Time-translation noninvariance of the propagator in the $A_0=0$ gauge

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We show that within the framework of stochastic mechanics, the quantization of a free electromagnetic or Yang-Mills field in the temporal gauge can be consistently carried out. The resulting longitudinal component of the photon or gluon propagator is time-translation noninvariant. The exact form of the propagator depends on the additional boundary condition which fully fixes the temporal gauge. This stochastic formalism not only provides a simple and unified derivation of the various forms of the gauge field propagator in the temporal gauge, it also shows that the time-translation noninvariance is an inherent property of the propagator.

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The free gauge field propagator in the temporal gauge obtained by conventional quantization methods is given by

$$D_{ij}(k)=\left[\delta_{ij}-\frac{k_ik_j}{k_0^2}\right]\frac{1}{k^2}\ , \tag{1}$$

with $D_{\mu 0}=0=D_{0\mu}$. The presence of the double pole at $k_0=0$ is usually attributed to the fact that the temporal gauge condition $A_0=0$ does not fix the gauge completely as there still exists a residual time-independent gauge freedom. Prior to the work of Caracciolo, Curci, and Menotti (CCM) [1], it was commonly believed that this singularity is just a gauge artifact and any prescription to the double pole which leads to a finite result is acceptable. However, CCM showed that the widely accepted Cauchy principal-value prescription

$$P(1/k_0^2)=[(k_0+i\varepsilon)^{-2}+(k_0-i\varepsilon)^{-2}]/2$$

[or $D_p(t,t')=-|t-t'|/2$ in configuration space] does not reproduce the Feynman gauge result for a rectangular Wilson loop calculation up to g^4, the fourth order in the coupling constant. The desired result was obtained when they introduced the following time-translation noninvariant longitudinal propagator:

$$D_{ij}^L(x,x')=\frac{\partial_i\partial_j}{\nabla^2}\delta(\mathbf{x}-\mathbf{x}')[-\tfrac{1}{2}|t-t'|\pm\tfrac{1}{2}(t+t')+\gamma]\ , \tag{2}$$

where γ is a constant determined by additional boundary conditions. Subsequently, there appeared various derivations of the CCM propagator or variations of it [2]. These propagators have been shown to give the correct Wilson loop calculations up to g^4 order. However, until now the temporal gauge has not been completely understood, particularly with regards to the relationship between various temporal gauge propagators regularized with different prescriptions, and it has not been determined whether time-translation noninvariance can be avoided.

The main aim of this paper is to give a unified treatment to the gauge field propagator in the temporal gauge based on stochastic mechanics [3]. We also show that the time-translation noninvariance is an intrinsic property of the gauge field propagation in the $A_0=0$ gauge. Since all the essential features of the problem to be discussed are already present at the level of a free electromagnetic field, it suffices to consider the case of a free photon propagator. Our results apply to the free gluon propagator.

The equation of motion for the free electromagnetic field in the temporal gauge is given by

$$\partial_t^2\mathbf{A}+\nabla\times\nabla\times\mathbf{A}=0\ . \tag{3}$$

By decomposing $\mathbf{A}$ into its transverse and longitudinal components

$$A_i^T=(\delta_{ij}-\nabla^{-2}\partial_i\partial_j)A_j\equiv T_{ij}A_j$$

and

$$A_i^L=\nabla^{-2}\partial_i\partial_jA_j\equiv L_{ij}A_j\ ,$$

(3) becomes

$$\partial_t^2\mathbf{A}^T=\nabla^2\mathbf{A}^T\ , \tag{4}$$

$$\partial_t^2\mathbf{A}^L=0\ . \tag{5}$$

Field quantization in stochastic mechanics requires that $t\to\mathbf{A}_t^T$ and $t\to\mathbf{A}_t^L$ each be promoted to become a Markov process. (For a mathematically more rigorous treatment in the case of scalar field see Ref. [4].) Since the usual time derivative ∂_t is not well defined for such processes, it is replaced by the mean forward and backward time derivatives $D_\pm$ defined by

$$D_+\mathbf{A}_t^{T/L}=\lim_{h\downarrow 0}E_t\left\{\frac{\mathbf{A}_{t+h}^{T/L}-\mathbf{A}_t^{T/L}}{h}\right\}\ , \tag{6}$$

