 \pm 0.0009$ or $\hat{\alpha}^{-1} = (4\pi/\hat{e}^2) = 127.9 \pm 0.1$. The expression for $(\alpha/\pi)\Delta_\gamma$ differs from the quantity $-2\delta e/e|_{\overline{\rm MS}}$ in Ref. [1] by the exclusion of the $\ln(m_t/m_Z)$ terms, a more accurate description of the QCD corrections, and the updated calculation of the five-flavor contribution. Numerically, however, $\hat{\alpha}^{-1}$ is very close to the value 127.8 ± 0.1 reported in Ref. [1] for $m_t = m_Z$, the small difference essentially arising from the change in the five-flavor contribution [10].

Concerning $\hat{s}^2 \equiv \sin^2\hat{\theta}_W(m_Z)$, we recall that in the neutral-current amplitudes this parameter is multiplied by the electroweak form factor $\hat{\kappa}$, which contains the γZ mixing term $-(\hat{c}/\hat{s})A_{\gamma Z}(q^2)/q^2$ [17, 40]. Here $A_{\gamma Z}(q^2)$ is the unrenormalized γZ transverse mixing amplitude as defined in Ref. [41], expressed in terms of the $\overline{\rm MS}$ couplings $\hat{e}$, $\hat{s}$, and $\hat{c}$. In order to implement the decoupling, we apply the Marciano-Rosner convention [32, 33], according to which the $\ln(m_t/m_Z)$ terms in ${\rm Re}A_{\gamma Z}(m_Z^2)/m_Z^2$ are subtracted for $m_t > m_Z$. At the two-loop level there is also an m_t-independent term, which must be subtracted, too. More generally, the idea is to subtract all contributions to ${\rm Re}A_{\gamma Z}(m_Z^2)/m_Z^2$ that involve particles of mass $m > m_Z$ and do not decouple in the limit $m \to \infty$. This can be implemented by adding a finite counterterm in the $\hat{s}^2$ renormalization, in analogy with Eq. (7d). Up to terms of $O(\alpha\alpha_s)$ we find, in the case of the top quark,

$$s_0^2 = \hat{s}^2\left\{1 + \frac{\rm const}{n-4} + \frac{\hat{c}}{\hat{s}}\left(A^{(t)}_{\gamma Z}\right)'(0)\Big|_{\overline{\rm MS}}\right\}, \tag{9a}$$

where s_0^2 is the bare parameter, $\left(A^{(t)}_{\gamma Z}\right)'(0) = (d/dq^2)A^{(t)}_{\gamma Z}(q^2)\Big|_{q^2=0}$, and $\overline{\rm MS}$ and the superscript (t) have the same meaning as in Eq. (7d). Again, const$/(n-4)$ is the divergent part of the counterterm and the last term is a finite contribution necessary to implement the decoupling of the $\ln(m_t/m_Z)$ and constant terms in the top contribution to ${\rm Re}A_{\gamma Z}(m_Z^2)/m_Z^2$. Specifically, we find:

$$\frac{\hat{c}}{\hat{s}}\left(A_{\gamma Z}^{(t)}\right)'(0)\Big|_{\overline{\rm MS}} = \frac{\hat{\alpha}}{\pi}\,d, \tag{9b}$$

where

$$d = \frac{1}{3}\left(\frac{1}{\hat{s}^2}-\frac{8}{3}\right)\left[\ln\frac{m_t}{m_Z}\left(1+\frac{\hat{\alpha}_s}{\pi}\right)-\frac{15\hat{\alpha}_s}{8\pi}\right]. \tag{9c}$$

In Eqs. (9a)–(9c) we have neglected all terms of $O(\alpha^2)$.

We must still discuss the mass scale employed in $\hat{\alpha}_s$. Following Refs. [22, 24–26, 31], in the present paper corrections arising from the (t,b) isodoublet are computed with $\hat{\alpha}_s(m_t)$. The reason is that the dominant contributions due to this isodoublet involve mass scales of $O(m_t)$ [31, 42]. This choice can also be justified by arguments based on effective-field theory [43]. On the other hand, the perturbative contributions from the two light-quark isodoublets involve self-energies evaluated at $q^2 = m_W^2$ or $q^2 = m_Z^2$ and, for that reason, are calculated with $\hat{\alpha}(m_Z)$. As an example, in Eqs. (7e) and (9c), which involve top-quark contributions, we identify $\hat{\alpha}_s$ with $\hat{\alpha}_s(m_t)$, while in the perturbative part of Eq. (8b), which includes light-quark isodoublets, we employ $\hat{\alpha}_s(m_Z)$. [In the latter equation we make a slight and negligible change to the above isodoublet rule by also evaluating the very small bottom contribution with $\hat{\alpha}_s(m_Z)$.] Numerically, the finite counterterms in Eqs. (7d) and (9a) are quite small: for $m_t = 150$ GeV, they are 1.0×10^{-3} in Eqs. (7d) and (7e) and 6.1×10^{-4} in Eqs. (9b) and (9c); for $m_t = 250$ GeV, the corresponding values are 2.2×10^{-3} and 1.5×10^{-3}, respectively. As we will see, when $\hat{s}^2$ is defined according to the decoupling convention explained above, the finite counterterm in Eq. (9a) introduces small compensatory shifts in $\Delta\hat{r}_W$ and $\Delta\hat{r}$. Similarly, $\hat{Z}_e$, defined in Eq. (7d), will introduce small compensatory changes in radiative corrections whenever $\hat{e}^2$ is employed as the zeroth-order parameter.

III. PERTURBATIVE $O(\alpha\alpha_s)$ CORRECTIONS TO $\Delta\hat{r}_W$, $\Delta\hat{r}$, AND Δr

When the decoupling of the top quark is implemented according to the discussion of Sec. II, the expression for $\Delta\hat{r}_W$ [cf. Eq. (2)] becomes (see Appendix B)

$$\Delta\hat{r}_W = \left[\frac{e^2}{\hat{s}^2}\mathrm{Re}\left(\frac{A_{WW}(m_W^2)-A_{WW}(0)}{m_W^2}\right)+e^2\Pi_{\gamma\gamma}^{(f)}(0)\right]_{\overline{\rm MS}}+\frac{\alpha}{\pi}\left(\frac{7}{4}\ln c^2-\frac{1}{6}\right)$$
$$+\frac{e^2}{16\pi^2\hat{s}^2}\left\{6+\frac{\ln c^2}{s^2}\left[\frac{7}{2}-\frac{5}{2}s^2-\hat{s}^2\left(5-\frac{3c^2}{2\hat{c}^2}\right)\right]\right\}-\frac{\alpha}{\pi}\,d, \tag{10a}$$

where $A_{WW}(q^2)$ is the unrenormalized WW transverse self-energy with the coupling $e^2/\hat{s}^2$ factored out [1], d is defined in Eq. (9c), and the $\overline{\rm MS}$ symbol has the same meaning as in Sec. II. The quantities $\Delta\hat{r}_W$, $\Delta\hat{r}$, and Δr are gauge invariant but some of their partial bosonic contributions are not. In Eq. (10a) and henceforth all explicit expressions and partial contributions are given in the 't Hooft–Feynman gauge. In particular, the expression involving curly brackets represents vertex- and box-diagram corrections to μ decay, evaluated in that gauge.

Except for the last term, Eq. (10a) is the same as Eq. (8b) in Ref. [1]. As $\hat{s}^2$ enters Eq. (2) as a zeroth-order parameter and is defined in the present paper according to the decoupling convention explained in Sec. II, there is now an additional contribution $-(\alpha/\pi)d$ arising from the finite contribution in Eqs. (9a)–(9c).

We now discuss the evaluation of Eq. (10a). The quantity $\left[e^2\Pi_{\gamma\gamma}^{(f)}(0)\right]_{\overline{\rm MS}}$ can be obtained from Eq. (7a). We have

$$\left[e^2\Pi_{\gamma\gamma}^{(f)}(0)\right]_{\overline{\rm MS}}+\frac{\alpha}{\pi}\left(\frac{7}{4}\ln c^2-\frac{1}{6}\right)=\frac{\alpha}{\pi}\left\{\Delta_\gamma+\frac{8}{9}\left[\ln\frac{m_Z}{m_t}\left(1+\frac{\hat{\alpha}_s}{\pi}+\frac{\hat{\alpha}}{3\pi}\right)+\frac{15}{8}\left(\frac{\hat{\alpha}_s}{\pi}+\frac{\hat{\alpha}}{3\pi}\right)\right]\right\}, \tag{10b}$$

where Δ_γ is defined in Eq. (8b) and the other terms on the right-hand side (rhs) represent the top contribution. Inserting Eq. (10b) into Eq. (10a) and neglecting small terms of $O(\alpha^2)$, we obtain

$$\Delta\hat{r}_W = \left[\frac{e^2}{\hat{s}^2}\mathrm{Re}\left(\frac{A_{WW}(m_W^2)-A_{WW}(0)}{m_W^2}\right)\right]_{\overline{\rm MS}}+\frac{\alpha}{\pi}\Delta_\gamma+\frac{e^2}{16\pi^2\hat{s}^2}\left\{6+\frac{\ln c^2}{s^2}\left[\frac{7}{2}-\frac{5}{2}s^2-\hat{s}^2\left(5-\frac{3c^2}{2\hat{c}^2}\right)\right]\right\}-\frac{\alpha}{\pi}\,\hat{d}, \tag{10c}$$

where

$$\hat{d} = \frac{d}{1-8\hat{s}^2/3}. \tag{10d}$$

The bosonic contribution $(e^2/\hat{s}^2m_W^2)[A_{WW}^{(b)}(m_W^2)-A_{WW}^{(b)}(0)]_{\overline{\rm MS}}$ is given in Eq. (A.6) of Ref. [1]. In order to study the fermionic contributions, denoted by a superscript (f), we define

$$B^{(f)} \equiv \frac{e^2}{\hat{s}^2}\mathrm{Re}\left[\frac{A_{WW}^{(f)}(m_W^2)-A_{WW}^{(f)}(0)}{m_W^2}\right]_{\overline{\rm MS}}, \tag{11a}$$

and write

$$B^{(f)} = B_0^{(f)} + B_{\rm QCD}^{(f)}, \tag{11b}$$

where $B_0^{(f)}$ and $B_{\rm QCD}^{(f)}$ stand for the $O(\alpha)$ and $O(\alpha\alpha_s)$ corrections. If we neglect very small terms proportional to $\alpha\, m_q^2/m_W^2$ $(q = d, s, b)$, the mixing angles in the quark sector are irrelevant [41] and to zeroth order in α_s we have (cf. Eq. (A.5) of Ref. [1])

$$B_0^{(f)} = \frac{\alpha}{2\pi\hat{s}^2}\left\{2\left(\ln c^2 - \frac{5}{3}\right) + \frac{\ln\omega_t}{2} + \frac{\omega_t}{8}(1+2\omega_t) + \frac{(\omega_t-1)^2}{2}\left(1+\frac{\omega_t}{2}\right)\ln\left(1-\frac{1}{\omega_t}\right)\right\}, \tag{11c}$$

where we have included the lepton and quark contributions and $\omega_t = m_t^2/m_W^2$. As a refinement, in Appendix C we discuss the contributions of $O(m^2/m_W^2)$, where m stands for a lepton or quark mass other than m_t. As these terms are very small, we may neglect in their evaluation the squares of the mixing angles. In that case one obtains a sum of isodoublet contributions, which, for arbitrary masses, is given in Appendix C. One finds that the corrections of $O(m^2/m_W^2)$ to $B_0^{(f)}$ are indeed very small, of order 10^{-5}.

The contribution of $O(\alpha\alpha_s)$ in Eq. (11b) is given by (see Appendix A)

$$B_{\rm QCD}^{(f)} = \frac{\alpha}{4\pi\hat{s}^2}\left\{\frac{2\hat{\alpha}_s(m_Z)}{\pi}\left(\ln c^2 + 4\zeta(3) - \frac{55}{12}\right) + \frac{\hat{\alpha}(m_t)}{\pi}\left[\ln c^2 + 4\zeta(3) - \frac{55}{12} - 4\omega_t\left(F_1(1/\omega_t) - F_1(0)\right) + \ln\omega_t\right]\right\}, \tag{11d}$$

where $F_1(x)$ is defined in Ref. [25]. In Eq. (11d) we have neglected all quark masses other than m_t. The first term in Eq. (11d) arises from the (u, d) and (c, s) isodoublets, while the second is associated with the (t, b) doublet. In Eqs. (10c), (10d), and (11d) the mass scale of $\hat{\alpha}_s$ has been chosen according to the prescription explained at the end of Sec. II.

The asymptotic behavior of $\Delta\hat{r}_W$ for large m_t can be obtained from Eqs. (9c), (10c), (10d), (11c), and (11d) and is given by

$$\Delta\hat{r}_W \sim \frac{\alpha}{6\pi\hat{s}^2}\left(1+\frac{\hat{\alpha}_s(m_t)}{\pi}\right)\ln\frac{m_t}{m_Z} \quad (m_t \gg m_Z). \tag{12}$$

This exhibits a smaller coefficient than the corresponding expression in Ref. [12], a feature that is due to a partial cancellation with the finite counterterm $-(\alpha/\pi)d$ in Eq. (10a). The asymptotic behavior for $m_H \gg m_Z$ is the same as in Ref. [12], namely

$$\Delta\hat{r}_W \sim \frac{\alpha}{24\pi\hat{s}^2}\ln\frac{m_H}{m_Z} \qquad (m_H \gg m_Z). \tag{13}$$

We now turn our attention to $\Delta\hat{r}$. When the top decoupling is implemented according to Sec. II, the expression for $\Delta\hat{r}$ becomes (see Appendix B)

$$\Delta\hat{r} = \Delta\hat{r}_W - \frac{\hat{e}^2[1-\Delta\hat{r}_W-(\alpha/\pi)\hat{d}]}{\hat{s}^2\, m_Z^2} \times{\rm Re}\left[\frac{A_{WW}(m_W^2)}{\hat{c}^2} - A_{ZZ}(m_Z^2)\right]_{\overline{\rm MS}} + \frac{\alpha}{\pi}\frac{\hat{s}^2}{\hat{c}^2}\, d, \tag{14}$$

where d and $\hat{d}$ are defined in Eqs. (9c) and (10d) and $A_{ZZ}(q^2)$ is the unrenormalized ZZ transverse self-energy with the coupling $e^2/\hat{s}^2$ factored out [1]. Except for the terms involving d and $\hat{d}$, which arise from the finite counterterms in Eqs. (7d) and (9a) associated with the decoupling of the top quark, Eq. (14) has the same form as Eq. (15b) of Ref. [1]. It is understood, however, that $\Delta\hat{r}_W$, $\hat{e}^2$, and $\hat{s}^2$ in Eq. (14) are defined according to the prescriptions of the present paper, namely Eqs. (10c), (8c), and (2). In Appendix B we show how Eqs. (10a)–(10c), and (14) can be derived from the results of Ref. [1] by neglecting very small contributions of $O(\alpha^2)$ without logarithmic or m_t^2/m_Z^2 enhancements, as well as terms of $O(\alpha^3)$. As pointed out in Ref. [1], if one neglects also subleading corrections of $O((\alpha/\pi\hat{s}^2)x_t)$, with x_t being defined in Eq. (17b), one can replace $\hat{e}^2[1-\Delta\hat{r}_W-(\alpha/\pi)\hat{d}] \to e^2$ in the second term of Eq. (14). As in Ref. [1], we have retained such subleading terms in Eq. (14) because the resulting expression describes very accurately the resummation of one-loop effects, which is particularly simple in the $\overline{\rm MS}$ framework.