$$D_-\mathbf{A}^{T/L}=\lim_{h\downarrow 0}E_t\left\{\frac{\mathbf{A}_t^{T/L}-\mathbf{A}_{t-h}^{T/L}}{h}\right\}\ , \tag{7}$$

where E_t is the conditional expectation with respect to the Markov processes $\mathbf{A}_t^{T/L}$ (i.e., $\mathbf{A}_t^T$ or $\mathbf{A}_t^L$). $\mathbf{b}_\pm(\mathbf{A}_t^{T/L},t)\equiv D_\pm\mathbf{A}_t^{T/L}$ are, respectively, the mean forward and backward velocities (or drifts). To describe the dynamics of $\mathbf{A}_t^{T/L}$, one uses the stochastic acceleration first introduced by Nelson [3]:

$$\mathbf{a}(\mathbf{A}_t^{T/L},t)=\tfrac{1}{2}(D_+D_- + D_-D_+)\mathbf{A}_t^{T/L} . \tag{8}$$

The stochastic versions of (4) and (5) are given by

$$\tfrac{1}{2}(D_+D_- + D_-D_+)\mathbf{A}_t^T=\nabla^2\mathbf{A}_t^T , \tag{9}$$

$$\tfrac{1}{2}(D_+D_- + D_-D_+)\mathbf{A}_t^L=0 . \tag{10}$$

$\mathbf{A}_t^{T/L}$ are also required to satisfy the following stochastic differential equations:

$$d\,\mathbf{A}_t^T=D_+\,\mathbf{A}_t^T dt + d\mathbf{W}_t^T , \tag{11}$$

$$d\,\mathbf{A}_t^L=D_+\,\mathbf{A}_t^L dt + d\mathbf{W}_t^L , \tag{12}$$

where the forward drifts $D_+\,\mathbf{A}_t^T$ and $D_+\,\mathbf{A}_t^L$ can be deduced from (9) and (10): $\mathbf{W}_t^T$ and $\mathbf{W}_t^L$ are Brownian motion processes (or Wiener processes) with zero means and the following covariances:

$$E\{W_i^T(t,\mathbf{x})W_j^T(t',\mathbf{x}')\}=2\nu T_{ij}\delta(\mathbf{x}-\mathbf{x}')C(t,t') , \tag{13}$$

$$E\{W_i^L(t,\mathbf{x})W_j^L(t',\mathbf{x}')\}=2\nu L_{ij}\delta(\mathbf{x}-\mathbf{x}')C(t,t') , \tag{14}$$

with ν the diffusion constant (its value will be taken as $\frac{1}{2}$ in this paper) and $C(t,t')$ given by

$$C(t,t')=\begin{cases}\min(|t|,|t'|) & \text{if } tt'\geq 0 , \\ 0 & \text{if } tt'<0 .\end{cases} \tag{15}$$

Furthermore, $d\mathbf{W}_t^{T/L}$ are, respectively, independent of $\mathbf{A}_t^{T/L}$ for $s\leq t$. One can also employ a backward description for $\mathbf{A}_t^{T/L}$, with forward drifts replaced by backward drifts $D_-\,\mathbf{A}_t^{T/L}$; $d\mathbf{W}_t^{T/L}$ have the same means and covariances except that $d\mathbf{W}_t^{T/L}$ are now, respectively, independent of $\mathbf{A}_t^{T/L}$ for $s\geq t$.

We shall not discuss the solution to (11) as it has been considered previously [5]. We remark that the physical component $\mathbf{A}_t^T$ is an Ornstein-Uhlenbeck process (or an oscillator process) similar to that for a scalar massive field. Equation (10) implies that $D_\pm\,\mathbf{A}_t^L$ are time independent, but from Maxwell equations one has $\partial_t\mathbf{A}^L=-\nabla A_0$, which together with the temporal gauge condition $A_0=0$ implies $D_\pm\,\mathbf{A}_t^L=0$. Now the solution to (12) subjected to the boundary condition $\mathbf{A}^L(t_0,\mathbf{x})=0$ is a Brownian motion process with mean zero and covariance given by the conditional expectation

$$E\{A_i^L(t,\mathbf{x})A_j^L(t',\mathbf{x}')|\,\mathbf{A}_{t_0}^L=0\}=L_{ij}\delta(\mathbf{x}-\mathbf{x}')D(t,t') , \tag{16a}$$

with

$$D(t,t')=\min(|t-t_0|,|t'-t_0|)$$
$$=\tfrac{1}{2}[-|t-t'|+|t-t_0|+|t'-t_0|] . \tag{16b}$$