We now turn our attention to the evaluation of the self-energies in the second term of Eq. (14). The bosonic contribution

$$(e^2/\hat{s}^2 m_Z^2){\rm Re}\left[A_{WW}^{(b)}(m_W^2)/\hat{c}^2 - A_{ZZ}^{(b)}(m_Z^2)\right]_{\overline{\rm MS}}$$

is given in Eq. (A.9) of Ref. [1]. To study the fermionic contributions, we write

$$C^{(f)} \equiv \frac{e^2}{\hat{s}^2 m_Z^2}{\rm Re}\left[\frac{A_{WW}^{(f)}(m_W^2)}{\hat{c}^2} - A_{ZZ}^{(f)}(m_Z^2)\right]_{\overline{\rm MS}}, \tag{15a}$$

and expand

$$C^{(f)} = C_0^{(f)} + C_{\rm QCD}^{(f)}, \tag{15b}$$

where $C_0^{(f)}$ and $C_{\rm QCD}^{(f)}$ are the $O(\alpha)$ and $O(\alpha\alpha_s)$ corrections, respectively. If the small terms proportional to $\alpha\, m_q^2/m_W^2$ $(q = d, s, b)$ are neglected, the mixing angles are once more irrelevant and, to zeroth order in α_s, we have (cf. Eq. (A.8) of Ref. [1])

$$C_0^{(f)} = \frac{\alpha}{2\pi\hat{s}^2}\frac{c^2}{\hat{c}^2}\left\{2\left(\ln c^2 - \frac{5}{3}\right) + \frac{\ln\omega_t}{2} + \frac{\omega_t}{8}(1+2\omega_t) + \frac{(\omega_t-1)^2}{2}(1+\omega_t/2)\ln\left(1-\frac{1}{\omega_t}\right)\right\}$$
$$+\frac{\alpha}{2\pi\hat{s}^2\hat{c}^2}\left\{\frac{5}{3}\left(\frac{7}{4}-\frac{10}{3}\hat{s}^2+\frac{40}{9}\hat{s}^4\right) - \frac{1}{8}\left[1+\left(1-\frac{8}{3}\hat{s}^2\right)^2\right]\left[\ln\mu_t + \frac{1}{3} + 2(1+2\mu_t)(\Lambda(D_t)-1)\right]\right\}$$
$$+\frac{3\alpha}{4\pi\hat{s}^2\hat{c}^2}\mu_t\left(\frac{1}{4}+\Lambda(D_t)-1\right), \tag{15c}$$

where $\mu_t = m_t^2/m_Z^2$, $D_t = 4\mu_t - 1$, and $\Lambda(D) = D^{1/2}\tan^{-1}(D^{-1/2})$ for $D > 0$. In analogy with our discussion of $B_0^{(f)}$, in Appendix C we give the expression for $C_0^{(f)}$ for arbitrary fermion masses in the approximation of neglecting the squares of mixing angles. We find that the corrections of $O(m^2/m_W^2)$ to $C_0^{(f)}$ are also of order 10^{-5}.

The contribution of $O(\alpha\alpha_s)$ in Eq. (15b) is given by

$$C_{\rm QCD}^{(f)} = \frac{\alpha}{4\pi\hat{s}^2\hat{c}^2}\left\{\frac{2\hat{\alpha}_s(m_Z)}{\pi}\left[c^2\ln c^2 + \left(4\zeta(3)-\frac{55}{12}\right)\left(-s^2+2\hat{s}^2\left(1-\frac{10}{9}\hat{s}^2\right)\right)\right]\right.$$
$$+\frac{\hat{\alpha}_s(m_t)}{\pi}\left[c^2\ln c^2 + \left(4\zeta(3)-\frac{55}{12}\right)\left(-s^2+2\hat{s}^2\left(1-\frac{10}{9}\hat{s}^2\right)\right) - c^2\left(4\omega_t F_1\left(\frac{1}{\omega_t}\right) - \ln\omega_t\right)\right.$$
$$\left.\left.+\left(1-\frac{8}{3}\hat{s}^2\right)^2\left(\mu_t V_1\left(\frac{1}{4\mu_t}\right) - \frac{\ln\mu_t}{4}\right) + \mu_t A_1\left(\frac{1}{4\mu_t}\right) - \frac{\ln\mu_t}{4}\right]\right\}, \tag{15d}$$

where the functions $F_1(x)$, $V_1(r)$, and $A_1(r)$ are defined in Ref. [25].

The leading asymptotic behavior of $C_0^{(f)}$ for large m_t is $(3\alpha/16\pi\hat{s}^2\hat{c}^2)(m_t^2/m_Z^2)$, which arises from the last term of Eq. (15c). In the $O(\alpha\alpha_s)$ corrections the leading contribution is contained in the combination

$$[\alpha\hat{\alpha}_s(m_t)/4\pi^2\hat{s}^2\hat{c}^2]\{\mu_t A_1[1/(4\mu_t)] - 4c^2\omega_t F_1(1/\omega_t)\},$$

which asymptotically becomes

$$-[\alpha\hat{\alpha}_s(m_t)/8\pi^2\hat{s}^2\hat{c}^2](\pi^2/3+1)(m_t^2/m_Z^2).$$

Combining these contributions, inserting the result in Eq. (14) and neglecting there the subleading $O(\alpha^2)$ contributions, we have

$$\Delta\hat{r} \sim -\frac{3\alpha}{16\pi\hat{s}^2\hat{c}^2}\frac{m_t^2}{m_Z^2}\left[1-\frac{2\hat{\alpha}_s(m_t)}{3\pi}\left(\frac{\pi^2}{3}+1\right)\right] \qquad (m_t \gg m_Z), \tag{16a}$$

while the leading asymptotic behavior for large m_H is the same as in Ref. [12], namely

$$\Delta\hat{r} \sim \frac{\alpha}{2\pi\hat{s}^2\hat{c}^2}\left(\frac{5}{6}-\frac{3}{4}\hat{c}^2\right)\ln\frac{m_H}{m_Z} \qquad (m_H \gg m_Z). \tag{16b}$$

The term involving $\hat{\alpha}_s$ in the last factor of Eq. (16a) represents the most important $O(\alpha\alpha_s)$ correction. Indeed, the contribution of Eq. (16a) depends very sensitively on m_t and the coefficient of $\hat{\alpha}_s/\pi$ in the last factor, namely -2.86, is quite large. The presence of this $\hat{\alpha}_s$ correction induces an increase in the value of m_t read from experiments of approximately

$$\frac{\Delta m_t}{m_t} \approx \frac{\hat{\alpha}_s(m_t)}{3\pi}\left(\frac{\pi^2}{3}+1\right) \approx 0.455\,\hat{\alpha}_s(m_t), \tag{16c}$$

which amounts to $\Delta m_t \approx (4.9, 7.5, 9.6, 11.7)$ GeV for $m_t = m_Z$, 150, 200, 250 GeV. It should be stressed that, as is obvious from the structure of Eq. (16a), these results depend sensitively on the precise definition of m_t. The quantity that appears in Eq. (16a) and the various expressions of this paper is the zero of the real part of the inverse propagator. In the literature it is variously referred to as the "physical," "on-shell," or "dressed" mass. In the approximation of neglecting the s dependence of the imaginary part of the top-quark self-energy, it coincides with the real part of the complex pole position [13–15]. It is also the mass that occurs in the Balmer formula for the toponium levels in the nonrelativistic bound-state picture and the parameter that governs the start of the $t\bar{t}$ cut in perturbation theory [44]. All the recent calculations of $O(\alpha\alpha_s)$ contributions and $t\bar{t}$ production [22–31] employ this definition or slight modifications thereof. It is worthwhile to notice that the $O(\hat{\alpha}_s)$ corrections become much smaller if one employs other definitions of mass [45]. For example, m_t and the running mass, $\hat{m}_t(\hat{m}_t)$, are related by [44]

$$m_t = \hat{m}_t(\hat{m}_t)[1 + 4\hat{\alpha}_s/(3\pi) + O(\hat{\alpha}_s^2)].$$

Inserting this into Eq. (16a), we get a contribution involving

$$[\hat{m}_t(\hat{m}_t)]^2[1 - 2(\hat{\alpha}_s/\pi)(\pi^2/9-1)]$$

and we see that the coefficient of $\hat{\alpha}_s/\pi$ has changed from -2.86 to -0.19. Similarly, if one expresses the cor-

rections in terms of the Georgi-Politzer mass, $M(-m^2)$ [46], which is gauge dependent and usually evaluated in the Landau gauge, the coefficient of $\hat{\alpha}_s/\pi$ becomes even smaller, namely $+0.09$. Because the perturbative evaluation of the (t,b)-isodoublet loops involves high mass scales, of $O(m_t)$ or $O(m_Z)$, both the on-shell and $\hat{m}_t(\hat{m}_t)$ definitions are in principle suitable, although the former is the natural choice in the DR approach [23–25, 30, 31]. The relevant question, of course, is what mass parametrization is more adequate to describe the physical issues at hand, namely the production and detection of the top quark. In this connection it also appears that the on-shell mass is the most appropriate parameter because, in the propagation of t and $\bar{t}$ between the "production" and "decay" vertices, configurations near the "mass shell" will be greatly enhanced kinematically (resonance effect). Another consideration, of a more practical nature, is that the m_t parametrization of the radiative corrections should be consistent with the one employed in the calculation of $t\bar{t}$ production [27–29] and, as mentioned before, this is again the "pole" or "on-shell" definition.

Returning to the evaluation of $\Delta\hat{r}$, we must still consider the irreducible contributions of $O(\alpha^2(m_t^2/m_W^2)^2)$. As mentioned before, these can be gleaned from the two-loop irreducible corrections to the ρ parameter. The very recent work of Ref. [7] leads to a significant change in the magnitude of these corrections. Indeed, these authors find for the leading high-m_t contributions to the ρ parameter an expansion of the form

$$\frac{1}{\rho} = 1 - x_t\left[1 + R\left(\frac{m_H}{m_t}\right)\frac{x_t}{3}\right] \equiv 1 - \Delta\bar{\rho}, \tag{17a}$$

where

$$x_t = \frac{3G_\mu m_t^2}{8\pi^2\sqrt{2}} \tag{17b}$$

is the one-loop term [47] and R is a negative function of m_H/m_t. When $m_H/m_t = 0$, R equals $19 - 2\pi^2 \approx -0.7392$, the result of Ref. [6], but as m_H/m_t increases, R rapidly becomes more negative, reaching a minimum of ≈ -11.8 for $m_H/m_t \approx 5.8$. In current discussions m_H/m_t ranges from ≈ 0.24 (corresponding to $m_H \approx 60$ GeV and $m_t \approx 250$ GeV) to ≈ 11 (corresponding to $m_H \approx 1$ TeV and $m_t \approx 91$ GeV). Noting that $R(0.24) \approx -3$ and $R(11) \approx -10$, it is clear that, although Refs. [6] and [7] agree in the limit $m_H/m_t = 0$, for realistic values of this ratio the results of Ref. [7] tell us that these corrections are considerably larger in magnitude.

Calling $\Delta\hat{r}^{(1)}$ the "one-loop" expression for $\Delta\hat{r}$ given in Eq. (14), we include the two-loop irreducible contributions by writing

$$\Delta\hat{r} = \Delta\hat{r}^{(1)} - \frac{1}{3}R\left(\frac{m_H}{m_t}\right)x_t^2\,\frac{\left(1-\Delta\hat{r}^{(1)}\right)^2}{1-\Delta\hat{r}_W}. \tag{18a}$$

The rationale is the following. In the $\overline{\mathrm{MS}}$ scheme the ρ parameter is naturally identified with $\hat{\rho} \equiv c^2/\hat{c}^2$ and, from Eqs. (2) and (3), we see that (cf. Eqs. (17a) and (17b) of Ref. [1])

$$\hat{\rho} = \frac{1-\Delta\hat{r}}{1-\Delta\hat{r}_W}. \tag{18b}$$

Neglecting very small terms of order $(Rx_t^2/3)^2$, one indeed verifies that when the second term in Eq. (18a) is inserted in Eq. (18b), it leads to an additional contribution of $-R(m_H/m_t)x_t^2/3$ to $1/\hat{\rho}$, in conformity with Eq. (17a).

We have given all the elements necessary to evaluate the basic radiative corrections $\Delta\hat{r}_W$ [cf. Eq. (10c)] and $\Delta\hat{r}$ [cf. Eqs. (14) and (18a)] including $O(\alpha\alpha_s)$ corrections. In conjunction with

$$\hat{s}^2 = \frac{1}{2}\left\{1 - \left[1 - \frac{4A^2}{m_Z^2(1-\Delta\hat{r})}\right]^{1/2}\right\}, \tag{19a}$$

which follows from Eq. (3), $\Delta\hat{r}$ can be employed to calculate $\hat{s}^2 \equiv \sin^2\hat{\theta}_W(m_Z)$ in terms of the accurately known quantities G_μ, m_Z, and α, as a function of m_t and m_H. The parameter $s^2 \equiv 1 - m_W^2/m_Z^2$ can be computed from (cf. Eq. (19) of Ref. [1])

$$s^2 = \hat{s}^2\left(1 - \frac{\hat{c}^2}{\hat{s}^2}\,\frac{\Delta\hat{r}_W - \Delta\hat{r}}{1-\Delta\hat{r}_W}\right) \tag{19b}$$

and Δr (cf. Eq. (22) of Ref. [1]) from

$$\Delta r = \Delta\hat{r}_W - \frac{\hat{c}^2}{\hat{s}^2}\,\frac{\Delta\hat{r}_W - \Delta\hat{r}}{1-(\hat{c}^2/\hat{s}^2)(\Delta\hat{r}_W - \Delta\hat{r})/(1-\Delta\hat{r}_W)}. \tag{19c}$$

Alternatively, writing $\hat{\rho} \equiv 1/(1-\Delta\hat{\rho})$ we have (cf. Eqs. (17a), (17b), and (20) of Ref. [1])

$$\Delta\hat{\rho} = \frac{\Delta\hat{r}_W - \Delta\hat{r}}{1-\Delta\hat{r}}, \tag{19d}$$

$$1 - \Delta r = \left(1 + \frac{c^2}{s^2}\Delta\hat{\rho}\right)(1-\Delta\hat{r}_W). \tag{19e}$$

Again, the m_W–m_Z interdependence can be expressed in two equivalent forms (cf. Eqs. (24) and (25) of Ref. [1]),

$$\frac{m_W^2}{m_Z^2} = \frac{1}{2}\left\{1 + \left[1 - \frac{4A^2}{m_Z^2(1-\Delta r)}\right]^{1/2}\right\}, \tag{19f}$$

$$\frac{m_W^2}{m_Z^2} = \frac{\hat{\rho}}{2}\left\{1 + \left[1 - \frac{4A^2}{m_Z^2\hat{\rho}(1-\Delta\hat{r}_W)}\right]^{1/2}\right\}. \tag{19g}$$

The W mass can be evaluated from (19b) or (19f) or (19g). Equations (19a)–(19g) have the same structure as in Ref. [1] because they follow from the same basic relations, namely Eqs. (1)–(3). In this paper, however, the explicit evaluation of $\Delta\hat{r}_W$ and $\Delta\hat{r}$ via Eqs. (10c), (14), and (18a) is somewhat different because we have included the $O(\alpha\alpha_s)$ corrections, updated the contributions of $e^2\mathrm{Re}[\Pi_{\gamma\gamma}^{(5)}(0) - \Pi_{\gamma\gamma}^{(5)}(m_Z^2)]$ and the two-loop irreducible parts, and implemented the decoupling of the top quark. By inference the same holds true for Δr and $\Delta\hat{\rho}$ when they are evaluated from $\Delta\hat{r}_W$ and $\Delta\hat{r}$ via Eqs. (19c) and (19d), respectively. It should be observed, moreover,

that, except for very small effects of $O(\alpha^2)$, i.e., of the same order as those we have neglected, the decoupling of the top quark should not affect physical observables such as the radiative correction Δr and the predicted value of m_W.

As shown in Appendix C, the corrections of $O(m_f^2/m_W^2)$ to Δr are also very small. However, they are enhanced relative to those of $\Delta\hat{r}_W$ and $\Delta\hat{r}$, and for $m_Z \leq m_t \leq 250$ GeV they vary from $\approx -7\times10^{-5}$ to $\approx 8\times10^{-5}$.

IV. RESIDUAL PARTS OF $\Delta\hat{r}_W$, $\Delta\hat{r}$, AND Δr

It is a simple matter to derive expressions relating m_W and m_Z to G_μ, $\hat{s}^2$, $\hat{c}^2$, and $\hat{e}^2$ (rather than e^2). To see this, we write Eq. (2) in the form

$$\hat{s}^2 = \frac{\pi\alpha}{\sqrt{2}G_\mu m_W^2}\frac{1}{1-(\alpha/\pi)\Delta_\gamma-[\Delta\hat{r}_W-(\alpha/\pi)\Delta_\gamma]}. \tag{20a}$$

Factoring out $[1-(\alpha/\pi)\Delta_\gamma]$ and recalling Eq. (8c), we have

$$\hat{s}^2 = \frac{\pi\hat{\alpha}}{\sqrt{2}G_\mu m_W^2}\frac{1}{1-(\Delta\hat{r}_W)_{\rm res}}, \tag{20b}$$

where

$$(\Delta\hat{r}_W)_{\rm res} = \left(\Delta\hat{r}_W - \frac{\alpha}{\pi}\Delta_\gamma\right)\frac{\hat{e}^2}{e^2}. \tag{20c}$$

The correction $(\Delta\hat{r}_W)_{\rm res}$ represents the "residual part" of $\Delta\hat{r}_W$ after we have subtracted the large contribution $(\alpha/\pi)\Delta_\gamma$, evaluated with the coupling $\hat{e}^2$ rather than e^2. As $(\Delta\hat{r}_W)_{\rm res} \ll \Delta\hat{r}_W$, we see that the dominant part of $\Delta\hat{r}_W$ can be absorbed by employing $\hat{\alpha}$ rather than α as the zeroth-order coupling.

Starting with Eq. (3), the analogous argument leads to

$$\hat{s}^2\hat{c}^2 = \frac{\pi\hat{\alpha}}{\sqrt{2}G_\mu m_Z^2}\frac{1}{1-(\Delta\hat{r})_{\rm res}}, \tag{21a}$$

where

$$(\Delta\hat{r})_{\rm res} = \left(\Delta\hat{r} - \frac{\alpha}{\pi}\Delta_\gamma\right)\frac{\hat{e}^2}{e^2}. \tag{21b}$$

Again $(\Delta\hat{r})_{\rm res}$ is the "residual part" of $\Delta\hat{r}$. In connection with the inclusion of the two-loop irreducible contributions of $O(\alpha^2(m_t^2/m_W^2)^2)$ [cf. Eq. (18a)], it is easy to see that the "one-loop" $\left(\Delta\hat{r}^{(1)}\right)_{\rm res}$ can be obtained by replacing $\Delta\hat{r}_W \to (\Delta\hat{r}_W)_{\rm res}$, $(\alpha/\pi)\hat{d} \to (\hat{\alpha}/\pi)\hat{d}$, and $(\alpha/\pi)d \to (\hat{\alpha}/\pi)d$ on the rhs of Eq. (14), and the final $(\Delta\hat{r})_{\rm res}$ follows by substituting $\Delta\hat{r}^{(1)} \to \left(\Delta\hat{r}^{(1)}\right)_{\rm res}$ and $\Delta\hat{r}_W \to (\Delta\hat{r}_W)_{\rm res}$ on the rhs of Eq. (18a). Of course, once $\Delta\hat{r}$ is known, one can directly use Eq. (21b) for the numerical evaluation of $(\Delta\hat{r})_{\rm res}$.

The corresponding expression from Eq. (1) is

$$s^2 = \frac{\pi\hat{\alpha}}{\sqrt{2}G_\mu m_W^2}\frac{1}{1-(\Delta r)_{\rm res}}, \tag{22a}$$

where

$$(\Delta r)_{\rm res} = \left(\Delta r - \frac{\alpha}{\pi}\Delta_\gamma\right)\frac{\hat{e}^2}{e^2} \tag{22b}$$

is the residual part of Δr. One readily finds that this quantity can be obtained by simply substituting $\Delta\hat{r}_W \to (\Delta\hat{r}_W)_{\rm res}$ and $\Delta\hat{r} \to (\Delta\hat{r})_{\rm res}$ everywhere on the rhs of Eq. (19c). Of course, Eq. (22b) can be directly used for numerical evaluations.

As illustrations, for $m_H = 250$ GeV and $m_t = 150$ GeV we have $\Delta\hat{r}_W = 7.02\times10^{-2}$, $\Delta\hat{r} = 6.34\times10^{-2}$, and $\Delta r = 4.74\times10^{-2}$, while $(\Delta\hat{r}_W)_{\rm res} = 3.6\times10^{-3}$, $(\Delta\hat{r})_{\rm res} = -3.6\times10^{-3}$, and $(\Delta r)_{\rm res} = -2.08\times10^{-2}$. The corresponding values for $m_H = 250$ GeV and $m_t = 200$ GeV are $\Delta\hat{r}_W = 7.08\times10^{-2}$, $\Delta\hat{r} = 5.88\times10^{-2}$, and $\Delta r = 2.90\times10^{-2}$, while $(\Delta\hat{r}_W)_{\rm res} = 4.3\times10^{-3}$, $(\Delta\hat{r})_{\rm res} = -8.5\times10^{-3}$, and $(\Delta r)_{\rm res} = -4.05\times10^{-2}$. Unlike $(\Delta\hat{r}_W)_{\rm res}$ or $(\Delta\hat{r})_{\rm res}$, $(\Delta r)_{\rm res}$ is quite large for $m_t \approx 200$ GeV.

A correction similar to $(\Delta r)_{\rm res}$ has been recently employed in Ref. [20]. The two quantities are, however, not identical because $\hat{\alpha}$, defined in the $\overline{\rm MS}$ scheme, differs somewhat from the effective parameter $\alpha(m_Z) = (128.8)^{-1}$ used in that work. This illustrates the rather obvious but important fact that running couplings are scheme dependent.

V. Δr IN THE ON-SHELL SCHEME

In this section we discuss the incorporation of the perturbative $O(\alpha\alpha_s)$ corrections to Δr [2, 3] in the on-shell scheme of renormalization [2]. We follow the formulation proposed recently in Ref. [8], based on the expression

$$\Delta r = \Delta\alpha - \frac{c^2}{s^2}\Delta\bar{\rho}\,(1-\Delta\alpha) + (\Delta r)_{\rm rem}, \tag{23a}$$

where $\Delta\alpha = 0.0597 \pm 0.0009$ represents the contribution of the charged leptons and the first five quark flavors to the photon vacuum-polarization function evaluated at $q^2 = m_Z^2$, i.e., $e^2{\rm Re}[\Pi_{\gamma\gamma}(0) - \Pi_{\gamma\gamma}(m_Z^2)]$,

$$\Delta\bar{\rho} = x_t\left[1 + R\left(\frac{m_H}{m_t}\right)\frac{x_t}{3}\right], \tag{23b}$$

x_t is defined in Eq. (17b), and $R(m_H/m_t)$ [7] is the function discussed after Eq. (17a). The result for $\Delta\alpha$ quoted above includes the recent calculation of the first five quark flavor contributions [10] and, for this reason, it slightly differs from the central value of 0.0602 employed in Ref. [8]. The second term in Eq. (23a) involves the leading m_t-dependent correction $\Delta\bar{\rho}$ to $1-1/\rho$ [cf. Eq. (17a)] and we see that, in the case of Δr, it is enhanced by a factor c^2/s^2. Its importance for large m_t in the m_W–m_Z interdependence was pointed out in 1980, in the work of Marciano and one of us (A.S.) [41]. Since that time, this potential effect has been discussed by several authors [4, 5, 22–26, 48, 49].

The separation into leading contributions [the first two terms in Eq. (23a)] and a "remainder" $(\Delta r)_{\rm rem}$ is the same as was proposed in Refs. [4, 5, 26], except that we have included the recent results of Ref. [7] on the two-

loop contribution to $\Delta\bar{\rho}$. It is important to note that $(\Delta r)_{\rm rem}$ differs from the quantity $(\Delta r)_{\rm res}$ introduced at the end of Sec. III. Whereas in the latter we subtract the large logarithmic corrections, in $(\Delta r)_{\rm rem}$ we also exclude the leading m_t-dependent contributions. The formulation of Ref. [8] provides also a very specific prescription to calculate $(\Delta r)_{\rm rem}$, namely

$$(\Delta r)_{\rm rem} = \Delta r^{(1)} - \Delta\alpha + \frac{c^2}{s^2}X + \frac{c^2}{s^2}(\tilde{x}_t - X)\frac{\sqrt{2}G_\mu m_W^2(1-\Delta\alpha)s^2}{\pi\alpha}, \tag{24a}$$

where $\Delta r^{(1)}$ is the familiar one-loop expression of Ref. [2], expressed in terms of α and α/s^2 as expansion parameters,

$$\tilde{x}_t = \frac{3\alpha}{16\pi s^2}\frac{m_t^2}{m_W^2}, \tag{24b}$$

$$X = \frac{e^2}{s^2}{\rm Re}\left[\frac{A_{WW}(m_W^2)}{m_W^2} - \frac{A_{ZZ}(m_Z^2)}{m_Z^2}\right]_{\rm fin}, \tag{24c}$$

and the subscript "fin" means "finite part," i.e., that the pole terms have been subtracted and μ' has been set equal to m_Z. We also note that X is a gauge-invariant quantity. In Eq. (24c) we follow the notation of Ref. [1], which differs from that of Refs. [2, 41] in that an explicit coupling e^2/s^2 has been factored out in the A_{WW} and A_{ZZ} self-energies. The term $-\Delta\alpha$ in Eq. (24a) subtracts from $\Delta r^{(1)}$ the large logarithmic corrections which are included as part of the leading contributions in Eq. (23a). The quantity $(c^2/s^2)X$ subtracts another important part of $\Delta r^{(1)}$, which is then treated more accurately in the second term of Eq. (23a) and the last term of Eq. (24a), according to the following prescription. Decomposing $-(c^2/s^2)X = -(c^2/s^2)\tilde{x}_t + (c^2/s^2)(\tilde{x}_t - X)$, the dominant part, $-(c^2/s^2)\tilde{x}_t$, is included in the second term of Eq. (23a) with the effective coupling constant changed according to $\alpha/s^2 \to \sqrt{2}G_\mu m_W^2(1-\Delta\alpha)/\pi$. The nondominant part, $(c^2/s^2)(\tilde{x}_t - X)$, is treated with the same coupling modification but it is retained as part of $(\Delta r)_{\rm rem}$ [the last term of Eq. (24a)]. The rationale for this treatment of $(c^2/s^2)(\tilde{x}_t - X)$ was explained in Refs. [8, 50] and reflects the fact that a careful analysis of the resummation of one-loop effects leads to an expression of the form of Eq. (23a) in which X, rather than its dominant part x_t, is multiplied by $\sqrt{2}G_\mu s^2 m_W^2(1-\Delta\alpha)/(\pi\alpha) = (1-\Delta\alpha)/(1-\Delta r)$. This is natural because, since Δr is the radiative correction in the relation between m_W, m_Z, G_μ, and α, it should involve quantities evaluated at $q^2 = m_Z^2$ and $q^2 = m_W^2$ such as X, rather than amplitudes evaluated at $q^2 = 0$ such as $\Delta\bar{\rho}$. We also point out that the neglect of this effect in Eq. (24a), namely the replacement $\sqrt{2}G_\mu m_W^2 s^2(1-\Delta\alpha)/(\pi\alpha) \to 1$ in the last term, would induce a change $\approx (c^2/s^2)(\tilde{x}_t - X)(c^2/s^2)\Delta\bar{\rho} = O((\alpha/\pi s^2)(c^2/s^2)^2\Delta\bar{\rho})$; although formally subleading, this is enhanced by two powers of c^2/s^2 and is, therefore, significantly larger than the expected theoretical error. It was already pointed out in Ref. [8] that Eqs. (23a) and (24a) include correctly not only the leading terms of $O(\alpha^2(m_t^2/m_W^2)^2)$ and $O(\alpha^2\ln^2(m_Z/m_f))$, where m_f is a generic fermion mass, but also the subleading contributions of $O(\alpha^2\ln(m_Z/m_f))$.

We now turn our attention to the incorporation of the perturbative $O(\alpha\alpha_s)$ contributions. We first consider Eq. (24a). The amplitudes modified by the QCD corrections are $X^{(f)}$ in the last term and the self-energy contributions $(e^2/s^2m_W^2){\rm Re}[A_{WW}^{(f)}(m_W^2) - A_{WW}^{(f)}(0)]_{\rm fin}$ and $e^2\left[\Pi_{\gamma\gamma}^{(f)}(0)\right]_{\rm fin}$ contained in $\Delta r^{(1)}$, where the superscript (f) denotes again fermionic contributions. The first two are obtained from $C^{(f)}$ [cf. Eqs. (15a)–(15d)] and $B^{(f)}$ [cf. Eqs. (11a)–(11d)], respectively, by simply changing everywhere $\hat{s}^2 \to s^2$ and $\hat{c}^2 \to c^2$. In particular, their QCD corrections are derived from Eqs. (15d) and (11d), respectively. The correction $e^2\left[\Pi_{\gamma\gamma}^{(f)}(0)\right]_{\rm fin}$ can be read from Eq. (7a) by removing the pole terms, setting $\mu' = m_Z$, and substituting $\hat{\alpha} \to \alpha$. It is not necessary to consider $(c^2/s^2)X^{(f)}$ in the third term of Eq. (24a) because, as explained before, it cancels an identical contribution in $\Delta r^{(1)}$. The value of $\Delta\alpha$ is not modified, as QCD corrections have already been taken into account in its evaluation.

In the above discussion, the quantities $\tilde{x}_t$ in Eq. (24a) and $\Delta\bar{\rho}$ in Eq. (23a) have not been altered, so that, except for $\Delta\alpha$, all the QCD corrections are contained in $(\Delta r)_{\rm rem}$. We may wish, however, to incorporate the leading QCD corrections in the second term of Eq. (23a). To achieve this, we subtract them from Eq. (24a) by replacing

$$\tilde{x}_t \to \tilde{x}_t' = \frac{3\alpha}{16\pi s^2}\frac{m_t^2}{m_W^2}\left[1 - \frac{2\hat{\alpha}_s(m_t)}{3\pi}\left(\frac{\pi^2}{3}+1\right)\right] \tag{25a}$$

in the last term of that equation and, at the same time, we substitute

$$\Delta\bar{\rho} \to \Delta\bar{\rho}' = x_t\left[1 - \frac{2\hat{\alpha}_s(m_t)}{3\pi}\left(\frac{\pi^2}{3}+1\right) + R\left(\frac{m_H}{m_t}\right)\frac{x_t}{3}\right] \tag{25b}$$

in Eq. (23a). The overall evaluation of Δr is, of course, the same whether we employ $\Delta\bar{\rho}$ in Eq. (23a) and $\tilde{x}_t$ in Eq. (24a) or the modified quantities, $\Delta\bar{\rho}'$ and $\tilde{x}_t'$. In the second formulation, however, the first two terms of Eq. (23a), with $\Delta\bar{\rho} \to \Delta\bar{\rho}'$, describe more accurately the leading m_t-dependent corrections.

The contributions of $O(\alpha^2\hat{\alpha}_s, \alpha\hat{\alpha}_s^2)$ are unknown at the present time and for this reason we have not made any attempt to include them. However, the structure of Eq. (25b) gives a hint about what their magnitude might be. Suppose, for example, that the leading QCD effects are always very small when the electroweak corrections are expressed in terms of the running mass, $\hat{m}_t(\hat{m}_t)$, as it happens with the $O(\alpha\hat{\alpha}_s)$ corrections. In that hypothetical case, the discussion after Eq. (16c) indicates that the modified $\Delta\bar{\rho}$ parameter would be obtained approximately by appending a factor $\{1 - [2\hat{\alpha}_s(m_t)/3\pi](\pi^2/3+1)\}$ to each x_t in Eq. (23b). For $m_H = 600$ GeV and $m_t = 200$ GeV, the difference with Eq. (25b) would lead to an aditional contribution to Δr of -3.4×10^{-4}. This is of the same order of magnitude as the sublead-

ing terms of $O((\alpha/\pi s^2)(c^2/s^2)x_t)$, discussed in Sec. I. The surprisingly large size of these possible corrections of $O((c^2/s^2)\hat{\alpha}_s x_t^2)$ is due to the m_t^4 dependence and the considerable magnitude of the function R. This observation illustrates the fact that, for large m_t values, the theoretical error due to unknown higher-order corrections may arise from very different sources.