Just as in the scalar case [6], this covariance coincides with the Euclidean propagator of the longitudinal component of the electromagnetic potential in the temporal gauge with the additional boundary condition $\mathbf{A}^L(t_0,\mathbf{x})=0$. This propagator is of the CCM type and has been derived by Girotti and Rothe [2] and Leroy, Micheli, and Rossi [2] based on the path integral formalism. If the time strip is taken as finite with $T'\leq t,t'\leq T''$, then for $t_0=T'\leq 0$ and $t,t'\leq 0$ one gets exactly the CCM result with $\alpha=+1$ and $\gamma=-T'$. On the other hand, for $t_0=T''\geq 0$ and $t,t'\geq 0$, $\alpha=-1$ and $\gamma=T''$ again agrees with the CCM result. Note that for $t_0=0$ and $tt'\geq 0$, $\gamma=0$ and $\alpha=\pm 1$ correspond to the two independent Brownian motions on the parameter set $(-\infty,0]$ and $[0,\infty)$, respectively. In other words, $\{\mathbf{A}_t^L\}_{-\infty<t<\infty}$ is a two-sided Wiener (or two-sided Brownian motion) process with the pair of independent Brownian motions $\{\mathbf{A}_t^L\}_{-\infty<t\leq 0}$ and $\{\mathbf{A}_t^L\}_{0\leq t<\infty}$ pieced together at $t=0$.

We remark that $\{\mathbf{A}_t^L\}_{t_0\leq t<\infty}$ can be regarded as $\{\mathbf{A}^L\}_{0\leq t<\infty}$ having the same initial condition but with its time origin shifted from $t=0$ to $t=t_0$. When $\mathbf{A}^L\{t_0,\mathbf{x}\}\neq 0$, the mean is then given by this value, but the covariance remains unchanged. In fact, one can easily show that the two processes $\{\mathbf{A}^L\}_{t_0\leq t<\infty}$ and $\{\mathbf{A}'^L\}_{t_0\leq t<\infty}$ with different initial values are related by a time-independent local gauge transformation. Thus the additional boundary condition on $\mathbf{A}_t^L$ fixes the gauge completely and (16) is a general realization of the CCM propagator in the fully fixed temporal gauge.

Next, we consider the effect of the boundary condition on the dynamical description of the Brownian motion. For a Brownian motion process $\mathbf{A}_t^L$ conditioned to start from zero at time t_0, its initial distribution is the Dirac measure at zero [i.e., $\delta(\mathbf{A}_t^L)$]. The probability at time t, $\rho(\mathbf{A}_t^L,t)$, with respect to the Lebesgue measure is given by

$$\rho(\mathbf{A}_t^L,t)=\left[\frac{1}{2\pi(t-t_0)}\right]^{3/2}\exp\left[\frac{-\mathbf{A}_t^L\cdot\mathbf{A}_t^L}{2(t-t_0)}\right] . \tag{17}$$

Now, according to stochastic mechanics, the osmotic velocity $\mathbf{u}$ is given by $\mathbf{u}=\frac{1}{2}\nabla\ln\rho=-\frac{1}{2}\mathbf{A}_t^L/(t-t_0)$. The forward $\mathbf{b}_+$ drift remains zero, and the backward drift becomes $\mathbf{b}_-=-2\mathbf{u}=\mathbf{A}_t^L/(t-t_0)$. The stochastic acceleration is given by $\mathbf{a}=\frac{1}{2}D_+\mathbf{b}_-=-\frac{1}{2}\mathbf{A}_t^L/(t-t_0)^2$.