VI. THRESHOLD EFFECTS

The fermionic contribution to the vacuum polarization of the intermediate vector bosons can be expressed in terms of the amplitudes

$$\Pi^{V,A}_{\mu\nu}(q,m_1,m_2) = -i\int d^4x\, e^{iq\cdot x}\langle 0\,|T^*[J^{V,A}_\mu(x)J^{V,A\,\dagger}_\nu(0)]|\,0\rangle, \quad (26a)$$

where T^* is the covariant time-ordered product and $J^V_\mu = \bar{\psi}_1\gamma_\mu\psi_2$ and $J^A_\mu = \bar{\psi}_1\gamma_\mu\gamma_5\psi_2$ are the vector and axial-vector currents, constructed with the spinors fields ψ_1 and ψ_2 endowed with masses m_1 and m_2, respectively. Thus, except for vector currents with $m_1 = m_2$, the conservation of the currents is explicitly broken by mass terms. In turn, the tensors $\Pi^{V,A}_{\mu\nu}$ have the well-known structure

$$\Pi^{V,A}_{\mu\nu}(q,m_1,m_2) = \Pi^{V,A}(s,m_1,m_2)\,g_{\mu\nu} + \lambda^{V,A}(s,m_1,m_2)\,q_\mu q_\nu, \quad (26b)$$

where, throughout this section, $s \equiv q^2$.

Threshold effects involving the $t\bar{t}$, $t\bar{b}$, and $b\bar{b}$ channels can be expressed as contributions to the imaginary parts of the functions $\Pi^{V,A}$ and $\lambda^{V,A}$ [24, 27–31]. A number of papers [23, 24] have made use of DR's to express the physically important amplitudes $\Pi^{V,A}$ in terms of their imaginary parts. In Refs. [30, 31] two of us (B.K. and A.S.) proposed to use DR's directly constructed from the Ward identities. We recall the basic strategy: contracting both sides of Eq. (26a) with q^μ, one derives the relation

$$\Pi^{V,A}(s,m_1,m_2) = -s\lambda^{V,A}(s,m_1,m_2) + \Delta^{V,A}(s,m_1,m_2), \quad (27a)$$

where $\Delta^{V,A}(s,m_1,m_2)$ is defined by

$$\int d^4x\, e^{iq\cdot x}\langle 0\,|T[\partial^\mu J^{V,A}_\mu(x)J^{V,A\,\dagger}_\nu(0)]|\,0\rangle \equiv \Delta^{V,A}(s,m_1,m_2)\,q_\nu. \quad (27b)$$

The idea then is to write DR's for $\lambda^{V,A}$ and $\Delta^{V,A}$ and to obtain $\Pi^{V,A}$ by means of Eq. (27a). This approach has been employed in Refs. [30, 31] to discuss both the perturbative and the threshold contributions in the on-shell scheme of renormalization. The aim of this section is to extend the discussion, so that the threshold effects can also be included in the $\overline{\text{MS}}$ calculations.

We recall that in our analysis the threshold contributions to the imaginary parts are nonzero over a small, finite range, i.e., they have compact support. As a consequence, the corresponding unsubtracted DR integrals for $\lambda^{V,A}$ and $\Delta^{V,A}$ are convergent and, moreover, vanish as $|s| \to \infty$. In using the DR approach, we self-consistently assume that the threshold contributions to $\lambda^{V,A}$ and $\Delta^{V,A}$ tend to zero as $|s| \to \infty$, so that these quantities satisfy unsubtracted DR's. Thus,

$$\lambda^{V,A}(s,m_1,m_2) = \frac{1}{\pi}\int ds'\,\frac{\mathrm{Im}\lambda^{V,A}(s',m_1,m_2)}{s'-s-i\epsilon}, \quad (28a)$$

$$\Delta^{V,A}(s,m_1,m_2) = \frac{1}{\pi}\int ds'\,\frac{\mathrm{Im}\Delta^{V,A}(s',m_1,m_2)}{s'-s-i\epsilon}. \quad (28b)$$

Inserting these expressions into Eq. (27a) and using the same equation to relate the imaginary parts, one obtains the two equivalent representations [31]

$$\Pi^{V,A}(s,m_1,m_2) = \frac{1}{\pi}\int ds'\left[\frac{\mathrm{Im}\Pi^{V,A}(s',m_1,m_2)}{s'-s-i\epsilon} + \mathrm{Im}\lambda^{V,A}(s',m_1,m_2)\right], \quad (29a)$$

$$\Pi^{V,A}(s,m_1,m_2) = \frac{s}{\pi}\int \frac{ds'}{s'}\,\frac{\mathrm{Im}\Pi^{V,A}(s',m_1,m_2)}{s'-s-i\epsilon} + \frac{1}{\pi}\int \frac{ds'}{s'}\mathrm{Im}\Delta^{V,A}(s',m_1,m_2). \quad (29b)$$

In Eqs. (28a)–(29b) and henceforth it is understood that $\lambda^{V,A}$, $\Delta^{V,A}$, and $\Pi^{V,A}$ represent the excess threshold contributions relative to the perturbative $O(\alpha\alpha_s)$ corrections. As explained in Ref. [31], Eq. (29b) can be directly derived from the following assumptions: (*i*) $\Pi^{V,A}$ satisfies a once-subtracted DR; (*ii*) the subtraction constant is determined from the Ward identity (27a), so that $\Pi^{V,A}(0) = \Delta^{V,A}(0)$; (*iii*) $\Delta^{V,A}(s)$ satisfies an unsubtracted DR, so that $\Delta^{V,A}(0)$ can be calculated from the second integral in Eq. (29b). The fact that $\Pi^{V,A}$ must satisfy a subtracted DR can be clearly seen by considering the particular case of vector currents with equal masses. In that case $\lambda^V(s',m,m) = -\Pi^V(s',m,m)/s'$ and $\Delta^V(s',m,m) = 0$, so that Eqs. (29a) and (29b) reduce to

$$\Pi^V(s,m,m) = \frac{s}{\pi}\int \frac{ds'}{s'}\,\frac{\mathrm{Im}\Pi^V(s',m,m)}{s'-s-i\epsilon}, \quad (30a)$$

which vanishes at $s = 0$, in conformity with the Ward identity (27a). If, instead, $\Pi^V(s,m,m)$ were to satisfy an unsubtracted DR, the condition $\Pi^V(0,m,m) = 0$ would imply $\int ds'\,\mathrm{Im}\Pi^V(s',m,m)/s' = 0$, which is manifestly false, as $\mathrm{Im}\Pi^V(s',m,m) \geq 0$. In summary, Eqs. (29a) and (29b) are the simplest possible DR's consistent with the Ward identity (27a). As mentioned before, the latter is a crucial requirement. We also see from Eq. (29a) that the threshold effects modify the asymptotic behavior of the full $\Pi^{V,A}(s)$ as $|s| \to \infty$ by constants, i.e., by subleading terms. [We recall that the perturbative contributions to $\Pi^{V,A}(s)$ grow as s, modulo logarithms.]

Threshold effects associated with the $t\bar{b}$ channels are

greatly suppressed because they are proportional to the squared reduced mass of the quarks [24] and can be neglected. Those involving the $b\bar{b}$ channel, i.e., "bottomium resonances," give significant contributions only to $e^2\Pi_{\gamma\gamma}(0)$ and are already included in the evaluation of $e^2\mathrm{Re}[\Pi^{(5)}_{\gamma\gamma}(0) - \Pi^{(5)}_{\gamma\gamma}(m_Z^2)]$. Thus, only the case $m_1 = m_2 = m_t$ is significant. For vector currents with equal masses, the relevant DR is given in Eq. (30a). For axial-vector currents, we employ

$$\Pi^A(s,m,m) = \frac{1}{\pi}\int ds' \left[\frac{\mathrm{Im}\Pi^A(s',m,m)}{s'-s-i\epsilon} + \mathrm{Im}\lambda^A(s',m,m)\right], \quad (30b)$$

in accordance with Eq. (29a). The amplitude $\mathrm{Im}\Pi^A(s',m_t,m_t)$ receives contributions from $J^P = 1^+$ states, i.e., $l = 1$, which are suppressed near threshold by centrifugal barrier effects. On the other hand, $\mathrm{Im}\lambda^A(s',m_t,m_t)$ also receives significant contributions from 0^- states, i.e., $l = 0$. A detailed discussion of $\mathrm{Im}\Pi^V(s',m_t,m_t)$ and $\mathrm{Im}\lambda^A(s',m_t,m_t)$ in both the resonance [24] and Green-function [28] approaches is given in Ref. [31]. In both cases one finds [31]

$$\mathrm{Im}\lambda^A(s',m_t,m_t) \approx \mathrm{Im}\lambda^V(s',m_t,m_t) = -\frac{\mathrm{Im}\Pi^V(s',m_t,m_t)}{s'}. \quad (30c)$$

The second equality is, of course, an exact consequence of the Ward identity (27a).

We now discuss the specific threshold contributions to the basic radiative corrections studied in the paper. The corrections Δ_γ and $\Delta\hat{r}_W$ contain $e^2\Pi^{(f)}_{\gamma\gamma}(0)$ [cf. Eqs. (10a) and (10b)]. The top contribution to this amplitude is obtained from Eq. (A7),

$$e^2\Pi^{(t)}_{\gamma\gamma}(0) = e^2\tfrac{4}{9}(\Pi^V)'(0,m_t,m_t), \quad (31a)$$

where $(\Pi^V)'(0,m_t,m_t) = (\partial/\partial s)\Pi^V(s,m_t,m_t)|_{s=0}$. Recalling Eq. (30a), we have

$$(\Pi^V)'(0,m_t,m_t) = \frac{1}{\pi}\int \frac{ds'}{s'^2}\mathrm{Im}\Pi^V(s',m_t,m_t). \quad (31b)$$

There are no additional significant threshold effects in $\Delta\hat{r}_W$, because those involving $\mathrm{Re}[A_{WW}(m_W^2) - A_{WW}(0)]/m_W^2$ [cf. Eq. (10a)] are suppressed by reduced-mass effects. In the case of $\Delta\hat{r}$ [cf. Eq. (14)] the same holds true for the term involving $\mathrm{Re}A_{WW}(m_W^2)/(\hat{c}^2 m_Z^2)$; there are, however, significant threshold contributions to $\mathrm{Re}A_{ZZ}(m_Z^2)/m_Z^2$. According to Eq. (A13)

$$\frac{\mathrm{Re}A^{(t)}_{ZZ}(m_Z^2)}{m_Z^2} = -\frac{1}{16\hat{c}^2 m_Z^2}\mathrm{Re}\left[\left(1-\frac{8}{3}\hat{s}^2\right)^2\Pi^V(m_Z^2,m_t,m_t) + \Pi^A(m_Z^2,m_t,m_t)\right]. \quad (32a)$$

Using Eqs. (30a)–(30c), we have

$$\frac{\mathrm{Re}\Pi^V(m_Z^2,m_t,m_t)}{m_Z^2} = \frac{1}{\pi}\mathcal{P}\int\frac{ds'}{s'}\frac{\mathrm{Im}\Pi^V(s',m_t,m_t)}{s'-m_Z^2}, \quad (32b)$$

$$\frac{\mathrm{Re}\Pi^A(m_Z^2,m_t,m_t)}{m_Z^2} \approx \frac{1}{\pi m_Z^2}\int ds'\,\mathrm{Im}\lambda^A(s',m_t,m_t) \approx -\frac{1}{\pi m_Z^2}\int\frac{ds'}{s'}\mathrm{Im}\Pi^V(s',m_t,m_t), \quad (32c)$$

where $\mathcal{P}$ denotes the principal value of the integral. We note that, as explained earlier, the threshold behavior of $\mathrm{Im}\Pi^A(s',m_t,m_t)$ is suppressed by centrifugal barrier effects and its contribution has been neglected in Eq. (32c). As the support for the threshold contributions to $\mathrm{Im}\Pi^V(s',m_t,m_t)$ is located in the neighborhood of $s' \approx 4m_t^2$, the dominant effect for $m_t^2 \gg m_Z^2$ is given by Eq. (32c), with Eqs. (31b) and (32b) being relatively suppressed by a factor $m_Z^2/(4m_t^2)$. Furthermore, the term involving $\mathrm{Re}\Pi^V(m_Z^2,m_t,m_t)$ in Eq. (32a) has a small cofactor $(1-8\hat{s}^2/3)^2 \approx 0.14$. In the range $m_Z \le m_t \le 250$ GeV, Eq. (32c) increases more rapidly than linearly with m_t, while the contributions of Eqs. (31b) and (32b) to $\Delta\hat{r}$ and Δr remain very small, at most a few times 10^{-5}. As discussed in Ref. [31], the sign of Eq. (32c) can be understood with an argument reminiscent of the one employed in technicolor theories to explain the generation of the vector-boson masses: 0^- states contribute to the functions $\lambda^A(s)$ and $\Delta^A(s)$ and, via the Ward identity (27a), to $\Pi^A(s)$. In technicolor theories this gives rise to $m_Z^2 \sim A_{ZZ}(0)$. In our case, when Eq. (32c) is inserted in Eq. (32a), it leads to a positive contribution to the Z^0 mass shift, $\delta m_Z^2 = \mathrm{Re}A_{ZZ}(m_Z^2)$.

Inserting into Eqs. (31b), (32b), and (32c) the detailed evaluations of $\mathrm{Im}\Pi^V(s',m_t,m_t)$ and $\mathrm{Im}\lambda^A(s',m_t,m_t)$ [31] derived from the analyses of Refs. [24, 28], we obtain the additional contributions to $\Delta\hat{r}_W$ and $\Delta\hat{r}$ associated with the $t\bar{t}$ threshold. Because the threshold contributions in $e^2\Pi^{(t)}_{\gamma\gamma}(0)$ and $\mathrm{Re}A_{\gamma Z}(m_Z^2)/m_Z^2$ are very small, we have not subtracted them from Δ_γ or the latter amplitude. Consequently, they do not affect the counterterms of Eqs. (7d) and (9a).

Analogous considerations are valid in the on-shell evaluation of Δr (see Sec. V and Ref. [31]). The threshold corrections that are not suppressed by reduced-mass effects occur again in $\Pi^{(t)}_{\gamma\gamma}(0)$ and $\mathrm{Re}A^{(t)}_{ZZ}(m_Z^2)/m_Z^2$, the latter being by far the dominant contribution for large m_t. Here one sets $\hat{s}^2 \to s^2$ and $\hat{c}^2 \to c^2$ everywhere. We also recall that in Δr the contribution of Eq. (32a) is enhanced by a factor c^2/s^2.

We end this section with the observation that the magnitude of the $1S$ contribution to $\mathrm{Im}\Pi^V(s,m_t,m_t)/s$ can be roughly understood on the basis of a simplified "Bohr-atom" model. We recall that, neglecting hard-gluon corrections, this contribution is approximately given by [24, 31]

$$\frac{\mathrm{Im}\Pi^V(s,m_t,m_t)}{s} \approx 3\,\frac{|R_{1,0}(0)|^2}{M_\theta}\,\delta\left(s-M_\theta^2\right), \quad (33)$$

where $R_{1,0}(0)$ is the radial wave function at the origin and $M_\theta \approx 2m_t$ is the mass of the $1S$ resonance, θ. In Ref. [24], $|R_{1,0}(0)|^2$ and M_θ have been studied in detail on the basis of the Richardson potential. Suppose now that we attempt to estimate this effect using the "Bohr-atom" picture. In the case of the one-electron atom, $[R_{1,0}(0)]^2 = 4(\alpha\mu)^3$, where μ is the reduced mass. For toponium we set $\mu = m_t/2$, $M_\theta = 2m_t$, and replace $\alpha \to 4\hat{\alpha}_s(k_1)/3$, where $k_1 = 2\hat{\alpha}_s(k_1)m_t/3$ is the inverse Bohr radius [27]. This leads to $3|R_{1,0}(0)|^2/\left(M_\theta m_Z^2\right) = (16/9)(\hat{\alpha}_s(k_1))^3 m_t^2/m_Z^2$, which is to be compared with $3m_t\xi(m_t)/\left(8m_Z^2\right)$, obtained from the Richardson potential [24, 31]. Here $\xi(m_t)$ is a monotonically increasing function of m_t, which is evaluated numerically and varies from 1.95 to 3.08 GeV for $m_Z \le m_t \le 250$ GeV. For $m_t = m_Z$, 250 GeV, the Richardson potential gives 0.0080 and 0.0347, respectively, while in the Bohr-atom model the corresponding values are 0.0096 and 0.0428, which are about 20% larger. This may be partly due to the fact that the Richardson potential is softer than Coulombic near the origin. Interestingly, the ratio of the values at $m_t = 250$ GeV and $m_t = m_Z$, which gives an indication of the m_t dependence, is almost the same in both cases. In the Bohr-atom model, for the nS states, $|R_{n,0}(0)|^2$ scales as n^{-3}. However, as the radii of the higher orbits increase as n^2, the momentum k_n at which $\hat{\alpha}_s$ is to be evaluated becomes smaller. For sufficiently high n, the relevant $\hat{\alpha}_s$ falls in the nonperturbative regime and the "Coulombic" picture becomes increasingly doubtful. The evaluation based on a realistic, "confining" potential is clearly preferable. However, it is interesting that a simple Coulomb potential gives a similar answer, at least in the case of the $1S$ state.