A heuristic interpretation of the situation is as follows. The Brownian motion is conditioned to start from zero at time t_0 with the application of a very large force to the "free" Brownian particle, which is then released gradually. Here we also note that the backward drift is singular at $t=t_0$. This is consistent with the uncertainty principle since any exact specification of the position of the Brownian particle at $t=t_0$ renders the momentum undefined. One can also view the situation as a special case of a standard stochastic control problem for Markov diffusion processes, where the control is the backward drift in the backward stochastic differential equation for the time-reversed Brownian motion $\mathbf{A}_t^L$ on the time interval $[t_0,T]$:

$$d\,\mathbf{A}_t^L=-\mathbf{b}_-(A_t^L,T-(t-t_0))dt+d\mathbf{W}_t^L . \tag{18}$$

It can be easily verified that the solution of (18) has a covariance which coincides with (16) after taking the limit

$T \to \infty$.

There exists another class of longitudinal propagators obtained by Scholz and Steiner using the path integral method [7]. They considered the field on the time interval $[T_1, T_2]$, with boundary conditions $\mathbf{A}^L(T_1,\mathbf{x}) = \mathbf{A}^L(T_2,\mathbf{x}) = 0$ in addition to $A_0 = 0$. In our present treatment, these boundary conditions require one to consider a process called Brownian bridge (or pinned Brownian motion) [8], which is a Brownian motion conditioned to start from zero at $t = T_1$ and to arrive at zero again at $t = T_2$. We consider first the Brownian bridge on an asymmetric time interval $[0,T]$. Its distribution at time t is given by the density

$$\rho(\mathbf{A}_t^L, t) = \left[\frac{T}{2\pi t(T-t)}\right]^{3/2} \exp\left[\frac{-\mathbf{A}_t^L \cdot \mathbf{A}_t^L T}{2t(T-t)}\right] . \tag{19}$$

One then finds that $\mathbf{b}_+ = -\mathbf{A}_t^L/(T-t)$ and $\mathbf{b}_- = \mathbf{A}_t^L/t$. Substituting $\mathbf{b}_+$ into the forward equation (12) and upon solving gives

$$\mathbf{A}_t^L = (T-t)\int_0^t \frac{d\mathbf{W}_s^L}{T-s} , \quad 0 \le t \le T . \tag{20}$$

One can also consider a Brownian bridge on $[-T, 0]$ in a similar way. Both these processes have zero means and the following time components for their covariances:

$$\begin{aligned} D_a(t,t') &= \min(|t|,|t'|) - \frac{tt'}{T} \\ &= -\frac{1}{2}|t-t'| \pm \frac{1}{2}(t+t') - \frac{tt'}{T} , \end{aligned} \tag{21}$$

where the positive and negative signs correspond, respectively, to $\{\mathbf{A}_t^L\}_{0 \le t \le T}$ and $\{\mathbf{A}_t^T\}_{-T \le t \le 0}$. The same results can be obtained by working with $\mathbf{b}_-$ in the backward equation (18).

The Brownian bridge on the symmetric time interval $[-T/2, T/2]$ and that on $[T_1, T_2]$ with $T_2 - T_1 = T$ can both be regarded as $\{\mathbf{A}_t^L\}_{0 \le t \le T}$ with the time origin shifted accordingly. The time components of their covariances can be easily computed and are, respectively,

$$D_s(t,t') = \frac{1}{2}\left[-|t-t'| - \frac{tt'}{T} + \frac{T}{4}\right] , \tag{22}$$

$$\begin{aligned} D_{T_1,T_2}(t,t') = \frac{1}{2}\Bigg[&-|t-t'| + \frac{T_1+T_2}{T}(t+t') \\ &- \frac{2tt'}{T} - \frac{2T_1T_2}{T}\Bigg] . \end{aligned} \tag{23}$$

These results agree with those of Scholz and Steiner [7]. Note that the $(t+t')$ term does not appear in (22) as it cancels out when one shifts the time origin from $t=0$ (or $t=-T$) to $t=-T/2$ in (21). It is clear from our discussion that the violation of time-translation invariance is not a direct consequence of the boundary conditions as suggested by these authors. In fact, any nonzero values of $\mathbf{A}^L(T_1,\mathbf{x})$ and $\mathbf{A}^L(T_2,\mathbf{x})$ do not alter the covariance of the Brownian bridge.

Another class of one-parameter family of longitudinal propagators obtained by Leroy, Micheli, and Rossi [9] based on finite time perturbation theory has the time component

$$\begin{aligned} D_\rho(t,t') = \tfrac{1}{2}[&-|t-t'| + (1+2\rho)(t+t') \\ &+ (1+2\rho+2\rho^2)T] . \end{aligned} \tag{24}$$

For $\rho = 0$ and -1, (24) gives the CCM propagator (2) with $\gamma = T/2$. When $\rho = -\frac{1}{2}$, (24) becomes $[-|t-t'| + T/2]/2$, which corresponds essentially to the time-translation invariant longitudinal propagator with the principal-value prescription (except for the constant $T/4$). However, such a propagator cannot be the covariance of a Brownian motion process or a Brownian bridge. In fact, the principal-value prescription $D_p(t,t') = -|t-t'|/2$ fails to satisfy the Schwarz inequality

$$|D_p(t,t')| \le \sqrt{D_p(t,t) D_p(t',t')} ,$$

and so it cannot be non-negative definite; hence, it cannot be associated with the covariance of any second order stochastic process. As for $D_{\rho=-1/2}(t,t')$, it can only be non-negative definite provided $|t-t'| < T/2$. If one further imposes the condition that $D_{\rho=-1/2}(t,t') = 0$ for $|t-t'| \ge T/2$, then the associated process is the Poisson increment process, which is neither Gaussian nor Markovian.

Finally, we comment briefly about attempts to obtain prescriptions to the $k_0 = 0$ singularity that preserve time-translation invariance [10]. Such prescriptions are usually implemented at the expense of the temporal gauge condition. For example, in the case of the Mandelstam-Leibbrandt (ML) prescription with $1/k_0^2 \to k_3^2/(k_0 k_3 + i\eta)^2$, which is time-translation invariant, the temporal gauge condition needs to be replaced by the nonlocal condition $A_0 = \eta \nabla \cdot \mathbf{A}/(\nabla^2 \partial_3)$. One also notes that the Euclidean gauge field associated with the propagator having the ML prescription fails to satisfy the Osterwalder-Schrader (OS) reflection positivity condition [11]. This remark applies to all prescriptions that violate the $A_0 = 0$ condition. Since the gauge field in the fully fixed temporal gauge contains only physical degrees of freedom, one expects Euclidean theory to satisfy the OS reflection positivity, which guarantees that the underlying Hilbert space for the corresponding gauge field in Minkowskian space-time has a positive-definite metric. We remark that it has been shown that the Euclidean transverse photon field satisfies the OS reflection positivity [12]. As for the longitudinal field component, one can easily show that it is OS reflection non-negative based on the property of its covariance.

From the above discussion it is clear that as far as the free gauge field is concerned, the field propagator in the temporal gauge obtained by the quantization method based on stochastic mechanics is free from the $k_0 = 0$ singularity and, hence, ambiguities regarding prescriptions. This propagator is intrinsically time-translation noninvariant as its longitudinal component is associated with the covariance of a Brownian motion process which is a nonstationary process. However, time-translation invariance is restored in gauge-invariant quantities as shown in the explicit Wilson loop calculation [1,2]. It needs to be stressed that although stochastic mechanics does provide a first step toward a better understanding of the gauge field in the temporal gauge, more work has to be carried out in order to obtain a more complete theory, in particular to include interactions. One also notes that in the case of a non-Abelian theory, the problem of Faddeev-Popov ghosts decoupling in the fully fixed temporal gauge remains incompletely understood. There exists arguments to show that, at a finite time t_0 with the vanishing of the longitudinal field component, there still remains a coupling between Faddeev-Popov ghosts and the transverse fields. Although the ghosts may finally decouple if the limit $t_0 \to \pm\infty$ is taken, such limits are not allowed in the stochastic formalism. Finally, we mention that the formulation of the temporal gauge in which Gauss' law is recovered through an average over time-independent gauge transformations appears to be a promising alternative approach that needs to be considered. According to this approach, ghosts only appear in the scalar products of gauge-invariant state functionals. One may carry out the stochastic quantization based on the vacuum state functional [13].

Note added in proof. It has come to our knowledge that to properly deal with Markov diffusion processes pinned down at both ends of a fixed time interval (i.e., to deal with two-point boundary value problems for diffusion processes related to stochastic mechanics), one needs to consider reciprocal processes or Bernstein diffusion processes and Bernstein bridges [14]. Zambrini has applied such processes extensively in the formulation of Euclidean quantum mechanics [15].

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