VII. NUMERICAL CALCULATIONS

In the previous sections we have discussed the theoretical background necessary for the incorporation of the leading QCD effects in the basic corrections $\Delta\hat{r}_W$, $\Delta\hat{r}$, and Δr. At the same time, as explained in Sec. I, we have introduced a number of refinements in the analysis of the electroweak corrections.

In this section we apply the previous results to numerically evaluate $\Delta\hat{r}_W$, $\Delta\hat{r}$, and Δr, and, most importantly, to derive precise values for m_W and $\hat{s}^2 \equiv \sin^2\hat{\theta}_W(m_Z)$, as functions of m_t and m_H. Working first in the $\overline{\text{MS}}$ scheme, the basic strategy is the same as in Ref. [1]. We employ Eq. (10c) and Eqs. (14) and (18a) as the basic expressions for $\Delta\hat{r}_W$ and $\Delta\hat{r}$, respectively, and iteratively evaluate these corrections in conjunction with Eq. (19a), leading to accurate values for $\hat{s}^2$. Then Δr can be evaluated from Eqs. (19c) or (19e), s^2 from Eq. (19b), and m_W from either s^2 or Eq. (19f). We use as input values $\alpha = (137.0359895)^{-1}$, $G_\mu = 1.16639 \times 10^{-5}$ GeV^{-2} [38], $m_Z = 91.187$ GeV [16], leading to $A = (\pi\alpha/\sqrt{2}G_\mu)^{1/2} = 37.2802$ GeV, and $\hat{\alpha}_s(m_Z) = 0.118$ [39]. As shown in Appendix C, the effects arising from finite fermion masses are very minor. Nonetheless, we include them as follows: the u, d, and s quarks are treated as massless, while we employ $m_c = 1.5$ GeV and $m_b = 4.5$ GeV. The leptons are given their physical masses [38], including the recent value $m_\tau = 1.777$ GeV [51]. In order to incorporate the new results on the leading irreducible corrections of $O(\alpha^2)$ [7], it is very convenient to use a precise analytical representation of the function $R(m_H/m_t)$ in Eqs. (17a), (18a), and (23b). In the range $r = m_H/m_t > 4$, the authors of Ref. [7] give the accurate asymptotic expansion

$$\begin{aligned} R(r) = {} & \frac{49}{4} + \pi^2 - 27\ln r + 6\ln^2 r \\ & + \frac{1}{3r^2}(2 - 12\pi^2 - 24\ln r - 108\ln^2 r) \\ & + \frac{1}{48r^4}(1613 - 240\pi^2 + 3000\ln r - 2880\ln^2 r). \end{aligned} \tag{34a}$$

In the complementary domain $0 < r \le 4$, only numerical values are available [7], which we have fitted with the expression

$$\begin{aligned} R(r) = {} & -0.7392088 + r(-11.5315 + 0.382497\ln r) \\ & + r^2(5.31338 - 3.055\ln r + 0.523039\ln^2 r). \end{aligned} \tag{34b}$$

The maximum deviation of Eq. (34b) from the original data [7] is $\lesssim 0.025$ and occurs near the matching point, $r = 4$.

In Tables I–V we display the calculated values of m_W and $\hat{s}^2$, as functions of m_t for $m_H = (60, 100, 250, 600, 1000)$ GeV. Currently, a fit to all data, using the electroweak radiative corrections of the SM and $\alpha_s(m_Z) = 0.120 \pm 0.006$, gives $m_t = 145^{+17+17}_{-19-19}$ GeV [51]. The central value corresponds to $m_H = 300$ GeV and the last error reflects the theoretical uncertainty associated with the range 60 GeV $\le m_H \le$ 1 TeV. This implies $m_t \le 173^{+17}_{-19}$ GeV at the 95% confidence level, where the error is again due to the m_H uncertainty. Although this strongly suggests that $m_t \lesssim 200$ GeV, in the tables we present values up to $m_t = 250$ GeV. One of the reasons is that it is theoretically interesting to study and compare the high-m_t dependence of the various radiative corrections. Moreover, there is always the possibility of a statistical surprise or that unknown new physics may alter the predictions of the SM, so that scenarios in which m_t is found at higher values are not completely excluded. For comparison purposes, we list in the tables the results of four different calculations: (*i*) only electroweak corrections, with all QCD corrections turned off (columns labeled EW for "electroweak"); (*ii*) electroweak plus perturbative $O(\alpha\alpha_s)$ corrections (columns labeled P for "perturbative"); (*iii*) the above, plus threshold effects calculated in the resonance approach according to Sec. VI (columns labeled P+R for "perturbative plus resonance"); (*iv*) the same as in (*iii*) but with threshold effects evaluated in the Green-function approach (columns labeled P+G for "perturbative plus Green function"). In order to keep the tables compact, we have displayed only the quantities of greatest physical interest, namely m_W and $\hat{s}^2$, rather than the radiative corrections $\Delta\hat{r}_W$, $\Delta\hat{r}$, and Δr, or the derived parameter s^2 of the on-shell scheme. The interested reader can readily glean these important quantities from the tables. Thus, inserting

TABLE I. Calculated values of $\sin^2\hat{\theta}_W(m_Z)$ and m_W, as a function of m_t, for $m_Z = 91.187$ GeV, $\hat{\alpha}_s(m_Z) = 0.118$, and $m_H = 60$ GeV. The EW column includes only electroweak radiative corrections. The P column incorporates perturbative $O(\alpha\alpha_s)$ contributions (see Sec. III). The P+R and P+G columns contain also $t\bar{t}$ threshold effects (see Sec. VI) in the resonance and Green-function approaches, respectively. The on-shell parameter $\sin^2\theta_W$ and the radiative corrections $\Delta\hat{r}_W$, $\Delta\hat{r}$, and Δr can be gleaned from this table, as explained in Sec. VII.

m_t (GeV)	$\hat{s}^2$				m_W (GeV)			
	EW	EW + QCD			EW	EW + QCD		
		P	P+R	P+G		P	P+R	P+G
m_Z	0.23307	0.23321	0.23327	0.23326	79.979	79.947	79.938	79.941
100	0.23287	0.23303	0.23309	0.23307	80.029	79.996	79.986	79.989
110	0.23262	0.23280	0.23287	0.23285	80.085	80.050	80.038	80.041
120	0.23235	0.23255	0.23263	0.23261	80.141	80.103	80.091	80.094
130	0.23207	0.23229	0.23238	0.23236	80.200	80.158	80.144	80.148
140	0.23176	0.23201	0.23211	0.23209	80.260	80.215	80.199	80.203
150	0.23144	0.23172	0.23182	0.23180	80.323	80.274	80.256	80.260
160	0.23110	0.23140	0.23152	0.23150	80.389	80.335	80.316	80.319
170	0.23074	0.23108	0.23120	0.23118	80.457	80.398	80.377	80.380
180	0.23036	0.23073	0.23087	0.23085	80.528	80.463	80.441	80.444
190	0.22997	0.23037	0.23052	0.23050	80.603	80.532	80.507	80.510
200	0.22955	0.22999	0.23015	0.23013	80.680	80.603	80.576	80.578
210	0.22912	0.22959	0.22976	0.22976	80.760	80.677	80.648	80.650
220	0.22867	0.22918	0.22936	0.22936	80.844	80.754	80.723	80.723
230	0.22820	0.22875	0.22895	0.22895	80.930	80.833	80.800	80.799
240	0.22772	0.22831	0.22851	0.22853	81.020	80.916	80.881	80.878
250	0.22721	0.22785	0.22806	0.22809	81.114	81.001	80.964	80.959

the value of $\hat{s}^2$ in Eq. (3) and those of $\hat{s}^2$ and m_W in Eq. (2), one finds $\Delta\hat{r}$ and $\Delta\hat{r}_W$, respectively. Using m_W, one calculates $s^2 \equiv 1 - m_W^2/m_Z^2$ and, in conjunction with Eq. (1), Δr. We have kept enough decimal figures in $\hat{s}^2$ and m_W, so that $\Delta\hat{r}_W$, $\Delta\hat{r}$, Δr, and s^2 can be accurately evaluated. The tables allow us to separate the threshold effects from the more established perturbative $O(\alpha\alpha_s)$ contributions. Although the resonance (R) and Green-function (G) approaches are quite different conceptually and technically, the tables reveal the welcome and rather surprising result that their effect on m_W and $\hat{s}^2$ is very similar over the entire range $m_Z \leq m_t \leq 250$ GeV, 60 GeV $\leq m_H \leq$ 1 TeV. Nonetheless, for reasons explained in Ref. [31] and Sec. VI, in our specific applications we use the resonance method for $m_t \lesssim 130$ GeV and the Green-function approach for $m_t \gtrsim 130$ GeV.

It is also a curious and rather surprising fact that most of the new effects we have considered relative to Ref.

TABLE II. As Table I, for $m_H = 100$ GeV.

m_t (GeV)	$\hat{s}^2$				m_W (GeV)			
	EW	EW + QCD			EW	EW + QCD		
		P	P+R	P+G		P	P+R	P+G
m_Z	0.23332	0.23346	0.23352	0.23351	79.952	79.921	79.912	79.914
100	0.23312	0.23328	0.23334	0.23332	80.002	79.969	79.960	79.962
110	0.23287	0.23305	0.23312	0.23310	80.058	80.023	80.012	80.015
120	0.23260	0.23280	0.23289	0.23287	80.115	80.077	80.064	80.068
130	0.23232	0.23254	0.23263	0.23261	80.173	80.132	80.118	80.121
140	0.23201	0.23227	0.23236	0.23234	80.234	80.188	80.172	80.176
150	0.23169	0.23197	0.23208	0.23206	80.296	80.246	80.229	80.233
160	0.23135	0.23166	0.23178	0.23175	80.361	80.307	80.288	80.292
170	0.23100	0.23133	0.23146	0.23144	80.429	80.370	80.349	80.352
180	0.23062	0.23099	0.23113	0.23111	80.500	80.435	80.412	80.416
190	0.23023	0.23063	0.23078	0.23076	80.574	80.503	80.478	80.481
200	0.22982	0.23025	0.23041	0.23040	80.651	80.574	80.547	80.549
210	0.22939	0.22986	0.23003	0.23002	80.730	80.647	80.618	80.619
220	0.22894	0.22945	0.22964	0.22963	80.813	80.723	80.692	80.692
230	0.22848	0.22903	0.22922	0.22923	80.899	80.801	80.769	80.767
240	0.22800	0.22859	0.22879	0.22881	80.988	80.883	80.848	80.845
250	0.22750	0.22814	0.22835	0.22838	81.080	80.967	80.930	80.925

TABLE III. As Table I, for $m_H = 250$ GeV.

m_t (GeV)	$\hat{s}^2$				m_W (GeV)			
	EW	EW + QCD			EW	EW + QCD		
		P	P+R	P+G		P	P+R	P+G
m_Z	0.23380	0.23394	0.23401	0.23399	79.894	79.862	79.854	79.856
100	0.23360	0.23376	0.23383	0.23381	79.944	79.911	79.901	79.904
110	0.23335	0.23353	0.23361	0.23359	80.000	79.965	79.953	79.956
120	0.23308	0.23329	0.23337	0.23335	80.056	80.018	80.005	80.009
130	0.23280	0.23303	0.23312	0.23310	80.114	80.072	80.058	80.062
140	0.23250	0.23275	0.23285	0.23283	80.174	80.128	80.113	80.116
150	0.23218	0.23246	0.23257	0.23255	80.236	80.186	80.169	80.172
160	0.23185	0.23215	0.23227	0.23225	80.301	80.246	80.227	80.230
170	0.23149	0.23183	0.23196	0.23194	80.368	80.308	80.287	80.290
180	0.23112	0.23149	0.23163	0.23161	80.437	80.372	80.349	80.352
190	0.23073	0.23114	0.23129	0.23127	80.510	80.439	80.414	80.417
200	0.23033	0.23077	0.23093	0.23092	80.585	80.508	80.481	80.483
210	0.22991	0.23039	0.23056	0.23055	80.663	80.579	80.550	80.552
220	0.22947	0.22999	0.23017	0.23017	80.744	80.653	80.622	80.622
230	0.22902	0.22958	0.22977	0.22978	80.827	80.730	80.697	80.695
240	0.22855	0.22915	0.22936	0.22937	80.914	80.809	80.773	80.771
250	0.22807	0.22871	0.22893	0.22896	81.003	80.890	80.853	80.848

[1], namely (a) the incorporation of the recent results on the leading irreducible corrections of $O(\alpha^2)$, (b) the perturbative $O(\alpha\alpha_s)$ corrections, and (c) the threshold contributions in the formulation of Sec. VI, increase the values of $\Delta\hat{r}$ and Δr for given m_t and m_H. Thus, they have a sign opposite to that of the leading m_t-dependent part of the one-loop corrections and, therefore, they induce an increase in the m_t upper bound. For large m_t, they are only partially compensated by the shift of -5×10^{-4} in $\Delta\hat{r}$ and Δr arising from the new calculation of $e^2\mathrm{Re}[\Pi^{(5)}_{\gamma\gamma}(0) - \Pi^{(5)}_{\gamma\gamma}(m_Z^2)]$ [10]. It should also be remembered that most of the corresponding shifts in $\Delta\hat{r}$ and Δr increase with m_t: (a) behaves as m_t^4 [cf. Eqs. (17a), (18a), and (23b)], the dominant (b) contributions as m_t^2 [cf. Eqs. (16a) and (19c)], and (c) more rapidly than linearly in m_t [31].

The QCD effects on m_W and $\hat{s}^2$ are visible in the tables. For example, for the intermediate value $m_H = 250$ GeV and $m_t = m_Z$, 130, 150, 180, 200, 250 GeV, the perturbative $O(\alpha\alpha_s)$ corrections lead to the shifts

$$\left.\begin{aligned} \Delta m_W &= -(32, 42, 50, 65, 77, 113)\ \mathrm{MeV} \\ \Delta\hat{s}^2 &= +(1.4, 2.3, 2.8, 3.7, 4.4, 6.4) \times 10^{-4} \end{aligned}\right\} O(\alpha\alpha_s). \tag{35a}$$

The threshold contributions lead to further shifts

TABLE IV. As Table I, for $m_H = 600$ GeV.

m_t (GeV)	$\hat{s}^2$				m_W (GeV)			
	EW	EW + QCD			EW	EW + QCD		
		P	P+R	P+G		P	P+R	P+G
m_Z	0.23430	0.23444	0.23450	0.23448	79.828	79.796	79.787	79.790
100	0.23409	0.23425	0.23432	0.23430	79.878	79.844	79.835	79.837
110	0.23384	0.23403	0.23410	0.23408	79.933	79.898	79.887	79.890
120	0.23358	0.23378	0.23386	0.23384	79.990	79.951	79.939	79.942
130	0.23329	0.23352	0.23361	0.23359	80.047	80.005	79.991	79.995
140	0.23299	0.23325	0.23335	0.23333	80.107	80.061	80.045	80.049
150	0.23268	0.23296	0.23307	0.23305	80.168	80.118	80.101	80.104
160	0.23234	0.23265	0.23277	0.23275	80.232	80.177	80.158	80.162
170	0.23200	0.23234	0.23247	0.23244	80.298	80.238	80.217	80.221
180	0.23163	0.23200	0.23214	0.23212	80.367	80.301	80.278	80.282
190	0.23125	0.23166	0.23181	0.23179	80.438	80.367	80.342	80.344
200	0.23086	0.23130	0.23146	0.23145	80.511	80.434	80.407	80.409
210	0.23045	0.23093	0.23110	0.23109	80.587	80.503	80.475	80.476
220	0.23002	0.23054	0.23072	0.23072	80.666	80.575	80.544	80.544
230	0.22958	0.23014	0.23034	0.23034	80.747	80.649	80.616	80.615
240	0.22913	0.22973	0.22994	0.22995	80.830	80.725	80.689	80.687
250	0.22867	0.22931	0.22953	0.22956	80.916	80.803	80.765	80.760

TABLE V. As Table I, for $m_H = 1000$ GeV.

m_t (GeV)	$\hat{s}^2$				m_W (GeV)			
	EW	EW + QCD			EW	EW + QCD		
		P	P+R	P+G		P	P+R	P+G
m_Z	0.23460	0.23474	0.23480	0.23479	79.786	79.754	79.745	79.748
100	0.23439	0.23455	0.23462	0.23460	79.836	79.803	79.793	79.796
110	0.23414	0.23432	0.23440	0.23438	79.892	79.856	79.845	79.848
120	0.23388	0.23408	0.23416	0.23414	79.948	79.910	79.897	79.900
130	0.23359	0.23382	0.23391	0.23389	80.006	79.964	79.950	79.953
140	0.23329	0.23355	0.23365	0.23362	80.065	80.019	80.004	80.007
150	0.23297	0.23326	0.23337	0.23334	80.127	80.077	80.059	80.063
160	0.23264	0.23295	0.23307	0.23305	80.190	80.136	80.116	80.120
170	0.23229	0.23264	0.23276	0.23274	80.256	80.196	80.175	80.179
180	0.23193	0.23230	0.23244	0.23242	80.325	80.259	80.236	80.239
190	0.23155	0.23196	0.23211	0.23209	80.395	80.324	80.299	80.302
200	0.23116	0.23160	0.23176	0.23175	80.468	80.390	80.363	80.366
210	0.23075	0.23123	0.23141	0.23140	80.543	80.459	80.430	80.431
220	0.23033	0.23085	0.23104	0.23104	80.621	80.530	80.499	80.499
230	0.22990	0.23046	0.23066	0.23066	80.700	80.602	80.569	80.568
240	0.22946	0.23006	0.23027	0.23028	80.782	80.677	80.641	80.638
250	0.22900	0.22965	0.22987	0.22990	80.866	80.753	80.715	80.710

$$\left.\begin{aligned}\Delta m_W &= -(8,14,14,20,25,42)\ \text{MeV}\\ \Delta\hat{s}^2 &= +(0.7,0.9,0.9,1.2,1.5,2.5)\times 10^{-4}\end{aligned}\right\}\ \text{(threshold)}. \tag{35b}$$

The departure from the monotonic behavior between $m_t = 130$ GeV and $m_t = 150$ GeV is due to the transition from the resonance to the Green-function approach. The effect of the leading irreducible corrections of $O((\alpha\, m_t^2/m_W^2)^2)$ cannot be seen in the tables because they have been included in every column. However, they can be estimated from the relations

$$\frac{\Delta m_W}{m_W} \approx \frac{c^2}{c^2 - s^2 - 2c^2 x_t}\, R\left(\frac{m_H}{m_t}\right)\frac{x_t^2}{6}, \tag{36a}$$

$$\Delta\hat{s}^2 \approx -\frac{\hat{s}^2\hat{c}^2}{\hat{c}^2 - \hat{s}^2}\, R\left(\frac{m_H}{m_t}\right)\frac{x_t^2}{3(1-x_t)^2}, \tag{36b}$$

where x_t is defined in Eq. (17b). The term $2c^2x_t$ in the first denominator of Eq. (36a) and the factor $(1-x_t)^{-2}$ in Eq. (36b) take into account the fact that some of the leading one-loop contributions to Δr and $\Delta\hat{r}$ depend on s^2 and $\hat{s}^2$, respectively, and are therefore affected by the shifts in these parameters. For the same values of m_H and m_t as employed before, Eqs. (36a) and (36b) give

$$\left.\begin{aligned}\Delta m_W &= -(1,5,8,16,24,52)\ \text{MeV}\\ \Delta\hat{s}^2 &= +(0.1,0.3,0.5,0.9,1.4,3.0)\times 10^{-4}\end{aligned}\right\}\ O(\alpha^2). \tag{36c}$$

Equations (35a), (35b), and (36c) can be compared with the experimental uncertainties $(\delta m_W)_{\text{expt}} \approx 100$ MeV and $(\delta\hat{s}^2)_{\text{expt}} \approx 4\times 10^{-4}$ expected at the end of 1993 [51]. We see that, in general terms, they are of the same order of magnitude. Of course, in the long run, even better experimental accuracies are envisaged, reaching perhaps $(\delta m_W)_{\text{expt}} \approx 50$ MeV. The above shifts can also be compared with the theoretical uncertainties

$$\begin{aligned}(\delta m_W)_{\text{th}} &= -\frac{m_W}{2}\,\frac{s^2}{c^2 - s^2 - 2c^2x_t}\,\frac{\delta\Delta r}{1-\Delta\alpha}\\ &\approx \pm 16\ \text{MeV},\end{aligned} \tag{37a}$$

$$(\delta\hat{s}^2)_{\text{th}} = \frac{\hat{c}^2\hat{s}^2}{(\hat{c}^2 - \hat{s}^2)}\,\frac{\delta\Delta\hat{r}}{1-\Delta_\gamma} \approx \pm 3\times 10^{-4}, \tag{37b}$$

arising from the error $(\delta\Delta r)_{\text{th}} \approx (\delta\Delta\hat{r})_{\text{th}} = \pm 9\times 10^{-4}$ in the calculation of $e^2\text{Re}[\Pi^{(5)}_{\gamma\gamma}(0) - \Pi^{(5)}_{\gamma\gamma}(m_Z^2)]$ [10]. We recall that the theoretical error arising from the neglect of higher-order electroweak corrections is expected to be of $O((\alpha/\pi s^2)x_t)$ in $\Delta\hat{r}_W$ and $\Delta\hat{r}$ and further enhanced by a factor c^2/s^2 in Δr (see Sec. I). Moreover, the error in the overall QCD corrections (perturbative and threshold effects) has been estimated to be $\pm 20\%$ [31]. Incidentally, Eq. (37b) shows that, if the experimental accuracy in $\hat{s}^2$ is improved in the future well beyond 4×10^{-4}, a meaningful theoretical interpretation will require a decrease in the above-mentioned theoretical errors.

As pointed out before, the higher-order corrections we have considered lead to an increase in the m_t values obtained from experiments. As an example, we consider the case $m_H = 250$ GeV and call m_t the parameter derived from $(m_W)_{\text{expt}}$ when the perturbative $O(\alpha\alpha_s)$ corrections are included. Table III shows that the possible values $m_t = m_Z$, 130, 150, 180, 200 GeV are larger than those obtained in the purely EW calculation by

$$\Delta m_t = +(5.8, 7.5, 8.3, 9.7, 10.8)\ \text{GeV}\quad [O(\alpha\alpha_s)]. \tag{38a}$$

The values of m_t derived from $(\hat{s}^2)_{\text{expt}}$ are shifted by slightly higher amounts, the differences with Eq.(38a) being $\lesssim 0.8$ GeV. The bulk of Δm_t arising from the perturbative $O(\alpha\alpha_s)$ corrections can be understood with the simple formula (16c), which describes the dominant

TABLE VI. Comparison between the values of Δr and m_W obtained using the $\overline{\text{MS}}$ approach of Sec. III [$(\Delta r)_{\text{I}}$, $(m_W)_{\text{I}}$] and the on-shell formulation of Sec. V [$(\Delta r)_{\text{II}}$, $(m_W)_{\text{II}}$], as a function of m_t, for $m_Z = 91.187$ GeV and $m_H = 60$ GeV. Nonperturbative $t\bar{t}$ threshold effects are not included here.

m_t (GeV)	$(\Delta r)_{\text{I}}$	$(\Delta r)_{\text{II}}$	$(m_W)_{\text{I}}$	$(m_W)_{\text{II}}$
	$\times 10^2$		(GeV)	
m_Z	6.003	6.006	79.947	79.946
100	5.735	5.738	79.996	79.995
110	5.437	5.440	80.050	80.049
120	5.135	5.138	80.103	80.103
130	4.823	4.826	80.158	80.158
140	4.498	4.502	80.215	80.214
150	4.159	4.162	80.274	80.273
160	3.801	3.805	80.334	80.334
170	3.425	3.429	80.398	80.397
180	3.029	3.033	80.463	80.463
190	2.611	2.615	80.532	80.531
200	2.170	2.173	80.603	80.602
210	1.705	1.708	80.677	80.676
220	1.214	1.217	80.754	80.753
230	0.697	0.699	80.833	80.833
240	0.151	0.153	80.916	80.915
250	-0.424	-0.423	81.001	81.001

contribution. Similarly, we see from Table III that, when threshold contributions are included, the values of m_t derived from $(m_W)_{\text{expt}}$ are larger than those obtained in the EW+P calculation by additional shifts of

$$\Delta m_t = +(1.7, 2.7, 2.7, 3.3, 4.1) \text{ GeV} \quad [\text{threshold}]. \tag{38b}$$

The corresponding variations arising from the irreducible $O(\alpha^2)$ corrections are

$$\Delta m_t = +(0.6, 1.2, 1.6, 2.5, 3.3) \text{ GeV} \quad [O(\alpha^2)]. \tag{38c}$$

TABLE VII. As Table VI, for $m_H = 250$ GeV.

m_t (GeV)	$(\Delta r)_{\text{I}}$	$(\Delta r)_{\text{II}}$	$(m_W)_{\text{I}}$	$(m_W)_{\text{II}}$
	$\times 10^2$		(GeV)	
m_Z	6.461	6.462	79.862	79.862
100	6.199	6.200	79.911	79.911
110	5.907	5.908	79.965	79.964
120	5.612	5.613	80.018	80.018
130	5.309	5.309	80.072	80.072
140	4.994	4.994	80.128	80.128
150	4.665	4.665	80.186	80.186
160	4.320	4.320	80.246	80.246
170	3.959	3.957	80.308	80.308
180	3.579	3.577	80.372	80.372
190	3.179	3.176	80.439	80.439
200	2.760	2.756	80.508	80.508
210	2.319	2.313	80.579	80.580
220	1.855	1.848	80.653	80.654
230	1.369	1.360	80.730	80.731
240	0.858	0.846	80.809	80.810
250	0.322	0.307	80.890	80.892

TABLE VIII. As Table VI, for $m_H = 1000$ GeV.

m_t (GeV)	$(\Delta r)_{\text{I}}$	$(\Delta r)_{\text{II}}$	$(m_W)_{\text{I}}$	$(m_W)_{\text{II}}$
	$\times 10^2$		(GeV)	
m_Z	7.036	7.037	79.754	79.754
100	6.779	6.780	79.803	79.803
110	6.494	6.494	79.856	79.856
120	6.206	6.205	79.910	79.910
130	5.910	5.910	79.964	79.964
140	5.604	5.603	80.019	80.020
150	5.286	5.284	80.077	80.077
160	4.954	4.951	80.135	80.136
170	4.607	4.603	80.196	80.197
180	4.244	4.239	80.259	80.260
190	3.865	3.859	80.324	80.325
200	3.469	3.461	80.390	80.392
210	3.056	3.045	80.459	80.461
220	2.625	2.612	80.530	80.532
230	2.175	2.159	80.602	80.605
240	1.707	1.687	80.677	80.680
250	1.220	1.195	80.753	80.757

Thus, for $m_H = 250$ GeV and $m_t = 200$ GeV, the combination of $O(\alpha\alpha_s)$, threshold, and leading irreducible $O(\alpha^2)$ corrections increases the value of m_t derived from $(m_W)_{\text{expt}}$ by ≈ 16.8 GeV.

Another topic of considerable interest is the comparison of calculations carried out in the $\overline{\text{MS}}$ and on-shell methods of renormalization. This is illustrated for $m_H = 60$, 250, 1000 GeV in Tables VI–VIII, where we show the evaluation of Δr, a physical observable, obtained on the basis of the $\overline{\text{MS}}$ approach of Sec. III and the on-shell formulation of Sec. V. For the purposes of this study, we have included the electroweak and perturbative $O(\alpha\alpha_s)$ corrections, leaving aside the threshold effects. We also display the derived values of m_W and s^2. Inspection of the tables shows that the two calculations of Δr are in excellent agreement over the entire range 60 GeV $\leq m_H \leq$ 1 TeV, $m_Z \leq m_t \leq$ 250 GeV, with a maximum discrepancy of $\approx 2.5 \times 10^{-4}$ occurring at $m_t = 250$ GeV and $m_H = 1$ TeV. Although such accurate agreement may be somewhat fortuitous, it is roughly of the expected order of magnitude, i.e., $O((\alpha/\pi\hat{s}^2)(\hat{c}^2/\hat{s}^2)x_t)$.

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APPENDIX A

In this appendix we discuss the perturbative corrections of $O(\hat{\alpha}_s)$ to the vacuum polarization functions involving quarks. Defining $\Pi^{V,A}(s,m_1,m_2)$ according to Eqs. (26a) and (26b), we expand

$$\Pi^{V,A}(s,m_1,m_2)=\Pi_0^{V,A}(s,m_1,m_2)+\frac{\hat{\alpha}_s}{\pi}\Pi_1^{V,A}(s,m_1,m_2), \tag{A1}$$

where $s=q^2$ and m_1 and m_2 are the masses of the virtual quarks in the loop. The functions $\Pi_0^{V,A}(s,m_1,m_2)$ have been extensively discussed in the literature. They can be gleaned, for example, from Ref. [41]. In the $O(\hat{\alpha}_s)$ terms we consider two limiting cases: $m_1=m_2=m$ (as occurs in the $\gamma\gamma$, ZZ, and $Z\gamma$ self-energies) and $m_1=m, m_2=0$ [as applies, to a very good approximation, to the (t,b) contribution to the WW self-energy].

Comparison of Refs. [22] and [25] leads to the following expressions:

$$\frac{\pi^2}{m^2}\Pi_1^V(s,m,m)=r\left(-\frac{1}{n-4}-l-4\zeta(3)+\frac{55}{12}\right)+V_1(r), \tag{A2}$$

$$\frac{\pi^2}{m^2}\Pi_1^A(s,m,m)=\frac{6}{(n-4)^2}+\frac{2}{n-4}\left(3l-\frac{r}{2}-\frac{11}{4}\right)+3l^2-\left(r+\frac{11}{2}\right)l+r\left(-4\zeta(3)+\frac{55}{12}\right)+6\zeta(3)+3\zeta(2)-\frac{11}{8}+A_1(r), \tag{A3}$$

$$\frac{\pi^2}{m^2}\Pi_1^{V,A}(s,m,0)=\frac{1}{4}\left[\frac{6}{(n-4)^2}+\frac{2}{n-4}\left(3l-\frac{x}{2}-\frac{11}{4}\right)+3l^2-\left(x+\frac{11}{2}\right)l+x\left(-4\zeta(3)+\frac{55}{12}\right)+6\zeta(3)+3\zeta(2)-\frac{11}{8}\right]+F_1(x), \tag{A4}$$

where $r\equiv s/(4m^2)$, $x\equiv s/m^2$, $l\equiv\ln(m^2/\mu')$ (μ' is the rescaled 't Hooft mass discussed in Sec. II), $V_1(r)$, $A_1(r)$, and $F_1(x)$ are complicated functions studied in Ref. [25], and the color factor appropriate to $N_c=3$ has been explicitly included. We recall that $\zeta(2)=\pi^2/6$ and $\zeta(3)=1.202\,06\ldots$. The above expressions can be used to evaluate the $O(\hat{\alpha}_s)$ corrections employed in the text. Thus, the quantity $\Pi_{\gamma\gamma}^{(f)}(0)$ in Sec. II can be written as

$$\Pi_{\gamma\gamma}^{(f)}(0)=\Pi_{\gamma\gamma}^{(l)}(0)+\Pi_{\gamma\gamma}^{(t)}(0)+\mathrm{Re}\Pi_{\gamma\gamma}^{(5)}(m_Z^2)+\mathrm{Re}[\Pi_{\gamma\gamma}^{(5)}(0)-\Pi_{\gamma\gamma}^{(5)}(m_Z^2)], \tag{A5}$$

where the superscripts (l), (t), and (5) refer to the contributions of the leptons, the top quark, and the first five quark flavors, respectively. It is easy to see that the contribution $\Pi_{\gamma\gamma}^{(q)}(s)$ of quark q to $\Pi_{\gamma\gamma}(s)$ is

$$\Pi_{\gamma\gamma}^{(q)}(s)=\frac{Q_q^2\Pi^V(s,m_q,m_q)}{s}, \tag{A6}$$

where Q_q is the charge of the quark in units of the positron charge e, and m_q is its mass. In particular,

$$\Pi_{\gamma\gamma}^{(t)}(0)=Q_t^2(\Pi^V)'(0,m_t,m_t), \tag{A7}$$

where the prime on the rhs denotes differentiation with respect to s. Using our Eq. (A2) and Eq. (18) of Ref. [25], we find that the $O(\hat{\alpha}_s)$ part of $\Pi_{\gamma\gamma}^{(t)}(0)$ is

$$(2\hat{\alpha}_s/9\pi^3)\{\ln(\mu'/m_t)+\tfrac{15}{8}-[2(n-4)]^{-1}\}.$$

The finite part of this result is included in the second term of Eq. (7a), while the pole contributes to the last. Similarly,

$$\mathrm{Re}\Pi_{\gamma\gamma}^{(5)}(m_Z^2)=\sum_{q\neq t}Q_q^2\frac{\mathrm{Re}\Pi^V(m_Z^2,m_q,m_q)}{m_Z^2}. \tag{A8}$$

Using Eq. (A2), we see that the $O(\hat{\alpha}_s)$ part of Eq. (A8) is

$$(\hat{\alpha}_s/4\pi^3)\sum_{q\neq t}Q_q^2[2\ln(\mu'/m_Z)+f_2(r_q)-(n-4)^{-1}],$$

where r_q and $f_2(r)$ are defined in Sec. II. The finite and pole parts have been included in the third and last terms of Eq. (7a), respectively. As mentioned in Sec. II, for $e^2\mathrm{Re}[\Pi_{\gamma\gamma}^{(5)}(0)-\Pi_{\gamma\gamma}^{(5)}(m_Z^2)]$ we employ a recent evaluation [10]. The contributions of $O(\hat{\alpha})$ within the square brackets in the second and third terms of Eq. (7a) are simply obtained from the $O(\hat{\alpha}_s)$ ones by dividing by the quadratic Casimir coefficient 4/3 for the fundamental representation of $\mathrm{SU}(3)_\mathrm{C}$, multiplying by an additional factor of Q_q^2, and replacing $\hat{\alpha}_s\to\hat{\alpha}$. The contributions of $O(\hat{\alpha})$ in $\Pi_{\gamma\gamma}^{(l)}(0)$ [first term in Eq. (7a)] can be obtained from those in $\Pi_{\gamma\gamma}^{(t)}(0)$ by changing $m_t\to m_l$ and dividing by $3Q_t^4$, where the 3 stands for the color factor.

In the approximation of neglecting the squares of mixing angles, the quark contribution to $A_{WW}^{(f)}$ is

$$\left[A_{WW}^{(f)}(s)\right]_{\text{quarks}}=-\frac{1}{8}\sum_{\text{doublets}}[\Pi^V(s,m_1,m_2)+\Pi^A(s,m_1,m_2)]. \tag{A9}$$

In the $O(\hat{\alpha}_s)$ part we approximate $m_1=m_2=0$ in the (u,d) and (c,s) contributions and $m_1=m_t$, $m_2=0$ in the (t,b) contribution. In this limit, $\Pi^A(s,m_1,m_2)=$

$\Pi^V(s,m_1,m_2)$ and we see that the $O(\hat{\alpha}_s)$ part of Eq. (A8) is

$$\left[A^{(f)}_{WW}(s)\right]_{\rm QCD} = -\frac{1}{4\pi}\,[2\hat{\alpha}_s(m_Z)\Pi^V_1(s,0,0) + \hat{\alpha}_s(m_t)\Pi^V_1(s,m_t,0)], \qquad \text{(A10)}$$

where the first and the second terms correspond to the contributions of the light and (t,b) isodoublets, respectively. The choice of renormalization scale in $\hat{\alpha}_s$ was explained at the end of Sec. II. Employing our Eq. (A4) and Eq. (20) of Ref. [25], we find

$$\mathrm{Re}\Pi^{V,A}_1(s,0,0) = -\frac{s}{4\pi^2}\left(\frac{1}{n-4} + \ln\frac{s}{\mu'^2} + 4\zeta(3) - \frac{55}{12}\right) \qquad \text{(A11)}$$

and

$$\mathrm{Re}\left[\frac{A^{(f)}_{WW}(m_W^2) - A^{(f)}_{WW}(0)}{m_W^2}\right]_{\rm QCD} = \frac{\hat{\alpha}_s(m_Z)}{8\pi^3}\left(\frac{1}{n-4} + \ln\frac{m_W^2}{\mu'^2} + 4\zeta(3) - \frac{55}{12}\right) + \frac{\hat{\alpha}_s(m_t)}{16\pi^3}\left[\frac{1}{n-4} + \ln\frac{m_W^2}{\mu'^2} + 4\zeta(3) - \frac{55}{12} - 4\omega_t\left\{F_1\left(\frac{1}{\omega_t}\right) - F_1(0)\right\} + \ln\omega_t\right]. \qquad \text{(A12)}$$

Subtracting the pole term, setting $\mu' = m_Z$, and multiplying by $e^2/\hat{s}^2$, we obtain Eq. (11d).

Similarly, the quark contribution to $A^{(f)}_{ZZ}(s)$ is

$$\left[A^{(f)}_{ZZ}(s)\right]_{\rm quarks} = -\frac{1}{16\hat{c}^2}\sum_q\left[(1-4\hat{s}^2C_{3q}Q_q)^2\Pi^V(s,m_q,m_q) + \Pi^A(s,m_q,m_q)\right], \qquad \text{(A13)}$$

where $C_{3q} = +1\,(-1)$ for up (down) members of the doublet and the sum is over quarks. In the $O(\hat{\alpha}_s)$ part we neglect all masses other than m_t and find

$$\left[A^{(f)}_{ZZ}(s)\right]_{\rm QCD} = -\frac{\hat{\alpha}_s(m_Z)}{16\pi\hat{c}^2}\sum_{q=1}^{4}\left[(1-4\hat{s}^2C_{3q}Q_q)^2+1\right]\Pi^V_1(s,0,0) - \frac{\hat{\alpha}_s(m_t)}{16\pi\hat{c}^2}\left\{\left(1-\frac{8}{3}\hat{s}^2\right)^2\Pi^V_1(s,m_t,m_t) + \Pi^A_1(s,m_t,m_t) + \left[\left(1-\frac{4}{3}\hat{s}^2\right)^2+1\right]\Pi^V_1(s,0,0)\right\}, \qquad \text{(A14)}$$

where the first and second terms are the contributions of the light and (t,b) isodoublets, respectively. Combining Eqs. (A10) and (A14) and recalling Eqs. (11d) and (A2)–(A4), we obtain

$$\frac{1}{m_Z^2}\mathrm{Re}\left[\frac{A^{(f)}_{WW}(m_W^2)}{\hat{c}^2} - A^{(f)}_{ZZ}(m_Z^2)\right]_{\rm QCD} = \frac{\hat{\alpha}_s(m_Z)}{8\pi^3\hat{c}^2}\left[c^2\ln c^2 + \left(\frac{1}{n-4} + \ln\frac{m_Z^2}{\mu'^2} + 4\zeta(3) - \frac{55}{12}\right)\left(-s^2 + 2\hat{s}^2\left(1-\frac{10}{9}\hat{s}^2\right)\right)\right] + \frac{\hat{\alpha}_s(m_t)}{16\pi^3\hat{c}^2}\left\{c^2\ln c^2 + \left(\frac{1}{n-4} + \ln\frac{m_Z^2}{\mu'^2} + 4\zeta(3) - \frac{55}{12}\right)\left(-s^2 + 2\hat{s}^2\left(1-\frac{10}{9}\hat{s}^2\right)\right) - c^2\left(4\omega_tF_1\left(\frac{1}{\omega_t}\right) - \ln\omega_t\right) + \left(1-\frac{8}{3}\hat{s}^2\right)^2\left(\mu_tV_1\left(\frac{1}{4\mu_t}\right) - \frac{\ln\mu_t}{4}\right) + \mu_tA_1\left(\frac{1}{4\mu_t}\right) - \frac{\ln\mu_t}{4}\right\}. \qquad \text{(A15)}$$

Subtracting again the pole terms, setting $\mu' = m_Z$, and multiplying by $e^2/\hat{s}^2$, we obtain Eq. (15d).

Finally, in order to implement the decoupling of the top quark, we need the quantity $(\alpha/\pi)d$, where d is defined in Eq. (9b). From Ref. [41] we find

$$A^{(t)}_{\gamma Z}(s) = -\frac{\hat{e}^2}{3\hat{s}\hat{c}}\left(\frac{1}{2} - \frac{4}{3}\hat{s}^2\right)\Pi^V(s,m_t,m_t), \qquad \text{(A16)}$$

and, therefore,

$$\frac{\alpha}{\pi}d = -\frac{e^2}{3}\left(\frac{1}{2\hat{s}^2} - \frac{4}{3}\right)(\Pi^V)'(0,m_t,m_t)|_{\overline{\rm MS}}. \qquad \text{(A17)}$$

The $O(\hat{\alpha}_s)$ part of Eq. (A17) is evaluated as in the case of $\Pi^{(t)}_{\gamma\gamma}(0)$ [cf. the discussion after Eq. (A7)]. A relevant combination that occurs in $\Delta\hat{r}_W$ and $\Delta\hat{r}$ is

$$e^2\,\Pi^{(t)}_{\gamma\gamma}(0)\Big|_{\overline{\rm MS}} - \frac{\alpha}{\pi}d = \frac{e^2}{6\hat{s}^2}\,(\Pi^V)'(0,m_t,m_t)|_{\overline{\rm MS}} = -\frac{\alpha}{\pi}\hat{d}, \qquad \text{(A18)}$$

where $\hat{d}$ is defined in Eq. (10d).

APPENDIX B

In this appendix we outline the derivation of the expressions for $\Delta\hat{r}_W$ [cf. Eq. (10a)] and $\Delta\hat{r}$ [cf. Eq. (14)], corresponding to the strategy in which the decoupling of the top quark is implemented. As $\hat{s}^2(1-\Delta\hat{r}_W)$ and $\hat{s}^2\hat{c}^2(1-\Delta\hat{r})$ are physical observables [cf. Eqs. (2) and (3)], we define $\Delta\hat{r}_W$ and $\Delta\hat{r}$ in such a way that

$$\left[\hat{s}^2(1-\Delta\hat{r}_W)\right]_{\rm old} = \left[\hat{s}^2(1-\Delta\hat{r}_W)\right]_{\rm new}, \tag{B1}$$

$$\left[\hat{s}^2\hat{c}^2(1-\Delta\hat{r})\right]_{\rm old} = \left[\hat{s}^2\hat{c}^2(1-\Delta\hat{r})\right]_{\rm new}. \tag{B2}$$

Here the subscript "old" labels the quantities obtained when the $\overline{\rm MS}$ counterterms cancel only the divergent parts involving δ [cf. Eq. (5a)], as in Ref. [1], while "new" denotes the corrections employed in the present paper, where the $\overline{\rm MS}$ counterterms contain small finite parts necessary to implement the top-quark decoupling in relevant amplitudes. Our strategy is to retain, in Eqs. (B1) and (B2), terms of $O(\alpha^2)$ when they involve large logarithmic or m_t^2/m_Z^2 enhancements, but to neglect them otherwise. We also neglect certain corrections of $O(\alpha^3)$. Recalling $e_0^2 = \hat{e}^2/\hat{Z}_e$ and Eqs. (7d), (9a), (9b,) and (A18), we have

$$\hat{s}^2_{\rm old} = \hat{s}^2_{\rm new}\left(1+\frac{\hat{\alpha}}{\pi}d\right), \tag{B3}$$

$$\left[\frac{\hat{e}^2}{\hat{s}^2}\right]_{\rm old} = \left[\frac{\hat{e}^2}{\hat{s}^2}\right]_{\rm new}\left(1-\frac{\hat{\alpha}}{\pi}\hat{d}\right), \tag{B4}$$

where d and $\hat{d}$ are defined in Eqs. (9c) and (10d), respectively.

The derivation of Eq. (10a) follows almost immediately. Inserting Eq. (B3) into Eq. (B1), we find

$$\left(1+\frac{\hat{\alpha}}{\pi}d\right)[1-\Delta\hat{r}_W]_{\rm old} = [1-\Delta\hat{r}_W]_{\rm new}. \tag{B5}$$

As the only large correction in $\Delta\hat{r}_W$ is $(\alpha/\pi)\Delta_\gamma$ and $\hat{\alpha}(1-\alpha\Delta_\gamma/\pi)=\alpha$ [cf. Eq. (8c)], we obtain

$$(\Delta\hat{r}_W)_{\rm new} = (\Delta\hat{r}_W)_{\rm old} - \frac{\alpha}{\pi}d. \tag{B6}$$

The expression for $(\Delta\hat{r}_W)_{\rm old}$ in terms of $\Pi^{(f)}_{\gamma\gamma}(0)$, WW self-energies, and the vertex- and box-diagram corrections to μ decay, is given in Eqs. (7b) and (8b) of Ref. [1]. The only contribution to $(\Delta\hat{r}_W)_{\rm old}$ with large logarithmic enhancement is $e^2\,\Pi^{(f)}_{\gamma\gamma}(0)\,|_{\overline{\rm MS}}$ and, to $O(\alpha)$, this does not involve $\hat{s}^2$. Therefore, we can replace $\hat{s}^2_{\rm old}\to\hat{s}^2_{\rm new}$ everywhere in $(\Delta\hat{r}_W)_{\rm old}$, neglecting very small terms of $O(\alpha^2)$. Inserting the expression for $(\Delta\hat{r}_W)_{\rm old}$ in Eq. (B6), we obtain Eq. (10a).

The derivation of Eq. (14) is more subtle. Inserting Eq. (B3) and

$$\hat{c}^2_{\rm old} = \hat{c}^2_{\rm new}\left(1-\frac{\hat{\alpha}}{\pi}\left[\frac{\hat{s}^2}{\hat{c}^2}\right]_{\rm new}d\right) \tag{B7}$$

into Eq. (B2), we have

$$(\Delta\hat{r})_{\rm new} = (\Delta\hat{r})_{\rm old} - \frac{\hat{\alpha}}{\pi}d\left(1-\left[\frac{\hat{s}^2}{\hat{c}^2}\right]_{\rm new}\right)[1-\Delta\hat{r}]_{\rm old}. \tag{B8}$$

We first consider the second term in Eq. (B8). Using Eq. (18b), this contribution can be written as

$$-(\hat{\alpha}/\pi)d(1-[\hat{s}^2/\hat{c}^2]_{\rm new})[1-\Delta\hat{r}_W]_{\rm old}c^2/\hat{c}^2_{\rm old},$$

which, neglecting small $O(\alpha^2)$ terms, becomes

$$-(\alpha/\pi)d(1-[\hat{s}^2/\hat{c}^2]_{\rm new})c^2/\hat{c}^2_{\rm new}.$$

Next we analyze the first term on the rhs of Eq. (B8). According to Eq. (15b) of Ref. [1], it is given by

$$(\Delta\hat{r})_{\rm old} = (\Delta\hat{r}_W)_{\rm old} - \left[\frac{\hat{e}^2}{\hat{s}^2\hat{c}^2}(1-\Delta\hat{r}_W)\right]_{\rm old}{\rm Re}\left[\frac{A_{WW}(m_W^2)-\hat{c}^2A_{ZZ}(m_Z^2)}{m_Z^2}\right]^{\rm old}_{\overline{\rm MS}}. \tag{B9}$$

Examination of Eq. (A.8) of Ref. [1] shows that

$${\rm Re}\left[\frac{A_{WW}(m_W^2)-\hat{c}^2A_{ZZ}(m_Z^2)}{m_Z^2}\right]^{\rm old}_{\overline{\rm MS}} = \frac{1}{8\pi^2}\left(\frac{3m_t^2}{8m_Z^2}+\frac{1}{2}\ln\frac{m_t}{m_Z}+\cdots\right), \tag{B10}$$

where the ellipses represent nonleading terms [some of which involve $\ln(m_t/m_Z)$ with very small coefficients proportional to s^2, $\hat{s}^2$, or $\hat{s}^4$]. The significant point is that the leading contributions in Eq. (B10) are independent of $\hat{s}^2$. Therefore, in Eq. (B9) we can replace the last factor in the second term by an analogous expression with $\hat{s}^2_{\rm old}\to\hat{s}^2_{\rm new}$, the difference being again small terms of $O(\alpha^2)$. Using Eqs. (B4), (B6), and (B7), Eq. (B9) becomes

$$(\Delta\hat{r})_{\rm old} = (\Delta\hat{r}_W)_{\rm new} + \frac{\alpha}{\pi}d - \left[\frac{\hat{e}^2}{\hat{s}^2}\right]_{\rm new}\frac{1}{m_Z^2}\left[1-\Delta\hat{r}_W-\frac{\alpha}{\pi}\hat{d}-\frac{\alpha}{\pi}d\left(1-\frac{\hat{s}^2}{\hat{c}^2}\right)\right]_{\rm new}{\rm Re}\left[\frac{A_{WW}(m_W^2)}{\hat{c}^2}-A_{ZZ}(m_Z^2)\right]^{\rm new}_{\overline{\rm MS}}. \tag{B11}$$

Recalling Eqs. (10), (12b), and (15a) of Ref. [1] and neglecting again very small contributions of $O(\alpha^2,\alpha^3)$, we find that the term proportional to $(\alpha/\pi)d(1-\hat{s}^2/\hat{c}^2)$ in Eq. (B11) can be written as $(\alpha/\pi)d(1-\hat{s}^2/\hat{c}^2)(c^2/\hat{c}^2-1)$. Inserting the above results into Eq. (B8), we find

$$(\Delta\hat{r})_{\rm new} = (\Delta\hat{r}_W)_{\rm new} + \frac{\alpha}{\pi}\, d \left[1 + \left(1 - \frac{\hat{s}^2}{\hat{c}^2}\right)\left(\frac{c^2}{\hat{c}^2} - 1\right) - \left(1 - \frac{\hat{s}^2}{\hat{c}^2}\right)\frac{c^2}{\hat{c}^2}\right]_{\rm new}$$
$$- \left[\frac{\hat{e}^2}{\hat{s}^2}\right]_{\rm new} \frac{1}{m_Z^2}\left(1 - (\Delta\hat{r}_W)_{\rm new} - \frac{\alpha}{\pi}\,\hat{d}\right) {\rm Re}\left[\frac{A_{WW}(m_W^2)}{\hat{c}^2} - A_{ZZ}(m_Z^2)\right]_{\overline{\rm MS}}^{\rm new}. \tag{B12}$$

The coefficient of $(\alpha/\pi)d$ simplifies to $\hat{s}^2/\hat{c}^2$ and Eq. (B12) equals Eq. (14).

APPENDIX C

If terms proportional to squares of mixing angles are neglected, the fermionic contributions to the WW self-energy become a sum over independent isodoublet contributions and we obtain

$$B_0^{(f)} = \frac{\alpha}{12\pi\hat{s}^2} \sum_{\rm doublets} N_c f_1(\omega_+, \omega_-), \tag{C1}$$

$$f_1(\omega_+,\omega_-) = \ln[c^2(\omega_+\omega_-)^{1/2}] + \left[1 - \frac{\omega_+ + \omega_-}{2} - \frac{(\omega_+ - \omega_-)^2}{2}\right] 2\Omega(\omega_+,\omega_-)$$
$$+ \frac{3(\omega_+^2 + \omega_-^2) - (\omega_+ - \omega_-)^4}{4(\omega_+ - \omega_-)} \ln\frac{\omega_+}{\omega_-} - \frac{5}{3} + \frac{\omega_+ + \omega_-}{4} + \frac{(\omega_+ - \omega_-)^2}{2}, \tag{C2}$$

where $B_0^{(f)}$ is defined in Eqs. (11a) and (11b), $N_c = 3$ and $N_c = 1$ for quark and lepton isodoublets, respectively, and $\omega_\pm = m_\pm^2/m_W^2$, with m_+ and m_- being the masses of the "up" and "down" fermions in the isodoublet. Calling

$$C \equiv \frac{\omega_+ + \omega_-}{2} - \frac{(\omega_+ - \omega_-)^2}{4} - \frac{1}{4}, \tag{C3}$$

the function $\Omega(\omega_+,\omega_-)$ is given by

$$\Omega(\omega_+,\omega_-) = C^{1/2}\cos^{-1}\frac{\omega_+ + \omega_- - 1}{2(\omega_+\omega_-)^{1/2}} \qquad \text{for } C > 0, \tag{C4}$$

$$\Omega(\omega_+,\omega_-) = \frac{|C|^{1/2}}{2}\ln\left|\frac{\omega_+ + \omega_- - 1 - 2|C|^{1/2}}{\omega_+ + \omega_- - 1 + 2|C|^{1/2}}\right| \qquad \text{for } C < 0. \tag{C5}$$

Equations (C4) and (C5) are a simpler version of Eqs. (A10) and (A11) of Ref. [49]. As expected from the integral representation of the self-energies [41], $f_1(\omega_+,\omega_-)$ is a symmetric function of ω_+ and ω_-. As $\omega_- \to 0$,

$$f_1(\omega_+, 0) = \ln(c^2\omega_+) + (\omega_+ - 1)^2\left(1 + \frac{\omega_+}{2}\right)\ln\left|\frac{\omega_+ - 1}{\omega_+}\right| - \frac{5}{3} + \frac{\omega_+}{4} + \frac{\omega_+^2}{2}, \tag{C6}$$

while in the limit $\omega_+, \omega_- \to 0$,

$$f_1(0,0) = \ln c^2 - \frac{5}{3}. \tag{C7}$$

If all the fermion masses other than m_t are neglected, Eqs. (C1), (C6), and (C7) lead to Eq. (11c). In order to estimate the terms of $O(m_b^2/m_W^2)$, we consider the case $\omega_+ > 1$ and $\omega_- \ll 1$. Including corrections of $O(\omega_-)$, we have

$$f_1(\omega_+,\omega_-) = f_1(\omega_+, 0) + \frac{3}{2}\omega_-\left[(1 + \omega_+^2)\ln\frac{\omega_+}{\omega_+ - 1} - \omega_+ - \frac{1}{2}\right] + O(\omega_-^2). \tag{C8}$$

Inserting the second term of Eq. (C8) in the (t,b)-isodoublet contribution to Eq. (C1), and identifying $\omega_- = \omega_b = m_b^2/m_W^2$ and $\omega_+ = \omega_t = m_t^2/m_W^2$, we obtain a very small shift,

$$\left[\delta B_0^{(f)}\right]_{m_b} = \frac{3\alpha}{8\pi\hat{s}^2}\,\omega_b\left[(1 + \omega_t^2)\ln\frac{\omega_t}{\omega_t - 1} - \omega_t - \frac{1}{2}\right]. \tag{C9}$$

Using $m_b = 4.5$ GeV and $m_Z = 91.187$ GeV, we find that the $O(m_b^2/m_W^2)$ corrections to Eq. (C1) are $\approx 2.5\times10^{-5}$, 5.4×10^{-6}, 2.8×10^{-6}, 1.7×10^{-6} for $m_t = m_Z$, 150, 200, 250 GeV, respectively. We note that the cofactor of ω_b in Eq. (C9) tends to 0 as $m_t \to \infty$. To estimate the contributions of $O(m_c^2/m_W^2)$, we set $m_s = 0$ and keep terms of $O(\omega_+)$ for $\omega_+ \ll 1$ in Eq. (C6),

$$f_1(\omega_+, 0) = f_1(0,0) + \frac{3}{2}\omega_+\left(\ln\omega_+ - \frac{1}{2}\right) + O(\omega_+^2). \tag{C10}$$

Inserting the second term of Eq. (C10) in the (c,s)-isodoublet contribution to Eq. (C1), and identifying $\omega_+ = \omega_c = m_c^2/m_W^2$, we obtain

$$\left[\delta B_0^{(f)}\right]_{m_c} = \frac{3\alpha}{8\pi\hat{s}^2}\,\omega_c\left(\ln\omega_c - \frac{1}{2}\right). \tag{C11}$$

For $m_c = 1.5$ GeV, this amounts to a shift of -1.1×10^{-5} to Eq. (C1). For the corrections of $O(m_\tau^2/m_W^2)$, we replace $\omega_c \to \omega_\tau = m_\tau^2/m_W^2$ and divide by the color factor 3 in Eq. (C11). This leads to a further correction of -5×10^{-6} to Eq. (C1).

Turning our attention to $C_0^{(f)}$, defined in Eqs. (15a) and (15b), and neglecting again the squares of mixing angles, we have

$$C_0^{(f)} = \frac{\alpha}{12\pi\hat{s}^2\hat{c}^2}\left[c^2\sum_{\text{doublets}} N_c f_2(\omega_+,\omega_-) - \frac{1}{4}\sum_f N_c\, g(\mu_f)\right], \tag{C12}$$

$$\begin{aligned} f_2(\omega_+,\omega_-) = {} & \left[1-\frac{3}{2}(\omega_+ + \omega_-)\right]\ln[c^2(\omega_+\omega_-)^{1/2}] + \left[1-\frac{\omega_+ + \omega_-}{2} - \frac{(\omega_+ - \omega_-)^2}{2}\right]2\Omega(\omega_+,\omega_-) \\ & -\frac{(\omega_+ - \omega_-)^3}{4}\ln\frac{\omega_+}{\omega_-} - \frac{5}{3} + \omega_+ + \omega_- + \frac{(\omega_+ - \omega_-)^2}{2}, \end{aligned} \tag{C13}$$

$$g(\mu_f) = \left[(1-4\hat{s}^2C_{3f}Q_f)^2+1\right]\left[\ln\mu_f + \frac{1}{3} + 2(1+2\mu_f)[\Lambda(D_f)-1]\right] - 6\mu_f[\ln\mu_f + 2\Lambda(D_f) - 2], \tag{C14}$$

where the f sum is over quark and lepton flavors, N_c is again the color factor, Q_f is the electric charge in units of the positron charge e, $C_{3f} = +1\,(-1)$ for up (down) members of the doublet, $\mu_f = m_f^2/m_Z^2$, $D_f = 4\mu_f - 1$, and

$$\Lambda(D_f) = D_f^{1/2}\tan^{-1}\left(D_f^{-1/2}\right) \qquad \text{for } D_f > 0, \tag{C15}$$

$$\Lambda(D_f) = \frac{|D_f|^{1/2}}{2}\ln\left|\frac{1+|D_f|^{1/2}}{1-|D_f|^{1/2}}\right| \qquad \text{for } D_f < 0. \tag{C16}$$

As is clear from their structure, the contributions involving f_2 and g in Eq. (C12) arise from the first and second terms in Eq. (15a), respectively.

One readily finds the limiting values

$$f_2(\omega_+,0) = \left(1-\frac{3}{2}\omega_+\right)\ln(c^2\omega_+) + (\omega_+-1)^2\left(1+\frac{\omega_+}{2}\right)\ln\left|\frac{\omega_+-1}{\omega_+}\right| - \frac{5}{3} + \omega_+ + \frac{\omega_+^2}{2}, \tag{C17}$$

$$f_2(0,0) = \ln c^2 - \frac{5}{3}, \tag{C18}$$

$$g(0) = -\frac{5}{3}\left[(1-4\hat{s}^2C_{3f}Q_f)^2+1\right]. \tag{C19}$$

If the fermion masses other that m_t are neglected, Eqs. (C12), (C14), and (C17)–(C19) lead to Eq. (15c). To discuss the corrections of $O(m_b^2/m_W^2)$, we consider $f_2(\omega_+,\omega_-)$ for $\omega_+ > 1$, $\omega_- \ll 1$ and $g(\mu_f)$ for $\mu_f \ll 1$. One readily finds

$$f_2(\omega_+,\omega_-) = f_2(\omega_+,0) + \frac{3}{2}\omega_-\left[(1+\omega_+^2)\ln\frac{\omega_+}{\omega_+-1} - \omega_+ - \ln(c^2\omega_+)\right] + O(\omega_-^2), \tag{C20}$$

$$g(\mu_f) = g(0) - 48\mu_f\hat{s}^2Q_f(2\hat{s}^2Q_f - C_{3f}) + O(\mu_f^2). \tag{C21}$$

Identifying $\omega_- = \omega_b$, $\omega_+ = \omega_t$, $\mu_f = \mu_b = m_b^2/m_Z^2$ and inserting Eqs. (C20) and (C21) into Eq. (C12), we find for the leading correction of $O(\mu_b)$

$$\left[\delta C_0^{(f)}\right]_{m_b} = \frac{3\alpha}{8\pi\hat{s}^2\hat{c}^2}\,\mu_b\left[(1+\omega_t^2)\ln\frac{\omega_t}{\omega_t-1} - \omega_t - \ln(c^2\omega_t) - \frac{8}{3}\hat{s}^2\left(1-\frac{2}{3}\hat{s}^2\right)\right]. \tag{C22}$$

Using the same input values as in Eq. (C9), we find that the $O(m_b^2/m_W^2)$ corrections to Eq. (C12) are $\approx 2.5\times10^{-5}$, -6.7×10^{-6}, -1.6×10^{-5}, -2.3×10^{-5} for $m_t = m_Z$, 150, 200, 250 GeV, respectively. To estimate the corrections of $O(m_c^2/m_W^2)$, we set $\omega_- = 0$ and keep terms of $O(\omega_+)$ for $\omega_+ \ll 1$ in Eq. (C17). In conjunction with Eq. (C21), this gives

$$\left[\delta C_0^{(f)}\right]_{m_c} = \frac{3\alpha}{8\pi\hat{s}^2\hat{c}^2}\,\mu_c\left[\frac{16}{3}\hat{s}^2\left(\frac{4}{3}\hat{s}^2-1\right)-\ln c^2\right], \tag{C23}$$

where $\mu_c = m_c^2/m_Z^2$. For $m_c = 1.5$ GeV, this is $\approx -8\times10^{-7}$. The corrections of $O(m_\tau^2/m_W^2)$ are even smaller on account of the absence of the color factor.

There are also very small corrections of $O(m_b^2/m_Z^2)$ and $O(m_c^2/m_Z^2)$ associated with the $f_1(r_q)$ contribution to $e^2\Pi_{\gamma\gamma}^{(f)}(0)$ [cf. Eq. (7a)]. Their magnitude is $\approx +5\times10^{-6}$.

Putting all these small $O(m_f^2/m_W^2)$ corrections together, we see that for $m_t = m_Z$, 150, 200, 250 GeV they amount to $(1.4, -0.6, -0.8, -0.9)\times10^{-5}$ in the case of $\Delta\hat{r}_W$ and $(-1.0, +0.2, +0.9, +1.5)\times10^{-5}$ in the case of $\Delta\hat{r}$. On the other hand, $\Delta r = \Delta\hat{r}_W - (\hat{c}^2/s^2)(\Delta\hat{r}_W - \Delta\hat{r})$ and the small corrections are enhanced in the second term, leading to shifts of $(-6.5, +2.1, +5.1, +7.8)\times10^{-5}$ for the same values of m_t.

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